\section*{Def.}
- A, B are called \textit{independent} if \( P(A \cap B) = P(A)P(B) \).
- A, B are called \textit{conditionally independent}, given C if \( P(A \cap B | C) = P(A | C)P(B | C) \).
- \( A_1, ..., A_n \) are called \textit{independent} if \( P(A_1 \cup \ldots \cup A_n) = \prod_{i=1}^{n} P(A_i) \) \( \forall S \subseteq \{1, \ldots, n\} \).

\textbf{Remark.} Neither indep. \( \Rightarrow \) cond. indep. nor vice versa.
- Pairwise indep. \( \not\Rightarrow \) indep.

\section*{HW \#3.1}

If \( A, B, C \) are indep., \( \Rightarrow \) \( A^c, B, C \) are indep.

We have to prove that all the following relations hold:

1. \( P(A^c \cap B) = P(A^c)P(B) \),
2. \( P(B \cap C) = P(B)P(C) \),
3. \( P(A \cap C) = P(A)P(C) \),
4. \( P(A^c \cap B \cap C) = P(A^c)P(B)P(C) \).

\( \circ \): follows from the assumption.
- \( \circ \): \[ P(A^c \cap B) = P(B) - P(A \cap B) = P(B) - P(A)P(B) = (1 - P(A))P(B) = P(A^c)P(B) \]
- \( \circ, \circ \): follows in the same way.

\textbf{Remark.} Extending this argument, one can prove that if \( A_1, ..., A_n \) are indep., then any set of events \( A_1, ..., A_n \) with some of \( A_i \) replaced by \( A_i^c \) are also indep.

\section*{HW \#3.3}

(See the HW3 for word description)

\[ \Omega: \quad P_i \xrightarrow{i=1} \Phi: \quad i \xrightarrow{P_i} \quad \text{indep.} \]

What is the prob. that, after 3 jumps they end up on the same lily pad?
Let $X_\Omega = \#$ of left jumps, $X_\phi = \#$ of right jumps.

$X_\Omega$: binomial RV with params 3 and $P_1$.

$X_\phi$: " " with params 3 and $P_2$.

Now let us mark the position of lily pads as

... $G$ $O$ $G$ $G$ $O$ $G$ $G$ $G$ $G$ ...  
-4 -3 -2 -1 0 1 2 3 4

Then after 3 jumps, $\Omega$'s position is $-1 + (3 - 2X_\Omega) = 2 - 2X_\Omega$

$\phi$'s position is $1 + (3 - 2X_\phi) = 4 - 2X_\phi$.

So the possible final positions for $\Omega$ and $\phi$ are

$\Omega$:  
-4 -3 -2 -1 0 1 2 3 4

$\phi$:  
-4 -3 -2 -1 0 1 2 3 4

and there are exactly 3 possible occasions where they end up on the same lily pad. Thus denoting this event by $A$,

$P(A) = \Pr(2 - 2X_\Omega = 4 - 2X_\phi = -2)$

$+ \Pr(2 - 2X_\Omega = 4 - 2X_\phi = 0)$

$+ \Pr(2 - 2X_\Omega = 4 - 2X_\phi = 2)$

$= \Pr(X_\Omega = 2$ and $X_\phi = 3)$

$+ \Pr(X_\Omega = 1$ and $X_\phi = 2)$

$+ \Pr(X_\Omega = 0$ and $X_\phi = 1$)

$= \sum_{k=0}^{2} \binom{3}{k} P_i^k (1-P_1)^{3-k} + \binom{3}{1} P_1 (1-P_2)^{2} P_i^2 (1-P_1)^{3} + \binom{3}{0} (1-P_1)^3 P_2^3 (1-P_2)^{3}$

Or simply $\frac{3}{2} \sum_{k=0}^{2} \binom{3}{k} P_i^k (1-P_1)^{3-k} (3-k) P_2^3 (1-P_2)^{3}$.

