Please provide complete and well-written solutions to the following exercises.

Due December 11, in the discussion section.

**Assignment 9**

**Exercise 1.** Let \( a < b \) be real numbers, and let \( f, g: [a, b] \to \mathbb{R} \) be Riemann integrable functions on \([a, b]\). Then

(i) The function \( f + g \) is Riemann integrable on \([a, b]\), and \( \int_a^b (f + g) = (\int_a^b f) + (\int_a^b g) \).

(ii) For any real number \( c \), \( cf \) is Riemann integrable on \([a, b]\), and \( \int_a^b (cf) = c(\int_a^b f) \).

(iii) The function \( f - g \) is Riemann integrable on \([a, b]\), and \( \int_a^b (f - g) = (\int_a^b f) - (\int_a^b g) \).

(iv) If \( f(x) \geq 0 \) for all \( x \in [a, b] \), then \( \int_a^b f \geq 0 \).

(v) If \( f(x) \geq g(x) \) for all \( x \in [a, b] \), then \( \int_a^b f \geq \int_a^b g \).

(vi) If there exists a real number \( c \) such that \( f(x) = c \) for \( x \in [a, b] \), then \( \int_a^b f = c(b - a) \).

(vii) Let \( c, d \) be real numbers such that \( c \leq a < b \leq d \). Then \([c, d]\) contains \([a, b]\). Define \( F(x) := f(x) \) for \( x \in [a, b] \) and \( F(x) := 0 \) otherwise. Then \( F \) is Riemann integrable on \([c, d]\), and \( \int_c^d F = \int_a^b f \).

(viii) Let \( c \) be a real number such that \( a < c < b \). Then \( f|_{[a, c]} \) and \( f|_{[c, b]} \) are Riemann integrable on \([a, c]\) and \([c, b]\) respectively, and

\[
\int_a^b f = \int_a^c f|_{[a, c]} + \int_c^b f|_{[c, b]}.
\]

**Exercise 2.** Let \( a < b \) be real numbers. Let \( f: [a, b] \to \mathbb{R} \) be a bounded function. Let \( c \in [a, b] \). Assume that, for each \( \delta > 0 \), we know that \( f \) is Riemann integrable on the set \( \{ x \in [a, b] : |x - c| \geq \delta \} \). Then \( f \) is Riemann integrable on \([a, b]\).

**Exercise 3.** Find a function \( f: [0, 1] \to \mathbb{R} \) such that \( f \) is not Riemann integrable on \([0, 1]\), but such that \(|f| \) is Riemann integrable on \([0, 1]\).

**Exercise 4.** Let \( a < b \) be real numbers. Let \( f: [a, b] \to \mathbb{R} \) be a bounded function. So, there exists a real number \( M \) such that \(|f(x)| \leq M \) for all \( x \in [a, b] \). Let \( P \) be a partition of \([a, b]\).

- Using the identity \( \alpha^2 - \beta^2 = (\alpha + \beta)(\alpha - \beta) \), where \( \alpha, \beta \in \mathbb{R} \), show that
  \[
  U(f^2, P) - L(f^2, P) \leq 2M(U(f, P) - L(f, P)).
  \]
- Show that if \( f \) is Riemann integrable on \([a, b]\), then \( f^2 \) is also Riemann integrable on \([a, b]\).
- Let \( f, g: [a, b] \to \mathbb{R} \) be Riemann integrable functions on \([a, b]\). Using the identity \( 4\alpha\beta = (\alpha + \beta)^2 - (\alpha - \beta)^2 \), where \( \alpha, \beta \in \mathbb{R} \), show that \( fg \) is Riemann integrable on \([a, b]\).
Exercise 5. Let $f : [0, 1] \to [0, \infty)$ be a continuous function such that $\int_0^1 f = 0$. Prove that $f(x) = 0$ for all $x \in [0, 1]$.

Exercise 6. The following exercise deals with metric properties of the space of Riemann integrable functions.

- Let $\alpha, \beta$ be real numbers. Prove that $\alpha \beta \leq (\alpha^2 + \beta^2)/2$. Now, let $a < b$ be real numbers, and let $f, g : [a, b] \to \mathbb{R}$ be two Riemann integrable functions. Assume that $\int_a^b f^2 = 1$ and $\int_a^b g^2 = 1$. (Recall that since $f, g$ are Riemann integrable, we know that $f^2, g^2$ and $fg$ are also Riemann integrable by Exercise 4.) Prove that
  $$\int_a^b fg \leq 1.$$ 

- Let $a < b$ be real numbers, and let $f, g : [a, b] \to \mathbb{R}$ be two Riemann integrable functions. Prove the Cauchy-Schwarz inequality:
  $$\left| \int_a^b fg \right| \leq \left( \int_a^b f^2 \right)^{1/2} \left( \int_a^b g^2 \right)^{1/2}$$

- Let $a < b$ be real numbers, and let $f, g, h : [a, b] \to \mathbb{R}$ be Riemann integrable functions. Define
  $$d(f, g) := \left( \int_a^b (f - h)^2 \right)^{1/2}.$$ 
  Prove the triangle inequality for $d$. That is, show that
  $$d(f, g) \leq d(f, h) + d(h, g).$$