Review

- Properties of integral.
- Two quintessential examples: when $a$ is a pure step & $a'$ is integrable.
- Fundamental theorem of calculus
- Pointwise convergence / Uniform convergence
  - Space $C(X)$ of continuous functions w/ sup norm $\|f\| = \sup_{x \in X} |f(x)|$.
    - $C(X)$ is always complete.
    - $C(X)$ is the space of continuous functions if $X$ is compact.
- Uniform convergence and integration.

Ex

If $f \in R(a)$ on $[a,b]$, then $\forall \varepsilon > 0$, $\exists g \in C([a,b])$ s.t. $\int_a^b |f-g| \, dx < \varepsilon$.

Proof

There are tons of proofs for this, but we do as follows: $\forall \varepsilon > 0$, pick $P = \{a = x_0 < \ldots < x_n = b\}$ s.t. $U(P,f,a) - L(P,f,a) < \frac{\varepsilon}{2}$. Define $g$ as the linear interpolation of points $(x_0, f(x_0), \ldots, (x_n, f(x_n))$.

Then we find that, for $x \in [x_i, x_{i+1}]$,

$$\min \{ f(x_i), f(x_{i+1}) \} \leq g(x) \leq \max \{ f(x_i), f(x_{i+1}) \} \leq M_i$$

$$\Rightarrow |f(x) - g(x)| \leq |f(x) - f(x_i)| + |g(x) - m_i|$$

$$\leq (M_i - m_i) + (M_i - m_i)$$

$$\Rightarrow \int_a^b |f-g| \, dx \leq 2(U(P,f,a) - L(P,f,a)) < \varepsilon.$$
Let $f \in R(a)$ on $[a,b]$ be s.t. $\int_a^b fg \, dx = 0$ for any $g \in C([a,b])$.

Show that $\int_a^b |f| \, dx = 0$.

**Sol.**

- It suffices to show that $\int_a^b f^2 \, dx = 0$.
- $\forall \varepsilon > 0$, use the previous exercise to pick $g \in C([a,b])$ s.t. $\int_a^b |f-g| \, dx < \varepsilon$.
- Let $M$ be a bound for $f$. Then
  
  \[
  |\int_a^b f^2 \, dx| = |\int_a^b f^2 \, dx - \int_a^b fg \, dx| \\
  = |\int_a^b f(f-g) \, dx| \\
  \leq \int_a^b M|f-g| \, dx \\
  < ME.
  \]

Since $\varepsilon$ is arbitrary, take $\varepsilon \downarrow 0$.

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**Ex.**

Prove: (1) $\sum_{k=1}^n \frac{1}{k^s} = \frac{s}{s-1} - \frac{1}{s-1} \frac{1}{n^{s-1}} - s \int_1^n \frac{x-1}{x^{s+1}} \, dx \quad \forall n \geq 1, \forall s \neq 1$.

(2) $\sum_{k=1}^n \frac{1}{k} = 1 + \log n - \int_1^n \frac{x-1}{x^2} \, dx$.

**Sol.**

Let $\varepsilon \in (0,1)$. Then

\[
\sum_{k=1}^n \frac{1}{k^s} = \int_{1-\varepsilon}^{1+\varepsilon} \frac{dx}{x^s} = \left[ \frac{1}{x^{s-1}} \right]_{1-\varepsilon}^{1+\varepsilon} + s \int_{1-\varepsilon}^{1+\varepsilon} \frac{x-1}{x^{s+1}} \, dx \\
= \frac{n}{(n+\varepsilon)^s} + s \int_{1-\varepsilon}^{1+\varepsilon} \frac{x-1}{x^{s+1}} \, dx.
\]

Take $\varepsilon \downarrow 0$, then

\[
\frac{1}{n^{s-1}} + s \int_1^n \frac{x-1}{x^{s+1}} \, dx \\
= \frac{1}{n^{s-1}} - s \int_1^n \frac{x-1}{x^{s+1}} \, dx + s \int_1^n \frac{dx}{x^2}.
\]

Compute the last integral according to $s \neq 1$ or $s = 1$.

