Review

- **pointwise convergence**: \( \{ f_n \} \rightarrow f \) pointwise on \( E \) if
  \[
  f(x) = \lim_{n \to \infty} f_n(x) \quad \forall x \in E.
  \]

- **uniform convergence**: \( \{ f_n \} \rightarrow f \) uniformly on \( E \) if
  \[
  \sup_{x \in E} |f_n(x) - f(x)| \to 0 \quad \text{as} \quad n \to \infty.
  \]

- **Weierstrass M test**: If \( \exists (M_n) \) s.t. \( \sup_{x \in E} |f_n(x)| \leq M_n \) and \( \sum M_n < \infty \), then \( \sum f_n \) converges uniformly.

- **THM 7.12**: If \( \{ f_n \} \) sequence of cont. func. on \( E \) and \( f_n \to f \) uniformly on \( E \), then \( f \) is also continuous.

- If \( X \) is a metric space, \( C[0,1] \): bounded continuous functions \( X \to \mathbb{R} \).
  Define the supremum norm on \( C[0,1] \) by
  \[
  \|f\|_\infty = \sup_{x \in [0,1]} |f(x)|.
  \]

- \( C[0,1] \) is a metric space. Then \( \{ f_n \} \subseteq C[0,1] \) converges uniformly to \( f \)
  (hence \( f \in C[0,1] \) as well) if and only if \( \{ f_n \} \to f \) under \( \|\cdot\|_\infty \).

- \( C[0,1] \) is a complete metric space.

**Ex 1**

Consider \( f_n(x) = x^n \) on \([0,1]\).

1. Show that \( f_n \to 0 \) pointwise on \([0,1]\).

2. Show that \( f_n \) does not converge uniformly on \([0,1]\).
   (This contrasts with the fact that \( f_n \to 0 \) uniformly on \([0,c]\)
   for any \( 0 < c < 1 \).)

**Sol**

1. Trivial: \( \lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} x^n = 0 \) since \( 0 \leq x < 1 \).

2. Assume otherwise that \( \{ f_n \} \) converges uniformly. Then the limit must be 0. But

   \[
   \| f_n - 0 \| \geq |f_n(1-\frac{1}{n}) - 0| = (1-\frac{1}{n})^n \to \frac{1}{e}
   \]

   as \( n \to \infty \), a contradiction.
Ex 2] Show that \( \sum_{n=1}^{\infty} x^n (1-x) \) does not converge uniformly on \([0,1]\).

**Sol:**

Its pointwise limit can be easily calculated:

\[
\sum_{n=1}^{\infty} x^n (1-x) = \begin{cases} 
\frac{1}{x} & \text{if } 0<x<1, \\
0 & \text{if } x=1.
\end{cases}
\]

Since this sum is the limit of continuous functions and is discontinuous, it cannot be the limit under uniform convergence.

Ex 3]

Show that

1. \( \sum_{n=1}^{\infty} \frac{\cos(nx)}{n^2} \) is continuous on \( \mathbb{R} \) [Hint: establish the uniform convergence]
2. \( \sum_{n=1}^{\infty} \frac{\sin(nx)}{n} \) is continuous on \((0,2\pi)\). [Hint: show that this converges uniformly on any closed interval \([e,2\pi-e]\) for \( e \in (0,\pi)\).]

**Sol: (1)** Since \( \frac{\cos(nx)}{n^2} \) is continuous and \( \left| \frac{\cos(nx)}{n^2} \right| \leq \frac{1}{n^2} \) and \( \sum \frac{1}{n^2} < \infty \), by the Weierstrass M-test, \( \sum \frac{\cos(nx)}{n^2} \) converges uniformly. So this is continuous.

(2) We first prove the following (compare this with Thm 3.42)

**THM (Dirichlet test)** Suppose \((a_n(x))\) and \((b_n(x))\) are bounded this on \(E\), s.t.

(a) The partial sums \( A_n(x) = \sum_{k=0}^{n} a_k(x) \) form a uniformly bounded seq.,
(b) \( b_0(x) \geq b_1(x) \geq b_2(x) \geq \ldots \) for all \( x \in E \),
(c) \( b_n \to 0 \) uniformly on \( E \).

