**Review on Probability Space**

**Def.**

A probability space is a triple \((\Omega, \mathcal{F}, P)\) where

(i) \(\Omega\) is a set (called the **sample space**.)

(ii) \(\mathcal{F}\) is the collection of all possible events. Here,

- **Event** is a subset of \(\Omega\), (which we understand as a collection of possible outcomes.)

Technically, \(\mathcal{F}\) is defined as a **\(\sigma\)-algebra** , a mathematical object satisfying

\[
\begin{align*}
1) & \ \emptyset \text{ and } \Omega \in \mathcal{F} \\
2) & \text{ If } A \in \mathcal{F}, \text{ then } A^c = \Omega \setminus A \in \mathcal{F} \\
3) & \text{ If } A_1, A_2, \ldots \in \mathcal{F}, \text{ then } \bigcup_{n=1}^{\infty} A_n \in \mathcal{F}.
\end{align*}
\]

Once \(\mathcal{F}\) is chosen, then members of \(\mathcal{F}\) are called events. In many cases, \(\mathcal{F} = 2^\Omega = \text{power set of } \Omega\), which consists of all subsets of \(\Omega\).

However, due to some technical detail, sometimes different choices of \(\mathcal{F}\) are made.

(iii) \(P\) is a **probability law**, or equivalently, a function \(P: \mathcal{F} \to \mathbb{R}\) such that

\[
\begin{align*}
1) & \ P(\Omega) = 1 \\
2) & \text{ If } A_1, A_2, \ldots \in \mathcal{F} \text{ are disjoint, } \ P\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} P(A_n). \\
3) & \ P(A) \geq 0 \ \text{ for any } A \in \mathcal{F}.
\end{align*}
\]

(In mathematical terms, a probability space is a measure space with total measure \(1\).)

**Def.**

- A **random variable** is a function from \(\Omega\) to \(\mathbb{R}\).
- A r.v. \(X\) is called **discrete** if \(X\) can have at most countably many values.
- For any discrete r.v. \(X\), associate the **probability mass function**, defined by
  \[
  P_X(x) = P(\{\omega \in \Omega : X(\omega) = x\}).
  \]

We use the shorthand notation \(P(X = x)\) to denote the latter probability.
Let $X$ be a discrete r.v. with PMF $P_X$.

1. The expectation of $X$ is defined by $E[X] = \sum_x x P_X(x)$, if the sum exists.
2. The variance of $X$ is defined by $\text{Var}(X) = E[(X - EX)^2]$, if $EX$ exists.

**Prop**
- $g$ : function, then $E[g(X)] = \sum xg(x)P_X(x)$.
- $E[aX+b] = aEX + b$, where $a, b$: constants
- $\text{Var}(X) = E[X^2] - (EX)^2$.

(Exp. & Var. of Geom. dist)

**Book 2.20**

In an advertising campaign, a chocolate factory places golden tickets in their candy bars with prob $p$, independently for each bar. Find the mean & variance of the # of candy bars you eat until you get a golden ticket.

**Sol**

Let $X = \#$ of candy bars to eat. Then $X$ has geometric dist. w/ param $p$. So $P_X(k) = (1-p)k-1 p$, $(k = 1, 2, \ldots)$. Then

$$EX = \sum_{k=1}^\infty k P_X(k) = \sum_{k=1}^\infty k p (1-p)^{k-1} = p \cdot \frac{1}{(1-(1-p))} = \frac{1}{p},$$

and

$$\text{Var}(X) = E[X^2] - (EX)^2,$$

$$E[X^2] = \sum_{k=1}^\infty k^2 P_X(k) = \sum_{k=1}^\infty k^2 p (1-p)^{k-1}$$

$$= \frac{1}{p^2} - \frac{1}{p}.$$

(Interpretation of $EX$)

**Book 2.21**

*(St. Petersburg paradox)* Toss a fair coin independently until the first head appears. If the number is $n$, you are rewarded with $2^n \$. What is the expected amount that you receive?

**Sol**

Let $X = \#$ of coin toss until the 1st head appears. Then $X$ has geometric dist. w/ parameter $\frac{1}{2}$. Then your reward is $2^X$. So
\[ EX = \sum_{k=1}^{\infty} 2^k P_X(k) = \sum_{k=1}^{\infty} 2^k \cdot \frac{1}{2^k} = \infty. \]

**Supp. 2.10**

Let \( N : \Omega \rightarrow \{0, 1, 2, \ldots \} \) be a non-negative integer-valued r.v. Show that

\[ EN = \sum_{n=1}^{\infty} P(N \geq n). \]

**Sol.**

\[ EN = \sum_{k=0}^{\infty} k P(N=k) = \sum_{k=1}^{\infty} \binom{k}{k-1} P(N=k) \]

\[ = \sum_{k=1}^{\infty} P(N=k) \]

\[ = \sum_{n=1}^{\infty} \sum_{k=n}^{\infty} P(N=k) = \sum_{n=1}^{\infty} P(N \geq n). \]

(Binomial, Geometric)

**Supp 2.6**

You are travelling in the rainforest without any insect repellent. As a consequence, at each second a mosquito lands on your neck w/ prob. 0.5. If it lands, it will bite you w/ prob 0.2 and will never bother you w/ prob 0.8.

