Solution of the 9th Homework

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December 16, 2014

1 Preliminary

1.1 Properties of supremum infimum combined with arithmetic operations

**Lemma 1.1.** Let $A \subset \mathbb{R}$ be a non-empty subset and $c \in \mathbb{R}$. Define $c + A = \{c + a : a \in A\}$. Then
\[
\sup(c + A) = c + (\sup A) \quad \text{and} \quad \inf(c + A) = c + (\inf A).
\]

The proof is quite trivial, so we skip the proof.

**Lemma 1.2.** Let $E \subset \mathbb{R}$ be a non-empty subset and $c \in \mathbb{R}$. Define $cA = \{ca : a \in A\}$. Then
\begin{itemize}
  \item If $c > 0$, then $\sup(cA) = c(\sup A)$ and $\inf(cA) = c(\inf A)$.
  \item If $c < 0$, then $\sup(cA) = c(\inf A)$ and $\inf(cA) = c(\sup A)$.
  \item If $c = 0$, then $\sup(cA) = \inf(cA) = 0$.
\end{itemize}

**Proof.** Let us first assume that $c > 0$. To prove that $\sup(cA) = c(\sup A)$, we claim that
\[
\sup(cA) \leq c(\sup A) \quad \text{and} \quad \sup(cA) \geq c(\sup A).
\]
For the first inequality, let $a' \in cA$ be arbitrary. Then $a' = ca$ for some $a \in A$. But since $a \leq \sup A$, we have $a' = ca \leq c(\sup A)$. This shows that $c(\sup A)$ is an upper bound of $cA$, hence we have $\sup(cA) \leq c(\sup A)$. The reverse inequality also follows in a similar manner. (Or notice that $c \sup A = c(\sup c^{-1}cA) \leq cc^{-1} \sup(cA) = \sup(cA)$.) Then $\inf(cA) = c(\inf A)$ also follows in the same way.

When $c < 0$, the proof goes in almost the same way, but what changes now is that multiplying $c$ to an inequality reverses the order. I leave the detail of the proof to you. \qed

1.2 Refinement of partition

Suppose that a closed bounded interval $[a, b]$ is given. If $P, P' \subset [a, b]$ are partitions such that $P \subset P'$, then we call $P'$ a refinement of $P$. Thus any refinement of $P$ is obtained by adding finitely many points of $[a, b]$. The next lemma shows why this concept is useful in the context of Riemann sum.

**Lemma 1.3.** Let $P, P'$ be partitions of $[a, b]$ and $f : [a, b] \to \mathbb{R}$ be a bounded function. Then
\begin{itemize}
  \item If $P'$ is a refinement of $P$, then $U(f, P') \leq U(f, P)$,
  \item If $P'$ is a refinement of $P$, then $L(f, P') \geq L(f, P)$.
\end{itemize}

In other words, refining a partition makes the upper sum to become smaller and the lower sum to become bigger.
Proof. We only prove the first part, since the second part follows mutatis mutandis. Also let us first consider a very simple case where \( P = \{a, b\} \) consists of only two endpoints and \( P' = \{a = t_0 < \cdots < t_m = b\} \). Then it is easy to observe that, for \( 1 \leq i \leq m \),

\[
M_i := \sup_{x \in [t_{i-1}, t_i]} f(x) \leq \sup_{x \in [a, b]} f(x) =: M.
\]

Indeed, this follows since \( f(x) \leq M \) for any \( a \leq x \leq b \). Then it follows that

\[
U(f, P') = \sum_{i=1}^{m} M_i(t_i - t_{i-1}) \leq \sum_{i=1}^{m} M(t_i - t_{i-1}) = M(b-a) = U(f, P).
\]

This observation readily generalizes to arbitrary partition \( P \) and its refinement \( P' \), but a direct proof may require huge burden of notations. Instead we give a concise demonstration.

Let us write \( P = \{a = x_0 < \cdots < x_n = b\} \). Also we write \( I_i = [x_{i-1}, x_i] \) for simplicity. Then it is easy to observe that \( P' \cap I_i \) is a partition in \( I_i \) which is a refinement of \( \{x_{i-1}, x_i\} \). Consequently,

\[
U(f, P') = \sum_{i=1}^{n} U(f|_{I_i}, P' \cap I_i) \leq \sum_{i=1}^{n} U(f|_{I_i}, \{x_{i-1}, x_i\}) = U(f, P)
\]

and the proof is done. \( \square \)

### 1.3 Riemann integrability condition

**Lemma 1.4.** Let \( f : [a, b] \to \mathbb{R} \) be bounded. Then the followings are equivalent:

(i) \( f \) is Riemann integrable.

