Solution of the 7th Homework

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1 Preliminary

In this section we deal with some facts that are relevant to our problems but can be coped with only previous materials.

1.1 Maximum and Minimum of subsets of $\mathbb{R}$

Let $E$ be a non-empty subset of $\mathbb{R}$. If there is an element $M \in E$ such that $x \leq M$ for any element $x \in E$, we call $M$ the maximum of $E$ and denote $M = \max E$. Similarly, if there is an element $m \in E$ such that $x \geq m$ for any element $x \in E$, we call $m$ the minimum of $E$ and denote $m = \min E$.

This concept is quite close to that of supremum and infimum, but the difference is that maximum and minimum need not always exist.

**Proposition 1.1.** Let $E \subset \mathbb{R}$. Then
- $E$ have at most one maximum and at most one minimum.
- If $\max E$ exists, then $\max E = \sup E$.
- If $\min E$ exists, then $\min E = \sup E$.

**Remark.** The first assertion justifies our notation as well as our usage of the definite article ‘the’.

**Proof.** Suppose that $M, M'$ are maximums of $E$. Since $M, M' \in E$, we must have $M' \leq M$ and $M \leq M'$. This implies $M = M'$ and hence there cannot exist two or more maximums. The same argument applies for the uniqueness of minimum.

Now assume that $M = \max E$ exists. Then $M$ is an upper bound of $M$. Moreover, $M$ is also the least upper bound since any $x < M$ cannot be an upper bound of $E$. Therefore $M = \sup E$.

The third assertion follows in exactly the same manner. □

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**Exercise 1.** Let $(a_n)_{n=m}^{\infty}$ be a sequence of real numbers converging to 0. Show that $(|a_n|)_{n=m}^{\infty}$ also converges to zero.

**Solution.** By definition, for any $\varepsilon > 0$ there exists $N \geq m$ such that whenever $n \geq N$ we have $|a_n - 0| \leq \varepsilon$. But since $|a_n - 0| = |a_n| = ||a_n| - 0|$, we also have $||a_n| - 0| < \varepsilon$. By reading out this result using definition again, we have $\lim_{n \to \infty} |a_n| = 0$ as desired. □
**Exercise 2.** Let $a < b$ be real numbers. Let $I$ be any of the four intervals $(a, b)$, $(a, b]$, $[a, b)$ or $[a, b]$. Then the closure of $I$ is $[a, b]$.

**Solution.** Let $\bar{I}$ denote the closure of $I$. We prove $\bar{I} = [a, b]$ by showing that a real number $x$ lies in $\bar{I}$ exactly when $x \in [a, b]$. To this end, we divide the case according to whether $x$ lies inside $I$ or not.

- Suppose that $x \in (a, b)$. Then $x \in I$ and $x$ is an adherent point of $I$. Thus $x \in \bar{I}$.
- Suppose that $x$ is either $a$ or $b$. Let us first consider the case where $x = a$. Then for any $\varepsilon > 0$, there exists $y$ such that $a < y < \min\{a + \varepsilon, b\}$. Then $y \in I$ and $|a - y| < \varepsilon$. This proves that $a \in \bar{I}$. The proof of $b \in \bar{I}$ is quite the same.
- Suppose that $x \notin [a, b]$. That is, either $x < a$ or $x > b$. Let us examine the case $x < a$ first. Then for $\varepsilon = a - x$, we find that there is no $y \in I$ satisfying $|x - y| < \varepsilon$. Indeed, for any $y \in I$ we have $y \geq a > x$ and $|y - x| = y - x \geq a - x = \varepsilon$.

This shows that $x \notin \bar{I}$. The case $x > b$ can be treated in a similar way, proving that $x \notin \bar{I}$.

