Assignment 6

Exercise 1. Let $\sum_{n=m}^{\infty} a_n$ be a formal series of real numbers. Then $\sum_{n=m}^{\infty} a_n$ converges if and only if: for every real number $\varepsilon > 0$, there exists an integer $N \geq M$ such that, for all $p, q \geq N$,

$$\left| \sum_{n=p}^{q} a_n \right| < \varepsilon.$$

(Hint: recall that a sequence is convergent if and only if it is a Cauchy sequence.)

Exercise 2. Let $\sum_{n=m}^{\infty} a_n$ be a formal series of real numbers. If $\sum_{n=m}^{\infty} a_n$ converges, then $\lim_{n \to \infty} a_n = 0$. Note that the contrapositive says: if $a_n$ does not converge to zero as $n \to \infty$, then $\sum_{n=m}^{\infty} a_n$ does not converge. (Hint: use Exercise 1.)

Exercise 3. Let $\sum_{n=m}^{\infty} a_n$ be a formal series of real numbers. If this series is absolutely convergent, then it is convergent. Moreover,

$$\left| \sum_{n=m}^{\infty} a_n \right| \leq \sum_{n=m}^{\infty} |a_n|.$$

Exercise 4.

- Let $\sum_{n=m}^{\infty} a_n$ be a series of real numbers converging to $x$, and let $\sum_{n=m}^{\infty} b_n$ be a series of real numbers converging to $y$. Then $\sum_{n=m}^{\infty} (a_n + b_n)$ is a convergent series that converges to $x + y$. That is,

$$\sum_{n=m}^{\infty} (a_n + b_n) = \left( \sum_{n=m}^{\infty} a_n \right) + \left( \sum_{n=m}^{\infty} b_n \right).$$

- Let $\sum_{n=m}^{\infty} a_n$ be a series of real numbers converging to $x$, and let $c$ be a real number. Then $\sum_{n=m}^{\infty} (ca_n)$ is a convergent series that converges to $cx$. That is,

$$\sum_{n=m}^{\infty} (ca_n) = c \left( \sum_{n=m}^{\infty} a_n \right).$$

- Let $\sum_{n=m}^{\infty} a_n$ be a series of real numbers, and let $k$ be a natural number. If one of the two series $\sum_{n=m}^{\infty} a_n$ or $\sum_{n=m+k}^{\infty} a_n$ converges, then the other also converges, and we have

$$\sum_{n=m}^{\infty} a_n = \left( \sum_{n=m}^{m+k-1} a_n \right) + \left( \sum_{n=m+k}^{\infty} a_n \right).$$
• Let $\sum_{n=m}^{\infty} a_n$ be a series of real numbers converging to $x$, and let $k$ be an integer. Then $\sum_{n=m+k}^{\infty} a_{n-k}$ also converges to $x$.

**Exercise 5.** Let $\sum_{n=m}^{\infty} a_n$, $\sum_{n=m}^{\infty} b_n$ be formal series of real numbers. Assume that $|a_n| \leq b_n$ for all $n \geq m$. If $\sum_{n=m}^{\infty} b_n$ is convergent, then $\sum_{n=m}^{\infty} a_n$ is absolutely convergent. Moreover,

$$\left| \sum_{n=m}^{\infty} a_n \right| \leq \sum_{n=m}^{\infty} |a_n| \leq \sum_{n=m}^{\infty} b_n.$$  

**Exercise 6.** For any $n \in \mathbb{N}$, define $a_n := (-1)^{n+1}/(n+1)$. Find a bijection $g: \mathbb{N} \to \mathbb{N}$ such that the series $\sum_{n=0}^{\infty} a_g(n)$ diverges.

**Exercise 7.** Let $(b_n)_{n=m}^{\infty}$ be a sequence of positive numbers. Then

$$\liminf_{n \to \infty} \frac{b_{n+1}}{b_n} \leq \liminf_{n \to \infty} b_n^{1/n}.$$  

**Exercise 8.** Let $(a_n)_{n=0}^{\infty}$, $(b_n)_{n=0}^{\infty}$, $(c_n)_{n=0}^{\infty}$ be sequences of real numbers. Then $(a_n)_{n=0}^{\infty}$ is a subsequence of $(a_n)_{n=0}^{\infty}$. Also, if $(b_n)_{n=0}^{\infty}$ is a subsequence of $(a_n)_{n=0}^{\infty}$, and if $(c_n)_{n=0}^{\infty}$ is a subsequence of $(b_n)_{n=0}^{\infty}$, then $(c_n)_{n=0}^{\infty}$ is a subsequence of $(a_n)_{n=0}^{\infty}$.

**Exercise 9.** Give an example of two convergent series of real numbers $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ such that the series $\sum_{n=0}^{\infty} (a_n b_n)$ is not convergent.

**Exercise 10.** Let $(a_n)_{n=0}^{\infty}$ be a sequence of real numbers, and let $L$ be a real number.

- If the sequence $(a_n)_{n=0}^{\infty}$ converges to $L$, then every subsequence of $(a_n)_{n=0}^{\infty}$ converges to $L$.
- Conversely, if every subsequence of $(a_n)_{n=0}^{\infty}$ converges to $L$, then $(a_n)_{n=0}^{\infty}$ itself converges to $L$. 