(ii) For any \( \varepsilon > 0 \), there exists a partition \( P \) of \([a, b]\) such that \( U(f, P) - L(f, P) < \varepsilon \).

**Proof.** (i) \( \Rightarrow \) (ii): Let \( \varepsilon > 0 \) be arbitrary. Using property of infimum and supremum, pick partitions \( P_1 \) and \( P_2 \) such that

\[
U(f, P_1) < \int_{a}^{b} f + \frac{\varepsilon}{2} \quad \text{and} \quad L(f, P_1) > \int_{a}^{b} f - \frac{\varepsilon}{2}.
\]

Let \( P = P_1 \cup P_2 \) be the common refinement. Then by Lemma 1.3 we also have

\[
U(f, P) < \int_{a}^{b} f + \frac{\varepsilon}{2} \quad \text{and} \quad L(f, P) > \int_{a}^{b} f - \frac{\varepsilon}{2}.
\]
But since $f$ is Riemann integrable, both the upper Riemann integral and the lower Riemann integral coincide, and hence

$$ U(f, P) - L(f, P) < \left( \int_a^b f + \frac{\varepsilon}{2} \right) - \left( \int_a^b f - \frac{\varepsilon}{2} \right) = \varepsilon. $$

(ii) $\implies$ (i): For each $\varepsilon > 0$, pick a partition $P$ satisfying the condition of (ii). Then we have

$$ 0 \leq \left( \int_a^b f - \int_a^b f \right) \leq U(f, P) - L(f, P) < \varepsilon. $$

Now since $\varepsilon$ is arbitrary, taking $\varepsilon \to 0^+$ shows that $\int_a^b f = \int_a^b f$, which implies (i) as desired. \qed

## Solution

### Exercise 1.

(Statement omitted.)

**Remark.** Our general strategy is as follows: suppose that $f : [a, b] \to \mathbb{R}$ is bounded functions. If we can somehow figure out that there exists $I \in \mathbb{R}$ satisfying

$$ I \leq \int_a^b f \quad \text{and} \quad \int_a^b f \leq I, $$

then it follows that

$$ I \leq \int_a^b f \leq \int_a^b f \leq I. $$

Thus all these inequalities boil down to equalities, and we find that (1) $g$ is Riemann integrable and (2) $\int_a^b f = I$. In our actual proofs, our goal is to identify suitable number $I$.

### Solution.

(i) To this end, we show that

$$ \int_a^b f + \int_a^b g \leq \int_a^b (f + g) \quad \text{and} \quad \int_a^b (f + g) \leq \int_a^b f + \int_a^b g. \tag{2.1} $$

In fact, this holds for any bounded function $f, g : [a, b] \to \mathbb{R}$ as we will see from our proof. Once this is proved, then for Riemann integrable functions $f, g : [a, b] \to \mathbb{R}$, we obtain

$$ \int_a^b f + \int_a^b g \leq \int_a^b (f + g) \quad \text{and} \quad \int_a^b (f + g) \leq \int_a^b f + \int_a^b g. $$

Therefore the conclusion follows by the remark.

So it remains to prove (2.1). Let $P, Q$ be any partitions of $[a, b]$. Then $P \cup Q$ is also a partition of $[a, b]$, and thus we can write $P \cup Q = \{a = x_0 < \cdots < x_n = b\}$. Then we have

$$ \sup_{x \in [x_{i-1}, x_i]} (f(x) + g(x)) \leq \left( \sup_{x \in [x_{i-1}, x_i]} f(x) \right) + \left( \sup_{x \in [x_{i-1}, x_i]} g(x) \right). $$

This is a direct consequence of the following fact: for all $x \in [x_{i-1}, x_i]$,

$$ f(x) + g(x) \leq \left( \sup_{x \in [x_{i-1}, x_i]} f(x) \right) + \left( \sup_{x \in [x_{i-1}, x_i]} g(x) \right). $$
With this, we readily observe that
\[
\int_a^b (f + g) \leq U(f + g, P \cup Q)
\]
\[
\leq U(f, P \cup Q) + U(g, P \cup Q)
\]
\[
\leq U(f, P) + U(g, Q).
\]
where at the last inequality we exploited Lemma 1.3. By taking infimum for all \(P\) and for all \(Q\), respectively, we obtain
\[
\int_a^b (f + g) \leq \inf_P U(f, P) + \inf_Q U(g, Q) = \int_a^b f + \int_a^b g.
\]
Then the first inequality of (2.1) follows from this. The second inequality follows exactly in the same way. (All you have to do is to replace suprema by infima, upper sums by lower sums, upper integrals by lower integrals and reverse the order of every inequality.) Therefore the proof if (i) is finished.