Therefore $\bar{I} = [a, b]$. □

**Exercise 3.** Let $X$ be a subset of $\mathbb{R}$, let $f : X \to \mathbb{R}$ be a function, let $E$ be a subset of $X$, let $x_0$ be an adherent point of $E$, and let $L$ be a real number. Then the following two statements are equivalent. (That is, one statement is true if and only if the other statement is true.)

(i) $f$ converges to $L$ at $x_0$ in $E$.

(ii) For every sequence $(a_n)_{n=0}^{\infty}$ in $E$, and which converges to $x_0$, the sequence $(f(a_n))_{n=0}^{\infty}$ converges to $L$.

**Solution.** (i) $\implies$ (ii) : Let $(a_n)_{n=0}^{\infty}$ be any sequence in $E$ that converges to $x_0$. To prove that $\lim_{n \to \infty} f(a_n) = L$, let $\varepsilon > 0$ be arbitrary. Then

- Using Definition 2.14, pick $\delta = \delta(\varepsilon) > 0$ such that for any $x \in E$ with $|x - x_0| < \delta$ we have $|f(x) - L| < \varepsilon$.
- Using the definition of convergence of sequence, pick $N = N(\delta)$ such that for any $n \geq N$ we have $|a_n - x_0| < \delta$.

Combining these two facts, we find that $|f(a_n) - L| < \varepsilon$ holds whenever $n \geq N$. Therefore $(f(a_n))_{n=0}^{\infty}$ converges to $L$.

(ii) $\implies$ (i) : We prove the contrapositive. Assume that $f$ does not converge to $L$ as $x \to x_0$ in $E$. By negating Definition 2.14, we find that

- There exists $\varepsilon > 0$ such that for any $\delta > 0$, there exists $x \in E$ such that $|x - x_0| < \delta$ but $|f(x) - L| \geq \varepsilon$.

Now for each particular choice $\delta = n^{-1}$ (where $n \in \mathbb{N}$), we utilize this statement to pick some $x = a_n \in E$ such that $|a_n - x_0| < \delta = n^{-1}$ but $|f(a_n) - L| \geq \varepsilon$.

On the one hand, by this construction we clearly have $\lim_{n \to \infty} a_n = x_0$. (Just apply the squeezing theorem to $x_0 - n^{-1} < a_n < x_0 + n^{-1}$.) On the other hand, $(f(a_n))_{n=0}^{\infty}$ cannot converge to $L$. Indeed, assume otherwise so that $f(a_n)$ converges to $L$. Then there exists $N$ such that whenever $n \geq N$ we have $|f(a_n) - L| < \frac{1}{2014} \varepsilon$. But since $|f(a_n) - L| \geq \varepsilon$ always holds, we must have $0 < \varepsilon < \frac{1}{2014} \varepsilon$, a contradiction! This completes the proof. □

**Exercise 4.** Let $X$ be a subset of $\mathbb{R}$, let $f : X \to \mathbb{R}$ be a function, let $E$ be a subset of $X$, let $x_0$ be an adherent point of $E$, and let $L$ be a real number, and let $\delta$ be a positive real number. Then the following two statements are equivalent:

(i) $\lim_{x \to x_0; x \in E} f(x) = L$. 

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(ii) \( \lim_{x \to x_0 \in E \cap (x_0 - \delta, x_0 + \delta)} f(x) = L. \)

**Solution.** For the simplicity of notation, let us denote \( E_\delta = E \cap (x_0 - \delta, x_0 + \delta). \)

(i) \( \iff \) (ii): This direction is almost trivial. Assume that \( \lim_{x \to x_0 \in E} f(x) = L. \) Let \( \varepsilon > 0 \) be arbitrary. Then there exists \( \eta > 0 \) such that \([1]\) for any \( x \in E \) with \( |x - x_0| < \eta \) we have \( |f(x) - L| < \varepsilon. \) So if \( x \in E_\delta \) and \( |x - x_0| < \eta, \) then we have \( x \in E \) and hence \( |f(x) - L| < \varepsilon. \) From this we read out that \( \lim_{x \to x_0 \in E_\delta} f(x) = L. \)