1) What is the average # of bites in a minute (i.e., 60 seconds)?

2) What is the expected time between successive bites?

**Sol.**

1) \( X = \# \) of bites in a minute has distribution Binomial (60, 0.1). So

\[ EX = \sum_{k=0}^{60} k P_X(k) = \sum_{k=0}^{60} k \cdot \binom{60}{k} (0.1)^k (0.9)^{60-k}. \]

But since \( k \cdot \binom{60}{k} = \frac{60!}{k!(60-k)!} = \frac{60 \cdot 59!}{(k-1)! (69-(k-1))!} = 60 \frac{59!}{(k-1)!} \) for \( k \geq 1, \)

\[ EX = \sum_{k=1}^{60} 60 \cdot (0.1) \cdot \binom{59}{k-1} (0.1)^{k-1} (0.9)^{59-k} \]

\[ = 60 \cdot (0.1) \sum_{k=0}^{59} \frac{59!}{\ell!} (0.1)^\ell (0.9)^{59-\ell} \quad (\ell = k-1) \]

\[ = 60 \cdot (0.1) = 6. \]

(In general, if \( X \sim \text{Binomial}(n, p) \), then \( EX = np \).)
(2) Let \( X = \) time until the next mosquito bite. Understanding \( \text{bite} = \text{success} \), we see that \( X \sim \text{Geometric}(0.5) \). So
\[
\mathsf{E}X = \frac{1}{0.5} = 2.
\]

\textbf{ExHW 5.6}

Consider the problem \#5 of HW1 (The chessboard problem). Let each trajectory be equally likely and let \( X = \# \) of moves until the token first reaches the right edge of the chess board. (Ex:

\[
\implies X = 10,
\]

etc.) Calculate \( \mathsf{E}X \) and \( \text{var}(X) \).

\textbf{Sol.}

Consider the event \( \mathbb{X} = k \). Clearly \( 1 \leq k \leq 14 \). Also, any trajectory in this event must be given as
\[
\left[ \begin{array}{c} 6 \\ \text{(r-1)} \end{array} \right] \quad \left[ \begin{array}{c} \text{(r)} \\ \text{(s)} \end{array} \right] \quad \left[ \begin{array}{c} \text{(s)} \end{array} \right].
\]

So the total \# of such trajectories is \( \binom{k-1}{6} \) and
\[
\mathsf{P}(X = k) = \binom{k-1}{6} \binom{14}{k}.
\]

This allows us to calculate \( \mathsf{E}X \) and \( \text{var}(X) \) via:
\[
\mathsf{E}X = \sum_{k=7}^{14} k \mathsf{P}(X = k),
\]

\[
\text{Var}(X) = \mathsf{E}[X^2] - (\mathsf{E}X)^2 = \sum_{k=7}^{14} k^2 \mathsf{P}(X = k) - (\mathsf{E}X)^2.
\]

We can even make a further simplification using the last problem of HW1:
\[
\binom{k-1}{6} + \binom{m}{m-1} + \binom{n-1}{m-1} = \binom{n}{m}.
\]

For example, \( k \binom{k-1}{6} = k \frac{(k-1)!}{6!(k-1)!} = \frac{k!}{7!}(k-1)! = \frac{k}{7} \binom{k}{7} = \frac{7}{14} \binom{k}{7} \) and
\[
\mathsf{E}X = \sum_{k=7}^{14} \frac{7}{14} \binom{k}{7} = \frac{7}{14} \left( \binom{7}{7} + \binom{8}{7} + \ldots + \binom{14}{7} \right) = \frac{9}{14} \binom{15}{7}.
\]
\[
\frac{15^2}{8^2} = \frac{15}{8} \\
\]

Calculating \( \text{var}(X) \) is much trickier, and you can utilize

\[
k^2\binom{k-1}{6} = (k+1)k\binom{k-1}{6} - k\binom{k-1}{6} \\
= 7 \cdot 8 \cdot \binom{k+1}{8} - 7 \cdot \binom{k}{7}.
\]