(ii) We divide the cases according to the sign of \(c\).
- If \(c > 0\), then we claim that
  \[
c \int_a^b f = \int_a^b cf \quad \text{and} \quad \int_a^b cf = c \int_a^b f.
  \]
  Indeed, the first part of the Lemma 1.2 shows that, with any partition \(P = \{a = x_0 < \cdots < x_n = b\}\),
  \[
  U(cf, P) = \sum_{i=1}^n \left( \sup_{x \in [x_{i-1}, x_i]} cf(x) \right) (x_i - x_{i-1})
  = \sum_{i=1}^n c \left( \sup_{x \in [x_{i-1}, x_i]} f(x) \right) (x_i - x_{i-1}) = cU(f, P).
  \]
  Then by Lemma 1.2 again, it follows that
  \[
  \int_a^b cf = \inf_P U(cf, P) = \inf_P cU(f, P) = c \inf_P U(f, P) = c \int_a^b f.
  \]
The other equality follows in the same way. Thus if \(f\) is Riemann integrable, then by the remark \(cf\) is also Riemann integrable and we obtain the desired equality.
- If \(c < 0\), then we claim that
  \[
c \int_a^b f = \int_a^b cf \quad \text{and} \quad \int_a^b cf = c \int_a^b f.
  \]
  Indeed, the second part of the Lemma 1.2 shows that, with any partition \(P = \{a = x_0 < \cdots < x_n = b\}\),
  \[
  U(cf, P) = \sum_{i=1}^n \left( \sup_{x \in [x_{i-1}, x_i]} cf(x) \right) (x_i - x_{i-1})
  = \sum_{i=1}^n c \left( \inf_{x \in [x_{i-1}, x_i]} f(x) \right) (x_i - x_{i-1}) = cL(f, P).
  \]
  Then by Lemma 1.2 again, it follows that
  \[
  \int_a^b cf = \inf_P U(cf, P) = \inf_P cL(f, P) = c \sup_P L(f, P) = c \int_a^b f.
  \]
The other equality follows in the same way. Thus if \(f\) is Riemann integrable, then by the remark \(cf\) is also Riemann integrable and we obtain the desired equality.

\[\text{[1]}\]Now the reason why we introduced two partitions is clear. We want to take infimum of \(U(f, P)\) and \(U(g, Q)\), respectively. But if \(P\) and \(Q\) are somehow related, then it is not easy to tell whether the corresponding infimum becomes the sum of infima. Thankfully, our observations show that we can indeed decouple \(U(f, P)\) and \(U(g, Q)\).
Finally, if $c = 0$, then there is nothing to prove since $\int_a^b 0 = 0$. (This follows from (vi), which holds independent of this part.)

(iii) Using part (ii), we know that $-f$ is Riemann integrable and $\int_a^b (-f) = -\int_a^b f$. Then by (i) the claim follows.

(iv) If $f \geq 0$, then we have

$$\int_a^b f \geq L(f, [a, b]) = \left( \inf_{x \in [a, b]} f(x) \right) (b - a) \geq 0.$$

(v) Apply (iv) to $f - g \geq 0$ instead. Then by (i), we have

$$\int_a^b f = \int_a^b (f - g) + \int_a^b g \geq \int_a^b g.$$

(vi) If $f$ is a constant function with the common value $c$, then it is straightforward to see that $U(f, P) = c(b-a) = L(f, P)$ for any partition $P$ of $[a, b]$. Upon taking infimum and supremum respectively, it follows that

$$c(b-a) = \int_a^b f = \int_a^b f = c(b-a)$$

and hence the claim follows.

