Integration

Def.
- upper / lower Darboux sum
- (upper / lower) Darboux integral
- Refinement lemma,
- $\text{L}(f, Q) \leq \text{U}(f, P)$, $\text{L}(f) \leq \text{U}(f)$.

Thm.
1. $f : [a, b] \to \mathbb{R}$ integrable $\iff$ $\forall \varepsilon > 0, \exists P \text{ st. } \text{U}(f, P) - \text{L}(f, P) < \varepsilon$
   (with value $\int_a^b f$)
2. $\exists \varepsilon > 0, \exists S > 0 \text{ st. } \text{mesh}(P) < S$
   $\implies \text{U}(f, P) - \text{L}(f, P) < \varepsilon$
3. Riemann integrable (with value $\int_a^b f$), i.e.
   $\forall \varepsilon > 0, \exists S > 0 \text{ st. } \text{mesh}(P) < S$
   $\implies |S - \int_a^b f| < \varepsilon$
   and $\int_a^b f = \int_a^b f$.

Thm.
- monotonic function on $[a, b]$ is integrable.
- continuous $\implies [a, b]$ is integrable.

Thm.
If $f, g : [a, b] \to \mathbb{R}$ are integrable,

Linearity
(i) $\forall \alpha, \beta \in \mathbb{R}, \alpha f + \beta g$ is integrable and $\int_a^b (\alpha f + \beta g) = \alpha \int_a^b f + \beta \int_a^b g$.

Positivity
(ii) If $f \geq 0$ on $[a, b]$, $\int_a^b f \leq \int_a^b g$.
(iii) If $f \geq 0$ on $[a, b]$ and $\int_a^b f = 0$, continuous, then $f \equiv 0$.
(iv) $|\int_a^b f| \leq \int_a^b |f|$

Thm.
If $f : [a, b] \to \mathbb{R}$ and $a < c < b$ is st. $f$ is integrable on $[a, c]$ and $[c, b]$, then $f$ is integrable on $[a, b]$ and

$\int_a^b f = \int_a^c f + \int_c^b f$. 
BK 32.3 \[ f : [a,b] \to \mathbb{R} : \quad f(x) = \begin{cases} x^2, & x \in \mathbb{Q} \\ 0, & \text{otherwise} \end{cases} \]

(a) Find \( L(f) \), \( U(f) \).

(b) Is \( f \) integrable?

\[ \text{Sol} \]

(a) For any \( P = \{a = t_0 < t_1 < \cdots < t_n = b\} \),

\[ \forall k = 1, \ldots, n, \ \exists \ r_k \in [t_{k-1}, t_k] \cap \mathbb{Q} \text{ s.t. } \lim r_k = t_k. \]

Consequently

\[ t_k^2 \geq M(f, [t_{k-1}, t_k]) \geq r_k^2 \quad \forall k, \]

and by taking \( \varepsilon \to 0 \), we get

\[ M(f, [t_{k-1}, t_k]) = t_k^2 = M(x^2, [t_{k-1}, t_k]). \]

This shows that

\[ U(f, P) = U(x^2, P) \quad \forall P : \text{partition} \]

\[ \implies U(f) = U(x^2) = \int_a^b x^2 \, dx = \frac{b^3}{3}. \]

Similarly, we can show that

\[ L(f, P) = L(0, P) = 0 \quad \forall P : \text{partition}. \]

\[ \implies L(f) = 0. \]

(b) Since \( U(f) \neq L(f) \), \( f \) is \textbf{NOT} integrable.

BK 32.6 If \( f : [a,b] \to \mathbb{R} \) is s.t. \( \exists (U_n), (L_n) : \text{upper/lower Darboux sums for } f \)
satisfying \( \lim (U_n - L_n) = 0 \), then \( f \) is integrable and

\[ \int_a^b f = \lim U_n = \lim L_n. \]

\[ \text{Sol} \]

\[ \bullet \text{ f is integrable : } \forall \varepsilon > 0, \text{ choose } n \text{ s.t. } U_n - L_n < \varepsilon. \text{ Write } \]

\[ U_n = U(f, P_n), \quad L_n = L(f, Q_n). \]

Then for \( R_{f} = P_n \cup Q_n : \text{refinement}, \]

\[ \text{ Then for } R_{f} = P_n \cup Q_n, \text{ refinement,} \]
\[ U(f, R_n) - L(f, R_n) \leq U(f, P_n) - L(f, Q_n) \]
\[ = U_n - L_n < \varepsilon. \]

So \( f \) is integrable.

- Since \( L_n \leq L(f) = U(f) \leq U_n \), we get
  \[ |U_n - U(f)| \leq |U_n - L_n| \rightarrow 0 \]
  \[ |L_n - L(f)| \leq |U_n - L_n| \rightarrow 0 \]

and thus \( \int_a^b f = \lim U_n = \lim L_n. \)

\[ \square \]

**Example**

Assume that \( f: [a,b] \rightarrow \mathbb{R} \) is integrable

- \( \forall (c,d) \subset [a,b], \exists \alpha \in (c,d) \) s.t. \( f(\alpha) = 0. \)

(a) Show that \( \int_a^b f = 0. \)

(b) Is \( f \) need to be zero?