Then \( \sum n a_n(x) b_n(x) \) converges uniformly on \( E \). (Remark: \((a_n(x))\) can be complex-valued.

**Proof:** Choose \( M \) such that \( |A_n(x)| \leq M \) for all \( x \in E \) and \( n \). Given \( \epsilon > 0 \), choose \( N \) s.t. \( b_n(x) \leq \epsilon/2N \) for all \( x \in E \). Then for \( N \leq p \leq N \),
\[
\sum_{n=p}^{q} a_n(x) b_n(x) = \sum_{n=p}^{q-1} A_n b_n - b_{n+1} c_n + A_{n+1} c_n - A_{n+2} c_n \\
\leq M \left( \sum_{n=p}^{q-1} (b_n c_n - b_{n+1} c_n) + b_{n+1} c_n + b_{n+2} c_n \right) \\
= 2M b_p c_n \leq 2M b_N c_n \leq \varepsilon.
\]

By the Cauchy criterion for uniform convergence the claim follows. \(\Box\)

Now set \(a_n(x) = \sin(nx)\) and \(b_n(x) = \frac{1}{n}\). Then
\[(a) : \sum_{k=1}^{n} \sin(kx) = \frac{\sin(\frac{nx}{2}) \sin(\frac{nx+\pi}{2})}{\sin(\frac{x}{2})}\]
is bounded by \(\frac{1}{\sin(\frac{\pi}{2})}\) on \([\varepsilon, \pi - \varepsilon]\).
\[(b), (c)\] is trivial.

Thus by the Dirichlet test, \(\sum_{n=1}^{\infty} \frac{\sin nx}{n}\) converges uniformly on \([\varepsilon, \pi - \varepsilon]\) for any \(\varepsilon \in (0, \pi)\) and hence is continuous. \(\Box\)

**Ex 4**
\[f_n(x) = \frac{1}{x} + \frac{1}{n}\] on \([1, \infty)\) converges uniformly to \(f\), but
\[
\sup_{x \in [1, \infty)} |f_n(x) - f| = \sup_{x \in [1, \infty)} \frac{1}{x} = \infty.
\]
So \(f_n(x)\) does not converge uniformly on \([1, \infty)\).

**Ex 5** (Hw #5)
\[(a) (f_n) : [0, 1] \to \mathbb{R} \text{ converges pointwise to } f \text{ on } [0, 1]\]
\[(b) \int_0^1 |f_n - g|^2 \, dx \to 0 \text{ as } n \to \infty\]
\[(c) f, g \text{ continuous},\]
then \(f = g\).

**Sol**

**Lemma** Under the condition (b), \(\forall \varepsilon > 0\) and \(\forall s > 0\), \(\exists s > 0\) such that
\[
1. |f_n(x) - g(x)| \leq \varepsilon \text{ for } x \in K\]
\[
2. |K| > 1 - \varepsilon.
\]
Proof. \(^{\text{• }}\) Choose \(n \geq N\) such that \(\int_0^1 \vert f_n - g \vert^2 \, dx < \varepsilon^2 \delta.\)

\(^{\text{• }}\) Choose \(P: \text{partition s.t. } U(P, \vert f_n - g \vert^2) < \varepsilon^2 \delta.\)

Let \(P = \{x_0 < x_1 < \ldots < x_m\} \text{ and } I_i = [x_{i-1}, x_i].\)

\[M_i = \sup_{t \in I_i} \vert f_n(t) - g(t) \vert^2.\]

Then we know that

\[
\sum_{i=1}^{m} M_i \vert I_i \vert = U(P, \vert f_n - g \vert^2) < \varepsilon^2 \delta.
\]

Let \(K = \bigcup_{\bar{M} < \varepsilon^2} I_i.\) Since

\[
\varepsilon^2 \delta = \varepsilon^2 \sum_{i: M_i > \varepsilon^2} \vert I_i \vert \leq \sum_{i: M_i > \varepsilon^2} M_i \vert I_i \vert < \varepsilon^2 \delta,
\]

we have \(\vert K \vert < \delta.\) Moreover, by definition of \(K,
\[
\forall x \in K \implies \exists i \text{ s.t. } x \in I_i \text{ and } M_i < \varepsilon^2,
\]

\[
\Rightarrow \vert f_n(x) - g(x) \vert^2 \leq \varepsilon^2,
\]

\[
\Rightarrow \vert f_n(x) - g(x) \vert \leq \varepsilon.
\]