(vii) Let $P$ be a partition of $[c, d]$ and $Q$ be a partition of $[a, b]$. Then $P \cup Q$ is a partition of $[c, d]$ which is a refinement of $P$. If we write

$$P \cup Q = \{ c = x_0 < \cdots < x_p = a < x_{p+1} < \cdots < x_q = b < x_{q+1} < \cdots < x_n = d \},$$

then it follows that

$$U(F, P \cup Q) = \sum_{i=1}^{p} \left( \sup_{x \in [x_{i-1}, x_i]} F(x) \right) \left( x_i - x_{i-1} \right) + \sum_{i=p+1}^{q} \left( \sup_{x \in [x_{i-1}, x_i]} F(x) \right) \left( x_i - x_{i-1} \right) + \sum_{i=q+1}^{n} \left( \sup_{x \in [x_{i-1}, x_i]} F(x) \right) \left( x_i - x_{i-1} \right) = U(f, Q).$$

Since $\int_{c}^{d} F \leq U(F, P \cup Q)$, it follows that $U(f, Q)$ is also bounded below by $\int_{c}^{d} F$ and hence taking infimum over $Q$,

$$\int_c^d F \leq \inf_Q U(f, Q) = \int_a^b f.$$

Similar argument shows that $\int_a^b f \leq \int_c^d F$. So if $f$ is integrable, we obtain

$$\int_a^b f \leq \int_c^d F \leq \int_c^d F \leq \int_a^b f$$

and hence the claim follows.

(viii) For this problem, we utilize the equivalent formulation as in Lemma 1.4. Let $\epsilon > 0$ be arbitrary. Then there exists a partition $P$ on $[a, b]$ such that $U(f, P) - L(f, P) < \epsilon$. We may assume that $P$ contains $c$, otherwise we can replace $P$ by the refinement $P \cup \{ c \}$ with preserving the inequality. Now write

$$P = \{ a = x_0 < \cdots < x_m = c < x_{m+1} < \cdots < x_n = b \}.$$
Using this we can define \( P_1 = \{ a = x_0 < \cdots < x_m = c \} \) and likewise \( P_2 = \{ c = x_m < \cdots < x_n = b \} \). Then it follows that

\[
U(f|_{[a,c]}, P_1) - L(f|_{[a,c]}, P_1) = \sum_{i=1}^{m} \left( \sup_{[x_{i-1}, x_i]} f - \inf_{[x_{i-1}, x_i]} f \right) (x_i - x_{i-1}) \leq \sum_{i=1}^{n} \left( \sup_{[x_{i-1}, x_i]} f - \inf_{[x_{i-1}, x_i]} f \right) (x_i - x_{i-1}) = U(f, P) - L(f, P) < \varepsilon.
\]

Similar argument shows that \( U(f|_{[c,b]}, P_2) - L(f|_{[c,b]}, P_2) < \varepsilon \) as well. Therefore by Lemma 1.4 both \( f|_{[a,c]} \) and \( f|_{[c,b]} \) are integrable. Also, using the same setting as before, we find that

\[
\int_a^c f|_{[a,c]} + \int_c^b f|_{[c,b]} \leq U(f|_{[a,c]}, P_1) + U(f|_{[c,b]}, P_2) = U(f, P) \leq L(f, P) + \varepsilon \leq \left( \int_a^b f \right) + \varepsilon.
\]

Similar argument shows that

\[
\int_a^c f|_{[a,c]} + \int_c^b f|_{[c,b]} \geq \left( \int_a^b f \right) - \varepsilon.
\]

Thus by taking \( \varepsilon \to 0^+ \) we obtain the equality

\[
\int_a^b f = \int_a^c f|_{[a,c]} + \int_c^b f|_{[c,b]}
\]

as desired. \( \square \)

**Exercise 2.** Let \( a < b \) be real numbers. Let \( f : [a,b] \to \mathbb{R} \) be a bounded function. Let \( c \in [a,b] \). Assume that, for each \( \delta > 0 \), we know that \( f \) is Riemann integrable on the set \( \{ x \in [a,b] : |x - L| < \delta \} \). Then \( f \) is Riemann integrable on \([a,b]\).

**Solution.** For the simplicity of our proof, let us assume that \( a < c < b \). For the exceptional cases \( c = a \) and \( c = b \), only a minor modification is needed, so we only focus on the case \( a < c < b \).