(c) If \( f \) is continuous, is \( f \) zero?

**Solution**

(a) \( \forall \) any partition \( P = \{a = t_0 < \ldots < t_n = b\} \), we have

\[ m(f, [t_{k-1}, t_k]) \leq 0 \leq M(f, [t_{k-1}, t_k]) \]

since \( \exists \alpha \in [t_{k-1}, t_k] \) s.t. \( f(\alpha) = 0. \) So

\[ L(f, P) \leq 0 \leq U(f, P) \forall P \]

and hence

\[ L(f) \leq 0 \leq U(f). \]

Since \( L(f) = U(f) \), we get \( L(f) = U(f) = 0. \)

**Example**

(b) No! Consider \( f: [0,1] \rightarrow \mathbb{R} \) given by

\[ f(x) = \begin{cases} 
1 & \text{if } x = \frac{1}{n}, n \in \mathbb{N} \\
0 & \text{otherwise.} 
\end{cases} \]

Then it is easy to see that \( \int_0^1 f = 0 \) but \( f \neq 0. \)
(c) Yes! \( \forall x \in (a, b) \), choose \( c_n \), \( d_n \) such that

\[
\begin{array}{c}
c_n < x < d_n \\
\lim c_n = \lim d_n = x.
\end{array}
\]

Then \( \forall n, \exists x_n \in (c_n, d_n) \) such that \( f(x_n) = 0 \). Now by squeezing lemma,

\[
\lim x_n = \lim c_n = \lim d_n = x.
\]

So by continuity of \( f \),

\[
f(x) = \lim f(x_n) = 0.
\]

Similar argument proves that \( f(a) = f(b) = 0 \) as well.

---

**Ex**

From Ex 33.7, we know that \( f : [a, b] \to \mathbb{R} \) integrable \( \Rightarrow f^2 : [a, b] \to \mathbb{R} \) integrable.

Assume that \( f, g : [a, b] \to \mathbb{R} \) integrable and

\[
\int_a^b f^2 = \int_a^b g^2 = 1.
\]

Show that

\[
\left| \int_a^b fg \right| \leq 1.
\]

**Sol**

Since \( \left| fg \right| \leq \frac{f^2 + g^2}{2} \), we have

\[
\int_a^b \left| fg \right| \leq \frac{1}{2} \left( \int_a^b f^2 + \int_a^b g^2 \right) = 1.
\]

Now the conclusion follows from

\[
\left| \int_a^b fg \right| \leq \int_a^b \left| fg \right|. \]

---

**Ex**

(Fundamental Theorem of Calculus of Variation) If \( f : [a, b] \to \mathbb{R} \) is a continuous function sat.

\[
\int_a^b fg = 0 \quad \forall g : [a, b] \to \mathbb{R} \text{ cont, } g(a) = g(b) = 0.
\]

Then \( f = 0 \) on \([a, b] \).

**Sol**

Pick \( g(x) = (x-a)(b-x) \) for. Then \( fg = (x-a)(b-x)f(x^2) > 0 \) and
\[ \int_a^b fg = 0. \] So \( fg = 0 \) on \([a, b]\) and hence \( f = 0 \) on \((a, b)\).

By continuity, this extends to all of \([a, b]\).

**BK 33.7**

\( f : [a, b] \to \mathbb{R} \) be st. \( |f| \leq B \) on \([a, b]\).

(a) Show that

\[ U(f^2, P) - L(f^2, P) \leq 2B \left( U(f, P) - L(f, P) \right). \]

(b) Show that \( f \) integrable \(\Rightarrow\) \( f^2 \) integrable.

**Sol.**

Let \( P = \{a = t_0 < \cdots < t_n = b\} \) \(\forall k = 1, \ldots, n\), choose \((x_k), (y_k)\) st.

\[
\begin{align*}
\lim_{t \to t_k} f(y_k) &= M(f, [t_k, t_{k+1}]) \\
\lim_{t \to t_k} f(x_k) &= m(f, [t_k, t_{k+1}]).
\end{align*}
\]

Then

\[ f(y_k)^2 - f(x_k)^2 = (f(y_k) + f(x_k))(f(y_k) - f(x_k)) \]

\[ \leq |f(y_k) + f(x_k)| \cdot |f(y_k) - f(x_k)| \]

\[ \leq 2B |f(y_k) - f(x_k)|. \]

But in any cases,

\[ |f(y_k) - f(x_k)| = f(y_k) - f(x_k) \text{ or } f(x_k) - f(y_k) \]

\[ \leq M(f, [t_k, t_{k+1}]) - m(f, [t_k, t_{k+1}]). \]

This shows that, as \( t \to \infty \),

\[ M(f^2, I_k) - m(f^2, I_k) \leq 2B \left( M(f, I_k) - m(f, I_k) \right). \]

Summing over \( k \) gives the inequality.

**Ex.**

Show that \( f(x) = \int_0^1 \frac{dx}{1 + x^2 e^{\beta x}} \) is continuous for \( \beta > 0 \).