Recall $\sum_{k=0}^{n} \binom{n}{k} P^k (1-P)^{n-k} = 1$, or in general, $\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} = (x+y)^n$. 

\[ \]

\[ \]
What is the prob. that the entire system is operational?

\[ P \cdot \left( 1 - (1-p)^2 \right) \cdot \left( 1 - (1-p)^3 \right) \]

\[ = \frac{p \cdot \left( 1 - (1-p)^2 \right) \cdot \left( 1 - (1-p)^3 \right)}{1 - (1-p)^2} \]

\[ = p \cdot \left[ 1 - (1-p) \right] \cdot \left( 1 - P \left( 1 - (1-p)^3 \right) \right) \cdot \left( 1 - (1-p)^2 \right) \]

\[ \therefore P \times \left[ 1 - (1-p) \right] \cdot \left( 1 - P \left( 1 - (1-p)^3 \right) \right) \cdot \left( 1 - (1-p)^2 \right) \]

**HW#3.4**

- n classes offered, for each offered class you enroll it with prob p1,
- for each offered class you buy the textbook with prob p2,

where all the decisions are independent.

**Sol.**

Let \( A_i = \) [the event that you enroll exactly i classes], \(0 \leq i \leq n\)

\( B = \) [you own the textbook for every class enrolled].

Then the events \( A_i \) form a partition of \( \Omega \), and thus

\[ P(B) = \sum_{i=0}^{n} P(B | A_i) P(A_i). \]

Using binomial distribution, \( P(A_i) = \binom{n}{i} p^i (1-p)^{n-i} \). Also, given \( A_i \), the conditional probability that you buy all the textbooks for those i classes is \( P(B | A_i) = p_2^i \). So

\[ P(B) = \sum_{i=0}^{n} \binom{n}{i} (p_2)^i (1-p)^{n-i} = (p_2 + 1-p)^n. \]
\textbf{Book 2.9} \\

\[ X \text{ : binom. with params } n \text{ and } p. \]

\[ k^* = \lfloor (n+1)p \rfloor \]

\[ p_X(k) = \binom{n}{k} p^k (1-p)^{n-k} : \text{PMF for } X. \]

Show that \( \sum \frac{p_X(k)}{\binom{n}{k}} \leq p_X(1) \leq \ldots \leq p_X(k^*) \]

\[ p_X(k^*) > p_X(k^*+1) > \ldots > p_X(n). \]

\textbf{Sol.)} \quad \text{If } p = 0, \quad p_X(k) = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{if } k > 0 \end{cases} \quad \text{and } k^* = 0 \quad \text{trivial!} \]

\text{If } p = 1, \quad p_X(k) = \begin{cases} 0 & \text{if } k < n \\ 1 & \text{if } k = n \end{cases} \quad \text{and } k^* = n+1 \]

\text{If } 0 < p < 1, \text{ then for } 0 \leq k \leq n,

\[ \frac{p_X(k+1)}{p_X(k)} = \frac{\binom{n}{k+1} p^{k+1} (1-p)^{n-k-1}}{\binom{n}{k} p^k (1-p)^{n-k}} = \frac{n-k+1}{k+1} \cdot \frac{p}{1-p}. \]

We find that

\[ p_X(k+1) > p_X(k) \iff \frac{n-k+1}{k+1} \cdot \frac{p}{1-p} > 1 \]

\[ \iff (n-k) p > (k+1)(1-p) \]

\[ \iff (n+1) p > k+1 \]

\[ \iff k^* \geq k+1. \]

This shows that \( p_X(k+1) > p_X(k) \) for \( k^* > k \) and \( p_X(k+1) < p_X(k) \) for \( k^* \leq k \), as desired. \hfill \Box

\textbf{Q. Why we divide into 3 cases?} \\

\textbf{Book 1.38} \\

On a round of golf (18 holes), Telis & Wendy played 10 holes with the score 4 - 6 (in favor of Wendy). Due to an urgent event, they stopped the round and decided to split the stake in proportion to their probability of winning if they completed the round.

Assuming Telis wins a hole with prob. \( p \) & Wendy wins with prob. \( 1-p \),

Compute the proportion of the stake each of them get.
Telis would have won if he wins \( > 5 \) holes out of 8 holes:

\[
P_T = \sum_{k=6}^{8} \binom{8}{k} p^k (1-p)^{8-k}.
\]

Wendy wins if Telis wins \( < 5 \) holes out of 8 holes:

\[
P_W = \sum_{k=0}^{4} \binom{8}{k} p^k (1-p)^{8-k}.
\]

Therefore,

\[
\frac{P_T}{P_T + P_W} \text{ for Telis, } \frac{P_W}{P_T + P_W} \text{ for Wendy.}
\]