We use Lemma 1.4 for the proof. Choose a bound \( M > 0 \) of our function \( f \). Let \( \varepsilon > 0 \) be arbitrary. Now we pick \( \delta \) as follows:

\[
\delta = \min \left\{ \frac{\varepsilon}{2014M}, \frac{c-a}{2}, \frac{b-c}{2} \right\} > 0.
\]

By the assumption, we know that \( f \) is Riemann integrable on the set \( \{ x \in [a,b] : |x - c| \geq \delta \} \). Note that we can write

\[
\{ x \in [a,b] : |x - c| \geq \delta \} = I_1 \cup I_2,
\]

where \( I_1 = [a,c-\delta] \) and \( I_2 = [c+\delta, b] \) are disjoint intervals. Then by invoking Lemma 1.4 for each \( i = 1,2 \) we can find a partition \( P_i \) of \( I_i \) such that

\[
U(f|_{I_i}, P_i) - L(f|_{I_i}, P_i) < \frac{\varepsilon}{2014}.
\]

Now let \( P = P_1 \cup P_2 \cup [a,b] \). Then \( P \) is a partition of \([a,b]\) and we have

\[
U(f, P) - L(f, P) \leq \sum_{i=1}^{2} |U(f|_{I_i}, P_i) - L(f|_{I_i}, P_i)| + \left( \sup_{[c-\delta, c+\delta]} f - \inf_{[c-\delta, c+\delta]} f \right) 2\delta \\
\leq \frac{\varepsilon}{2014} + \frac{\varepsilon}{2014} + 2M\delta \leq \frac{4\varepsilon}{2014} < \varepsilon.
\]

Therefore \( f \) is integrable. \( \square \)

**Exercise 3.** Find a function \( f : [0,1] \to \mathbb{R} \) such that \( f \) is not Riemann integrable on \([0,1]\), but such that \(|f|\) is Riemann integrable on \([0,1]\).
Solution. Define our \( f \) by

\[
 f(x) = \begin{cases} 
 1, & x \in \mathbb{Q} \cap [0,1] \\
 -1, & \text{otherwise.} 
\end{cases}
\]

Since any interval of positive length contains both rational numbers and irrational numbers, for any partition \( P = \{a = x_0, \cdots < x_n = b\} \) we have

\[
 \sup_{x \in [x_{i-1},x_i]} f(x) = 1 \quad \text{and} \quad \inf_{x \in [x_{i-1},x_i]} f(x) = -1.
\]

Consequently it follows that \( U(f,P) = 1 \) and \( L(f,P) = -1 \) for any partition \( P \), hence we have

\[
 \int_0^1 f = 1 - (-1) = \int_0^1 f.
\]

On the other hand, \( |f| = 1 \) and hence is Riemann integrable on \([0,1]\). \( \square \)

**Exercise 4.** Let \( a < b \) be real numbers. Let \( f : [a,b] \to \mathbb{R} \) be a bounded function. So, there exists a real number \( M \) such that \( |f(x)| \leq M \) for all \( x \in [a,b] \). Let \( P \) be a partition of \([a,b]\).

(i) Using the identity \( \alpha^2 - \beta^2 = (\alpha - \beta)(\alpha + \beta) \), where \( \alpha, \beta \in \mathbb{R} \), show that

\[
 U(f^2,P) - L(f^2,P) \leq 2M(U(f,P) - L(f,P)).
\]

(ii) Show that if \( f \) is Riemann integrable on \([a,b]\), then \( f^2 \) is also Riemann integrable on \([a,b]\).

(iii) Let \( f, g : [a,b] \to \mathbb{R} \) be Riemann integrable functions on \([a,b]\). Using the identity \( 4\alpha \beta = (\alpha + \beta)^2 - (\alpha - \beta)^2 \), where \( \alpha, \beta \in \mathbb{R} \), show that \( fg \) is Riemann integrable on \([a,b]\).