(ii) \( \implies \) (i): Assume that \( \lim_{x \to x_0 \in E_\delta} f(x) = L. \) That is, for any \( \varepsilon > 0, \) there exists \( \eta > 0 \) such that whenever \( x \in E_\delta \) and \( |x - x_0| < \eta, \) we have \( |f(x) - L| < \varepsilon. \) To complete the proof, let \( \eta' = \min(\eta, \delta). \) Then whenever \( x \in E \) and \( |x - x_0| < \eta', \) we have both \( x \in E_\delta \) and \( |x - x_0| < \eta. \) Then \( |f(x) - L| < \varepsilon. \) From this we read out that \( \lim_{x \to x_0 \in E} f(x) = L. \)

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**Exercise 5.** Let \( X \) be a subset of \( \mathbb{R}, \) let \( f : X \to \mathbb{R} \) be a function, and let \( x_0 \in X. \) Then the following three statements are equivalent.

(i) \( f \) is continuous at \( x_0. \)

(ii) For every sequence \( (a_n)_{n=0}^{\infty} \) in \( X \) such that \( \lim_{n \to \infty} a_n = x_0, \) we have \( \lim_{n \to \infty} f(a_n) = f(x_0). \)

(iii) For every \( \varepsilon > 0, \) there exists a \( \delta = \delta(\varepsilon) > 0 \) such that, for all \( x \in X \) with \( |x - x_0| < \delta, \) we have \( |f(x) - f(x_0)| < \varepsilon. \)

**Solution.** (i) \( \iff \) (ii): \( f \) is continuous at \( x_0 \) if and only if \( \lim_{x \to x_0 \in X} f(x) = f(x_0). \) By Exercise 3, this is true if and only if (ii) is true.

(i) \( \iff \) (iii): The statement (iii), together with the choice \( E = X \) in Definition 2.14, exactly tells us that \( \lim_{x \to x_0 \in X} f(x) = f(x_0), \) which is the definition of the continuity of \( f \) at \( x_0. \)

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**Exercise 6.** Let \( X, Y \) be subsets of \( \mathbb{R}. \) Let \( f : X \to Y \) and \( g : Y \to \mathbb{R} \) be functions. Let \( x_0 \in X. \) If \( f \) is continuous at \( x_0, \) and if \( g \) is continuous at \( f(x_0), \) then \( g \circ f \) is continuous at \( x_0. \)

**Solution.** We have \( \lim_{x \to x_0 \in X} f(x) = f(x_0) \) and \( \lim_{y \to f(x_0) \in Y} g(y) = g(f(x_0)). \) Let \( \varepsilon > 0 \) be arbitrary. Using Definition 2.14,

- We can pick \( \eta = \eta(\varepsilon) > 0 \) such that whenever \( y \in Y \) and \( |y - f(x_0)| < \eta \) we have \( |g(y) - g(f(x_0))| < \varepsilon. \)
- We can pick \( \delta = \delta(\eta) > 0 \) such that whenever \( x \in X \) and \( |x - x_0| < \delta \) we have \( |f(x) - f(x_0)| < \eta. \)

Combining these two statements, we find that whenever \( |x - x_0| < \delta, \) we have \( f(x) \in Y \) and \( |f(x) - f(x_0)| < \eta, \) hence \( |g(f(x)) - g(f(x_0))| < \varepsilon. \) From this we read out that \( \lim_{x \to x_0 \in X} g(f(x)) = g(f(x_0)). \)

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**Exercise 7.** Let \( a < b \) be real numbers. Let \( f : [a, b] \to \mathbb{R} \) be a continuous function on \( [a, b]. \) Let \( M := \sup_{x \in [a, b]} f(x) \) be the maximum value of \( f \) on \( [a, b], \) and let \( m := \inf_{x \in [a, b]} f(x) \) be the minimum value of \( f \) on \( [a, b]. \) Let \( y \) be a real number such that \( m \leq y \leq M. \) Then there exists \( c \in [a, b] \) such that \( f(c) = y. \) Moreover, \( f([a, b]) = [m, M]. \)

