**RANDOM REMARKS on HW 1.**

**HW 2.5** Show that $(3+\sqrt{5})^{2/3}$ is not rational.

Remark. Many students attempted the following approach:

Let $r = (3+\sqrt{5})^{2/3}$, $\Rightarrow$ $r^{3/2} = 3 + \sqrt{5}$

$\Rightarrow$ $(r^{3/2} - 3)^2 = 2$

$\Rightarrow$ $r^3 - 6r^{3/2} + 7 = 0$ ...

This is wrong because $r^3 - 6r^{3/2} + 7$ is NOT a polynomial in $r$.

---

**THE SET IR of REAL NUMBERS**

**HW 3.8** Let $a,b \in \mathbb{R}$. If $a \leq b$, $\forall b_i > b$ then $a \leq b$.

Intuition. By the trichotomy of ordering, either $a < b$, $a = b$, $a > b$.

- $a < b$ 
  \[ \text{range of } b_i \]
  \[ a \quad b \text{ } \{ \text{No problem.} \} \]

- $a = b$
  \[ \text{range of } b_i \]
  \[ a = b \text{ } \]

- $a > b$
  \[ \text{range of } b_i \]
  \[ \text{Problem!} \]

**Sol.** Indeed, if $a > b$ then choose $b_i \in (b, a)$. (You may let $b_i = \frac{a + b}{2}$.) Then $b < b_i$ implies $a \leq b_i$, but we also have $a > b_i$. **Not a contradiction!**
§4. The Completeness Axiom

Keywords
- maximum / minimum
- upper bound, bounded above / lower bound, bounded below / bounded
- supremum (least upper bound) / infimum (greatest lower bound)
- completeness axiom
- Archimedean property / denseness of \( \mathbb{R} \).

Ex 4.7

S, T \subseteq \mathbb{R} be non-empty & bounded.

(a) If \( S \subseteq T \), then \( \inf T \leq \inf S \leq \sup S \leq \sup T \).

(b) \( \sup (S \cup T) = \max \{ \sup S, \sup T \} \).

Sol)

(a) \( \inf T \leq \inf S \): [It suffices to prove that \( \inf T \) is a lower bound of \( S \). Indeed,

\[
\begin{align*}
\inf T \leq s & \quad \text{for some } s \in S \\
\Rightarrow \inf T \leq s & \quad \Rightarrow \inf T \leq s
\end{align*}
\]

\( \Rightarrow \inf T \leq s \Rightarrow \inf T \leq s \).]

\( \Rightarrow \inf T \leq \inf S \).

(b) \( \inf S \leq \sup S \): Pick any \( s_0 \in S \). Then

\( \Rightarrow \inf S \leq s_0 \) and \( s_0 \leq \sup S \)

\( \Rightarrow \inf S \leq \sup S \).

(c) \( \sup S \leq \sup T \): Follow the same logic as above.

(b) We prove that \( L = \max \{ \sup S, \sup T \} \) is the least upper bound of \( S \cup T \):

1. \( L \) is an upper bound:

\[
\forall r \in S \cup T \Rightarrow r \leq \sup S \text{ or } r \leq \sup T
\]

\( \Rightarrow r \leq L \).
2. \( L \) is the least upper bound:

\[
\forall M, \quad M \text{ is an upper bound of } S \cup T \\
\Rightarrow \forall r \in S \cup T, \quad r \leq M \\
\Rightarrow \forall s \in S, \quad s \leq M \quad \text{and} \quad \forall t \in T, \quad t \leq M \\
\Rightarrow M \text{ is an upper bound of both } S \text{ and } T \\
\Rightarrow \sup S \leq M, \quad \sup T \leq M \\
\Rightarrow L = \max \{ \sup S, \sup T \} \leq M.
\]

Ex 4.3-4 Find \( \sup \) & \( \inf \) of the following sets and prove why.

(e) \( \{ \frac{1}{n} : n \in \mathbb{N} \} \)

Sol)

Let \( S = \{ \frac{1}{n} : n \in \mathbb{N} \} \). Then we guess

\[
\inf S = 0, \quad \sup S = 1.
\]

- \( \sup S = 1 \) follows from \( \max S = 1 \).
- \( \inf S = 0 \): We prove that 0 is the greatest lower bound.
  1. 0 is a lower bound. Trivial! (\( 0 \leq \frac{1}{n} \quad \forall n \in \mathbb{N} \)).
  2. 0 is the greatest lower bound.

\[ \iff \text{If } M > 0, \text{ then } M \text{ is NOT a lower bound.} \]

We prove the latter statement. If \( M > 0 \), then \( \exists n \in \mathbb{N} \) such that \( Mn > 1 \Rightarrow M > \frac{1}{n} \in S \). So, \( M \) cannot be a lower bound as desired.

Ex 4.15 If \( a \leq b + \frac{1}{n} \ \forall n \Rightarrow a \leq b. \)

Sol)

Assume otherwise. Then \( a > b \Rightarrow \exists n \in \mathbb{N} \) such that \( (a - b)n > 1 \)

\[ \Rightarrow a > b + \frac{1}{n} \quad \neg. \]
Proposition}

Let \( S \subseteq \mathbb{R} \) be nonempty & bounded above. Then TFAE:

(a) \( L = \sup S \)

(b) \( L \) is an upper bound of \( S \) and

\[
\forall \varepsilon > 0, \ \exists s_0 \in S, \ \ L - \varepsilon < s_0.
\]

\[ S \]

---

\( L - \varepsilon \)

---

\( S \)

---

\( L \)

---

Solution

(a) \( \implies \) (b): If \( L = \sup S \), then \( L \) is the least upper bound of \( S \), Consequently,

1. \( L \) is an upper bound of \( S \)

2. \( \forall \varepsilon > 0, \ L - \varepsilon \) is NOT an upper bound of \( S \). So

\( \exists s_0 \in S \) such that \( L - \varepsilon < s_0 \).

(b) \( \implies \) (a): If \( L \) satisfies (b), it suffices to prove that

\( L \) is the least among upper bounds. If \( M < L \), then

for \( \varepsilon = L - M > 0 \), \( M = L - \varepsilon \) and \( \exists s_0 \in S \), \( M < s_0 \). So \( M \) cannot be an upper bound of \( S \).

Exercise

Find \( \sup \) & \( \inf \) of

(a) \( \varepsilon \frac{m}{n} : n \in \mathbb{N}, \ 1 \leq m \leq n^2 \)

(b) \( \varepsilon \frac{mn}{m^2 + n^2} : \ m, n \in \mathbb{N} \)

(c) \( \varepsilon \sin n : \ n \in \mathbb{N} \)

Answer

(a) Supremum = 1, Infimum = 0

(b) Supremum = \( \frac{1}{2} \), Infimum = 0

(c) Supremum = 1, Infimum = -1.
**Prop**

Let \( f : A \times B \rightarrow \mathbb{R} \) be a function such that there exists a \( M \in \mathbb{R} \) satisfying \( f(a, b) \leq M \) for all \( a \in A \), \( b \in B \).

Show that

\[
\sup_{b \in B} \sup_{a \in A} f(a, b) = \sup_{a \in A} \sup_{b \in B} f(a, b) = \sup_{a \in A \times B} f(a, b).
\]

Using shorthand notation,

\[
\sup_{b \in B} \sup_{a \in A} f(a, b) = \sup_{a \in A \times B} f(a, b).
\]