**Solution.** (i) Let \( P = \{a = x_0 < \cdots < x_n = b\} \). Then it suffices to prove that, for each \( 1 \leq i \leq n \),

\[
 \left( \sup_{[x_{i-1},x_i]} f^2 - \inf_{[x_{i-1},x_i]} f \right) \leq 2M \left( \sup_{[x_{i-1},x_i]} f - \inf_{[x_{i-1},x_i]} f \right).
\]

Indeed, multiplying \( x_i - x_{i-1} \) to both sides and summing over \( i \) yields the desired inequality. So let us focus on (2.2). Let \( x, y \in [x_{i-1},x_i] \) be arbitrary. Then

\[
 f(y)^2 - f(x)^2 = (f(y) + f(x))(f(y) - f(x)) \leq |f(y) + f(x)||f(y) - f(x)|.
\]

But since \( |f(y) + f(x)| \leq |f(y)| + |f(x)| \leq 2M \) and

\[
 |f(y) - f(x)| \leq \left( \sup_{[x_{i-1},x_i]} f - \inf_{[x_{i-1},x_i]} f \right),
\]

we obtain the following inequality:

\[
 f(y)^2 - f(x)^2 \leq 2M \left( \sup_{[x_{i-1},x_i]} f - \inf_{[x_{i-1},x_i]} f \right).
\]

Thus by taking sup over \( y \in [x_{i-1},x_i] \) and inf over \( x \in [x_{i-1},x_i] \), we obtain (2.2) as desired.

(ii) For any \( \varepsilon > 0 \), use Lemma [1.4] to choose a partition \( P \) of \([a,b]\) such that \( U(f,P) - L(f,P) < \varepsilon/(2M + 1) \). Then by (i), we have

\[
 U(f^2,P) - L(f^2,P) \leq 2M(U(f,P) - L(f,P)) < \varepsilon.
\]

This proves that \( f \) is also Riemann integrable.

(iii) By the previous part, we know that both \( (f + g)^2 \) and \( (f - g)^2 \) are Riemann integrable. Thus by Exercise 9.1, their linear combination

\[
 fg = \frac{1}{4}(f + g)^2 - \frac{1}{4}(f - g)^2
\]
is also Riemann integrable.

\[ \delta > 0 \]

This shows that \( F \) defines a differentiable function such that \( \int_0^1 f = 0 \). Prove that \( f(x) = 0 \) for all \( x \in [0,1] \).

**Exercise 5.** Let \( f : [0,1] \to [0,\infty) \) be a continuous function such that \( \int_0^1 f = 0 \). Prove that \( f(x) = 0 \) for all \( x \in [0,1] \).

**Solution.** Assume otherwise. Then there exists \( c \in [a,b] \) such that \( f(c) > 0 \). Then by continuity, with \( \varepsilon = f(c)/2 \), we can find \( \delta > 0 \) such that \( |f(x) - f(c)| < \varepsilon \) whenever \( x \in [a,b] \) and \( |x - c| < \delta \). Let \( p = \max\{a,c - \delta/2\} \) and \( q = \min\{b,c + \delta/2\} \). Then it follows that

\[ x \in [p,q] \implies x \in [a,b] \text{ and } |x - c| < \delta \]
\[ \implies |f(x) - f(c)| < \varepsilon = \frac{1}{2} f(c) \]
\[ \implies f(x) > \frac{1}{2} f(c) = \varepsilon. \]

Now let \( P = \{a,p,q,b\} \) be a partition of \([a,b]\). Then we find that

\[ 0 = \int_a^b f \geq L(f,P) \geq \varepsilon(q-p) > 0, \]

a contradiction! Therefore we are done.

**Another proof.** Here is another proof which is in some sense a sledgehammer method. Since \( f \) is continuous, \( F(x) = \int_a^x f \) defines a differentiable function such that \( F' = f \) on \([a,b]\). Moreover, since \( f \) is non-negative, for \( a \leq x \leq b \) we have

\[ 0 \leq F(x) \leq \int_a^x f + \int_x^b f = \int_a^b f = 0. \]

This shows that \( F \) is identically zero, hence \( F' = 0 \) as well.

**Exercise 6.** The following exercise deals with metric properties of the space of Riemann integrable functions.

(i) Let \( \alpha, \beta \) be real numbers. Prove that \( \alpha \beta \leq (\alpha^2 + \beta^2)/2 \). Now, let \( a < b \) be real numbers, and let \( f, g : [a,b] \to \mathbb{R} \) be two Riemann integrable functions. Assume that \( \int_a^b f^2 = 1 \) and \( \int_a^b g^2 = 1 \). (Recall that since \( f, g \) are Riemann integrable, we know that \( f^2, g^2 \) and \( fg \) are also Riemann integrable by Exercise 4.) Prove that

\[ \int_a^b fg \leq 1. \]