**Rmk.**

For $s > 1$, (1) shows that $\zeta(s) = \sum_{n=1}^\infty \frac{1}{n^s} = \frac{s}{s-1} - s \int_1^\infty \frac{x-1}{x^{s+1}} \, dx$.

The latter integral converges even for $s > 0$. 

Ex) For \( u \in R(\alpha) \), define \( \|u\|_2 = \left( \int_a^b |u|^2 \, d\alpha \right)^{1/2} \). \( \forall f, g \in R(\alpha) \),

1. Show that \( \int_a^b |fg| \, d\alpha \leq \|f\|_2 \|g\|_2 \). (Cauchy-Schwarz)
2. Show that \( \|f + g\|_2 \leq \|f\|_2 + \|g\|_2 \). (Triangle Ineq.)

Sol) (1) Standard argument: May assume \( \int_a^b |g|^2 \, d\alpha \neq 0 \), otherwise both sides are zero. Let

\[
I(t) = \int_a^b (|f| + t|g|)^2 \, d\alpha, \quad t \in \mathbb{R}.
\]

Then \( I(t) \geq 0 \) and

\[
I(t) = t^2 \int_a^b |g|^2 \, d\alpha + 2t \int_a^b |f||g| \, d\alpha + \int_a^b |f|^2 \, d\alpha
\]

Is a quadratic polynomial. So it has no distinct real zero and

discriminant = \( \left( \int_a^b |f||g| \, d\alpha \right)^2 - \left( \int_a^b |g|^2 \, d\alpha \right) \left( \int_a^b |f|^2 \, d\alpha \right) \)

is non-positive.

(2) \( \|f + g\|_2^2 = \int_a^b |f + g|^2 \, d\alpha \)
\[
\leq \int_a^b (|f| + |g|)^2 \, d\alpha
\]
\[
= \int_a^b |f|^2 \, d\alpha + 2 \int_a^b |f||g| \, d\alpha + \int_a^b |g|^2 \, d\alpha
\]
\[
\leq \|f\|_2^2 + 2 \|f\|_2 \|g\|_2 + \|g\|_2^2
\]
\[
\leq (\|f\|_2 + \|g\|_2)^2.
\]

Ex) (1) Consider \( f_n(x) = x^n \) on \( [0, 1] \). Then

- \( \lim_{n \to \infty} f_n(x) = 0 \) pointwise,
- \( f_n \to 0 \) uniformly. Indeed,

\[
\|f_n - 0\| \geq |f_n(1 - \frac{1}{n}) - 0| = (1 - \frac{1}{n})^n \to \frac{1}{e}
\]

and thus \( \|f_n - 0\| \to 0 \) as \( n \to \infty \).

(2) Consider the series \( \sum_{n=1}^\infty x^n(1-x) \) on \( x \in [0, 1] \). By direct computation,
\[
\sum_{n=1}^{\infty} x^n (1-x) = \begin{cases} 
0 & \text{for } x \in [0,1) \\
1 & \text{for } x = 1
\end{cases}, \text{ pointwise.}
\]

If the convergence were uniform, then the limit should have been continuous as well, which is not the case. So the convergence is NOT uniform.

**Ex.** (Dini's theorem) Let \( K \) be a compact metric space, \( g, g_n \in C(K) \) be such that:

- \( g_1 \leq g_2 \leq \cdots \) (monotone increasing)
- \( g_n \to g \)
- \( \lim_{n \to \infty} g_n(x) = g(x) \) pointwise.

Show that \( g_n \to g \) uniformly.