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\([1]\) here we used the letter \( \eta \) instead of \( \delta \) since the letter \( \delta \) is already used for some other use and we want to avoid confusion.
**Solution.** By the Maximum Principle, both the values $M$ and $m$ are achieved at some different points in $[a, b]$. That is, there exists $a' < b'$ in $[a, b]$ such that $(m, M) = \{f(a'), f(b')\}$. [2] Now note that $f$ is continuous on $[a', b']$ as well. Thus if $m \leq y \leq M$, then by the Intermediate Value Theorem, there exists $c \in [a', b']$ such that $f(c) = y$. Since $c \in [a, b]$ as well, the first assertion follows.

For the second assertion, we prove that $[m, M]$ and $f([a, b])$ are subsets of each other. To this end, just observe that

- For any $y \in [m, M]$ we have $f(c) = y$ for some $c \in [a, b]$ by the first assertion. So we have $y \in f([a, b])$.
- If $y \in f([a, b])$, then $y = f(c)$ for some $c \in [a, b]$. Then

$$m = \inf_{x \in [a, b]} f(x) \leq f(c) \leq \sup_{x \in [a, b]} f(x) = M$$

and hence $y \in [m, M]$. This shows that $[m, M] \subset f([a, b])$ and $f([a, b]) \subset [m, M]$. Therefore they are equal to each other. \[\square\]

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**Exercise 8.** Let $a < b$ be real numbers, and let $f : [a, b] \to \mathbb{R}$ be a function which is both continuous and strictly monotone increasing. Then $f$ is a bijection from $[a, b]$ to $[f(a), f(b)]$, and the inverse function $f^{-1} : [f(a), f(b)] \to [a, b]$ is also continuous and strictly monotone increasing. (Hint: To prove that $f^{-1}$ is continuous, use the $\varepsilon - \delta$ definition of continuity.)

**Solution.** We first show the bijectivity of $f$ from $[a, b]$ to $[f(a), f(b)]$.

- **(Surjectivity)** Notice that $f(a)$ is the minimum of $f([a, b])$. Indeed, whenever $y \in f([a, b])$ we have $y = f(c)$ for some $c \in [a, b]$ and thus $f(a) \leq f(c) = y$. Likewise $f(b)$ is the maximum of $f([a, b])$. Then by the previous exercise we have

$$f([a, b]) = [\min f([a, b]), \max f([a, b])] = [f(a), f(b)].$$

This shows that $f$ is a surjection from $[a, b]$ to $[f(a), f(b)]$.

- **(Injectivity)** Injectivity of $f$ also follows easily. If $x, y \in [a, b]$ and $x \neq y$, we may assume without loss of generality that $x < y$. Then $f(x) < f(y)$ and thus $f(x) \neq f(y)$, proving the injectivity of $f$.

Next we show that the inverse function $f^{-1}$ is also continuous. To this end, we claim the following:

**Claim.** For any $y_0 \in [f(a), f(b)]$ and $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon, y_0) > 0$ such that for any $y \in [f(a), f(b)]$ satisfying $|y - y_0| < \delta$ we have $|f^{-1}(y) - f^{-1}(y_0)| < \varepsilon$.

Indeed, let $y_0 \in [f(a), f(b)]$ and $\varepsilon > 0$. We put

$$x^- = \max [f^{-1}(y_0) - \varepsilon, a], \quad y^- = f(x^-),$$
$$x^+ = \min [f^{-1}(y_0) + \varepsilon, b], \quad y^+ = f(x^+).$$

Notice that, from the construction, we have $x^- \leq f^{-1}(y_0) \leq x^+$ and hence $y^- \leq y_0 \leq y^+$. To complete the proof, we consider several cases to deal with general cases and some exceptional cases separately.