(ii) Let \( a < b \) be real numbers, and let \( f, g : [a,b] \to \mathbb{R} \) be two Riemann integrable functions. Prove the Cauchy-Schwarz inequality:

\[ \left| \int_a^b fg \right| \leq \left( \int_a^b f^2 \right)^{1/2} \left( \int_a^b g^2 \right)^{1/2} \]

(iii) Let \( a < b \) be real numbers, and let \( f, g, h : [a,b] \to \mathbb{R} \) be Riemann integrable functions. Define

\[ d(f,g) := \left( \int_a^b (f - g)^2 \right)^{1/2} \]

Prove the triangle inequality for \( d \). That is, show that

\[ d(f,g) \leq d(f,h) + d(h,g). \]

**Solution.** (i) By expanding the trivial inequality \((\alpha - \beta)^2/2 \geq 0\) we get the desired inequality. So if \( \int_a^b f^2 = 1 \) and
\( \int_a^b g^2 = 1 \), then
\[
\int_a^b f g \leq \int_a^b \frac{f^2 + g^2}{2} = 1.
\]

(ii) Let us first consider the case where \( \int_a^b f^2 > 0 \) and \( \int_a^b g^2 > 0 \). Plugging

\[
\bar{f} = f / \left( \int_a^b f^2 \right)^{1/2} \quad \text{and} \quad \bar{g} = g / \left( \int_a^b g^2 \right)^{1/2},
\]

it follows that \( \int_a^b \bar{f}^2 = 1 \) and \( \int_a^b \bar{g}^2 = 1 \). So by the previous exercise, we get
\[
\int_a^b f g = \left( \int_a^b \bar{f} \bar{g} \right) \left( \int_a^b f^2 \right)^{1/2} \left( \int_a^b g^2 \right)^{1/2} \leq \left( \int_a^b f^2 \right)^{1/2} \left( \int_a^b g^2 \right)^{1/2}.
\]

This almost proves the part (ii) but we need more works:

- We claim that (2.3) holds for any cases. To this end we have to handle the exceptional cases where \( \int_a^b f^2 = 0 \) or \( \int_a^b g^2 = 0 \). Without losing the generality, we may assume that \( \int_a^b f^2 = 0 \). Let \( \varepsilon > 0 \) be arbitrary. Then by the following inequality
\[
fg = (\varepsilon^{-1} f)(\varepsilon g) \leq \frac{(\varepsilon^{-1} f)^2 + (\varepsilon g)^2}{2},
\]

it follows that
\[
\int_a^b f g \leq \int_a^b \frac{(\varepsilon^{-1} f)^2 + (\varepsilon g)^2}{2} = \frac{\varepsilon^2}{2} \int_a^b g^2.
\]

But since \( \varepsilon \) is arbitrary, sending \( \varepsilon \to 0^+ \) we have \( \int_a^b f g \leq 0 \). This shows that (2.3) indeed holds for any cases.

- What we actually want to prove is not (2.3), but rather the inequality with the left-hand side of (2.3) replaced by its absolute value, as in the statement of (ii). To this end, let us plug \( -f \) instead of \( f \) into (2.3). (This is possible since we have proved that (2.3) holds for any Riemann integrable functions \( f, g : [a, b] \to \mathbb{R} \).)

Then
\[
- \int_a^b f g = \int_a^b (-f)g \leq \left( \int_a^b (-f)^2 \right)^{1/2} \left( \int_a^b g^2 \right)^{1/2} = \left( \int_a^b f^2 \right)^{1/2} \left( \int_a^b g^2 \right)^{1/2}.
\]

Therefore this inequality and (2.3) yield the desired result.

(iii) Let us consider \( d(f, g)^2 \) instead. With a bit of algebra, we have
\[
d(f, g)^2 = \int_a^b (f - g)^2 = \int_a^b (f - h + h - g)^2 = \int_a^b (f - h)^2 + 2 \int_a^b (f - h)(h - g) + \int_a^b (h - g)^2.
\]

Then by the Cauchy-Schwarz inequality,
\[
d(f, g)^2 \leq \int_a^b (f - h)^2 + 2 \left( \int_a^b (f - h)^2 \right)^{1/2} \left( \int_a^b (h - g)^2 \right)^{1/2} + \int_a^b (h - g)^2
\]
\[
= d(f, h)^2 + 2d(f, h)d(h, g) + d(h, g)^2 = d(f, h) + d(h, g). \]

Taking square root to both sides we get the desired inequality. \( \Box \)