**Sol.** \( \forall \varepsilon > 0 \), consider the set

\[
K_n := \{ x \in K : g(x) - g_n(x) \geq \varepsilon \}
\]

For each \( \forall n \geq 1 \). Since \( K_n \) is compact (it is a preimage of the closed set \( [\varepsilon, \infty) \) under the continuous function \( g - g_n \)) and \( K_1 \supseteq K_2 \supseteq K_3 \supseteq \cdots \) by monotonicity of \( (g_n) \). Also, by pointwise convergence,

\[
\bigcap_{n=1}^{\infty} K_n = \emptyset.
\]

By the nested set property, \( K_N = \emptyset \) for some \( N \). Then \( \forall n \geq N \), we have \( \|g - g_n\| \leq \varepsilon \).

**Rem.** We cannot weaken any of the highlighted assumptions:

- Consider \( g_n(x) = \frac{x}{n} \in C(\mathbb{R}) \) and \( g(x) = 1 \).

Then \( g_n, g \in C(\mathbb{R}) \), \( g_n \not\to g \) pointwise but \( \|g_n - g\| = 1 \).

- Consider \( g_n(x) = 1 - x^n \) on \([0,1]\). Then \( g_n \not\to g \) pointwise but \( g_n \to g \) uniformly.

- It can be shown that \( \sum_{n=1}^{\infty} \sin(nx) \) converges \( \forall x \in [-\pi, \pi] \), but the limit is discontinuous.
Ex

(1) Let \( I_n = \int_0^1 (1-x^n) \, dx \). Integration by parts shows \( I_n = \frac{2^n}{2n-1} I_{n-1} \).

Together with \( I_0 = 1 \), we get

\[
\int_0^1 (1-x^n) \, dx = \frac{(2n)(2n-2) \cdots 2}{(2n-1)(2n-3) \cdots 1}.
\]

(2) \( \forall n \geq N, \) notice that for \( x \in [0, \sqrt{N}] \),

\[
\left( n \frac{x^2}{n+1} \right)^{n+1} = \left( 1 - \frac{x^2}{n+1} \right)^n
\]

\[\geq \left( 1 - \frac{n+1}{n} \cdot \frac{x^2}{n+1} \right)^n = \left( 1 - \frac{x^2}{n} \right)^n.\]

By Dini's theorem, on \( [0, \sqrt{N}] \), \( \left( 1 - \frac{x^2}{n} \right)^n \to e^{-x^2} \) uniformly.

(3) Thus we have

\[
\int_0^{\sqrt{N}} e^{-x^2} \, dx = \lim_{n \to \infty} \int_0^{\sqrt{n}} \left( 1 - \frac{x^2}{n} \right)^n \, dx.
\]

On the other hand, since \( \left( 1 - \frac{x^2}{n} \right)^n \leq e^{-x^2} \) on \( [0, \sqrt{n}] \),

\[
\int_0^{\sqrt{n}} \left( 1 - \frac{x^2}{n} \right)^n \, dx \leq \int_0^{\sqrt{n}} e^{-x^2} \, dx \leq \int_0^{\infty} e^{-x^2} \, dx.
\]

Combining,

\[
\int_0^{\sqrt{n}} e^{-x^2} \, dx \leq \liminf_{n \to \infty} \int_0^{\sqrt{n}} \left( 1 - \frac{x^2}{n} \right)^n \, dx
\]

\[\leq \limsup_{n \to \infty} \int_0^{\sqrt{n}} \left( 1 - \frac{x^2}{n} \right)^n \, dx \leq \int_0^{\infty} e^{-x^2} \, dx.
\]

Take \( N \to \infty \) to conclude:

\[
\lim_{n \to \infty} \int_0^{\sqrt{n}} \left( 1 - \frac{x^2}{n} \right)^n \, dx = \int_0^{\infty} e^{-x^2} \, dx.
\]

But since \( \int_0^{\sqrt{n}} \left( 1 - \frac{x^2}{n} \right)^n \, dx = \sqrt{n} \frac{(2n)(2n-2) \cdots 2}{(2n-1)(2n-3) \cdots 1} \), we get

\[
\lim_{n \to \infty} \sqrt{n} \frac{(2n)(2n-2) \cdots 2}{(2n-1)(2n-3) \cdots 1} = \frac{\sqrt{\pi}}{2}.
\]