Now assume that \(f \neq g\) so that \(\exists x_0 \in [0, 1]\) s.t. \(f(x_0) \neq g(x_0).\)

Then \(\exists\) a closed interval \(I_0\) such that for \(\varepsilon = \frac{\vert f(x_0) - g(x_0) \vert}{2},\)

\(x_0 \in I_0 \subset [0, 1]\) and \(\vert f(x) - g(x) \vert \geq \varepsilon\) on \(I_0.\)

For each \(j = 1, 2, \ldots,\) pick \((K_j)\) and \(n_1 < n_2 < n_3 < \ldots\) using Lemma s.t.

\[
\delta \vert f_{n_j}(x) - g(x) \vert \leq \frac{\varepsilon}{2^n} \text{ for } x \in K_j,
\]

\[
\vert K_j \vert > 1 - \frac{\varepsilon}{2^n}.
\]

We claim that \(I_0 \cap \bigcup_{j=1}^{\infty} K_j\) is non-empty. Indeed, if this is empty, then by Thm 2.36, \(\exists j_1, \ldots, j_m\) s.t. \(I_0 \cap K_{j_1} \cap \ldots \cap K_{j_m} = \emptyset.\)

This implies \(I_0 \subset K_{j_1}^c \cup \ldots \cup K_{j_m}^c.\) But this would imply

\[
\vert I_0 \vert \leq \vert K_{j_1}^c \vert + \ldots + \vert K_{j_m}^c \vert
\]

\[
< \frac{\varepsilon}{2^n} + \ldots + \frac{\varepsilon}{2^{2m}}
\]

\[
< \sum_{j=1}^{\infty} \frac{\varepsilon}{2^j} = \frac{\varepsilon}{2},
\]

\(\Box.\)
This contradiction shows that \( \exists x^* \in I_0 \cap \bigcap_{j} K_j \). But since
\[
|f_{n_j}(x^*) - g(x^*)| \leq \frac{1}{2^j} \quad \forall j,
\]
we have
\[
\lim_{j \to \infty} f_{n_j}(x^*) = g(x^*).
\]

On the other hand, by (a),
\[
\lim_{j \to \infty} f_{n_j}(x^*) = f(x^*),
\]
This contradicts that \( |f(x^*) - g(x^*)| \geq \varepsilon \).

\[ \text{Ex 6} \text{ (HW #6)} \text{ Let } \psi : \mathbb{R} \to \mathbb{R} \text{ be as follows: } \psi(-1) = 0 \text{ and } \psi' \text{ looks like}
\]
\[
1 \quad k \quad x \sim \text{ of order } 1
\]
\[
\text{area} = \psi(1)
\]
\[-1 \quad -\frac{1}{2} \quad \frac{1}{2} \quad 1
\]

Now define \( \exists f_{n,k} : -n \cdot 2^n \leq k \leq n \cdot 2^n, \quad n=1,2,\ldots \) such that
\[
f_{n,k}(x) = \frac{1}{2^n} \psi(2^n(x - \frac{k}{2^n})).
\]

Then \( f_{n,k}' \) looks like
\[
1 \quad k \quad \text{ of order } 2^n.
\]
\[
\frac{k}{2^n} \quad \frac{k}{2^n} \quad \frac{k}{2^n}
\]

It is clear that, as elements of \( C(\mathbb{R}) \),
1. \( \| f_{n,k} \| \leq \frac{1}{2^n} \psi(1) \).
2. For any \( x \in \mathbb{R} \), \( \exists N \) s.t. whenever \( n \geq N \),
   \( \exists k_0 \) s.t. \( f_{n,k_0}(x) = 1 \) but
   \( \exists k_1 \) s.t. \( f_{n,k_1}(x) = 0 \).

Now rearrange \( (f_{n,k}) \) in lexicographical way to create a single-indexed sequence \( (f_n) \). This serves as a counter-example.