- **Case 1.** Suppose that $f(a) < y_0 < f(b)$. The inequality $a < f^{-1}(y_0) < b$ shows that $x^- < f^{-1}(y_0) < x^+$. Then this implies $y^- < y_0 < y^+$. No we put

$$\delta = \min \{y^+ - y_0, y_0 - y^-\} > 0.$$

Then whenever $y \in [f(a), f(b)]$ and $|y - y_0| < \delta$, we must have

$$y^- = y_0 - (y_0 - y^-) < y_0 - |y_0 - y| \leq y \quad \text{and} \quad y \leq y_0 + |y - y_0| < y_0 + |y^+ - y_0| = y^+.$$

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[2] This tricky demonstration is a technical, brief way of saying that ‘one of $a'$ and $b'$ is a maximum point and the other is a minimum point’.
Therefore $f^{-1}$ is continuous at $y_0$.

- **Case 2.** Suppose that $y_0 = f(a)$. Then $x^- = a = f^{-1}(y_0) < x^+$. Now let

  $$\delta = y^+ - y_0 > 0.$$ 

  Then for any $y \in [f(a), f(b)]$ satisfying $|y - y_0| < \delta$, we have $y_0 \leq y < y^+$ and hence $f^{-1}(y_0) \leq f^{-1}(y) < f^{-1}(y_0) + \epsilon$. This implies that $|f^{-1}(y) - f^{-1}(y_0)| < \epsilon$, proving that $f^{-1}$ is continuous at $y_0 = f(a)$.

- **Case 3.** The case $y_0 = f(b)$ is treated in a similar way as in Case 2. Therefore $f^{-1}$ is continuous at any point on $[f(a), f(b)]$, and hence a continuous function.

\[\square\]

**Exercise 9.** Let $(a_n)_{n=m}^{\infty}$ and $(b_n)_{n=m}^{\infty}$ be two sequences of real numbers. Then $(a_n)_{n=m}^{\infty}$ and $(b_n)_{n=m}^{\infty}$ are equivalent if and only if $\lim_{n \to \infty} (a_n - b_n) = 0$.

**Solution.** Carefully read out the definition of equivalent sequences to convince yourself that the observation $|a_n - b_n| = |a_n - b_n| - 0$ suffices to complete the proof.

**Exercise 10.** Let $a < b$ be real numbers, and let $f : [a, b] \to \mathbb{R}$ be a function. Assume that there exists a real number $L > 0$ such that, for all $x, y \in [a, b]$, we have $|f(x) - f(y)| \leq L|x - y|$. Such an $f$ is called **Lipschitz continuous**. Prove that $f$ is continuous. Then, find a continuous function that is not Lipschitz continuous.

**Solution.** For any $\epsilon > 0$, pick $\delta = \epsilon / (L + 1)$. Then whenever $x, y \in [a, b]$ and $|x - y| < \delta$, we have

$$|f(x) - f(y)| \leq L|x - y| \leq \frac{L \epsilon}{L + 1} < \epsilon.$$ 

This proves that $f$ is continuous at any point. (And also proves that $f$ is uniformly continuous.)

An example of a function which is continuous but not Lipschitz continuous is $f : [0, 1] \to \mathbb{R}$ given by $f(x) = \sqrt{x}$. To check this, notice that for any $n \in \mathbb{N}$,

$$|f(\frac{1}{n}) - f(0)| = \frac{1}{\sqrt{n}} = \frac{\sqrt{n}}{n} = \sqrt{n}|\frac{1}{n} - 0|.$$
Thus no number $L \geq 0$ cannot satisfy $|f(x) - f(y)| \leq L|x - y|$ for all $x, y \in [0, 1]$. (Otherwise we must be able to find some $L \geq 0$ satisfying $L \geq \sqrt{n}$ for all $n$, which is impossible.)