Out of the students in a class, 60% : genius, 70% : loves chocolate, 40% : both.

What is the probability that a randomly selected student is neither a genius nor loves chocolate?

Sol)

Let \( G \) = the set of geniuses (in the class).
\( C \) = the set of students who love chocolate.

The probability we want to know is \( P(G^c \cap C^c) \). Using various properties of probability law, together with set operations,

\[
P(G^c \cap C^c) = P(C^c \cup C^c)
\]
\[
= 1 - P(G \cup C)
\]
\[
= 1 - (P(G) + P(C) - P(G \cap C))
\]
\[
= 1 - (0.6 + 0.7 - 0.4) = 0.1.
\]

#1.8

Let \( A, B, C \) be opponents, arranged in the order of play.

Let \( P_A, P_B, P_C \) be probabilities that you win \( A, B, C \), respectively.

Let \( E_A, E_B, E_C \) be the events that you win \( A, B, C \), resp.

In particular, \( P(E_A) = P_A, \ P(E_B) = P_B, \ P(E_C) = P_C \), and clearly we can model that \( E_A, E_B, E_C \) are independent (you are not swayed by other games!).

Then winning two games in a row means that either \( E_A \cap E_B \) occurs or \( E_B \cap E_C \) occurs,

\[
\text{win both} \rightarrow E_B \cap E_C.
\]

\[
A, B, C
\]

\[
\text{win both} \rightarrow E_A \cap E_B
\]

So the event of winning two games in a row is \( (E_A \cap E_B) \cup (E_B \cap E_C) \)
This gives
\[ P(S \cup T) = P(S) + P(T) - P(S \cap T) = \frac{P(A \cap B)}{2} + \frac{P(B \cap C)}{2} - \frac{P(A \cap B \cap C)}{2} \]

We want to maximize this probability. By the observation above, this amounts
to minimizing \( P_A P_C \), which is achieved by choosing \( A \) and \( C \) as strong players.
(Or equivalently choosing \( B \) as the weakest.)

**#1.10** Show that \( P( (A \cap B^c) \cup (B \cap A^c) ) = P(A) + P(B) - 2 P(A \cap B) \).

**Sol.** In view of the Venn diagram,

\[
\begin{align*}
&A \cap B^c \quad B \cap A^c \\
&(A \cap B^c) \cup (B \cap A^c) = (A \cup B) \cap (A \cap B)^c.
\end{align*}
\]

Since \( A \cap B \subseteq A \cup B \),
\[ P( (A \cup B) \cap (A \cap B)^c ) = P( (A \cup B) \cap (A \cap B)^c ) + P( (A \cup B) \cap (A \cap B) ) = P( (A \cup B) ) \]

On the other hand,
\[ P( (A \cup B) ) = P(A) + P(B) - P(A \cap B) \]

Comparing two identities gives the desired result.
HW1.3

Car: model { A, B, C }, engine type { normal, hybrid }, transmission type { manual, auto }, color { 5 types, automatic transmission only }

How many ways can you configure your car?

SOL 1

- When engine type is normal,
  \[ \text{[# of models]} \times \text{[# of transmission types]} \times \text{[# of colors]} = 30. \]
- When engine type is hybrid,
  \[ \text{[# of models]} \times \text{[# of colors]} = 15. \]

Adding them together we have total 45 ways.

HW1.5

You have \( 8 \times 8 \) chessboard

Token at the low left corner.

How many ways of moving this token to the top right corner?

(a) If in each move you are only allowed to move the token one square to the right or one square up? I.e.,

\[ \begin{array}{c}
\text{\# of moves: } \# + \# \\
\text{[Diagonal movement]} \end{array} \]

(b) If you are additionally allowed to move diagonally? I.e.,

\[ \begin{array}{c}
\text{\# of moves: } \# + \# + \# \\
\text{[Diagonal movement]} \end{array} \]

SOL 1

(a) Denoting \( \rightarrow \): one square to the right, then any such trajectory of the token consists of exactly \( 7 \rightarrow 's \) and \( 7 \uparrow 's \). So the total \# of trajectories is \( \binom{14}{7} \).
(D) Denoting the diagonal move as \( \Delta \), the total number of moves in a trajectory is \( 14-k \), where \( k = \text{[# of } \Delta \text{s]} \).

- Also, clearly \( 0 \leq k \leq 7 \) as the maximum possible number of diagonal moves is 7.

- If we fix \( k \in \{0, \ldots, 7\} \), then the number of trajectories with \( k \) \( \Delta \)'s is

\[
\binom{14-k}{7k, 7-k, k},
\]

and hence the total number is

\[
\sum_{k=0}^{7} \binom{14-k}{7k, 7-k, k}.
\]