**Example**

Define $\log : (0, \infty) \rightarrow \mathbb{R}$ by $\log(x) = \int_1^x \frac{dt}{t}$. Prove:

(a) $\log$ is increasing.

(b) $\log(xy) = \log(x) + \log(y)$

(c) $\log'(x) = \frac{1}{x}$

(d) $\log(1) = 0$.

**Solution**

(a) Since $\log(x) = \frac{t}{\log x} > 0$, $\log$ is strictly increasing.

(b) $\log(xy) = \int_1^{xy} \frac{dt}{t} = \int_1^x \frac{dt}{t} + \int_x^{xy} \frac{dt}{t}$

\[= \log(x) + \int_x^{xy} \frac{1}{t} \cdot \frac{d(t/x)}{dt} dt.\]

So with $\phi(t) = \frac{t}{x}$, we have by Change of Variable,

\[= \log(x) + \int_{\phi(x)}^{\phi(xy)} \frac{dt}{t} = \log(x) + \int_1^{y} \frac{dt}{t} = \log(x) + \log(y).\]

(d) $\log(1) = \int_1^1 \frac{dt}{t} = 0$.

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**Example**

Assume that $f : [a, b] \rightarrow \mathbb{R}$ is Riemann integrable and $\forall [c,d] \subseteq [a,b] \exists x^* \in [c,d]$ satisfying $f(x^*) = 0$.

(a) Show that $\int_a^b f = 0$.

(b) Is $f$ need to be zero?

(c) If $f$ is continuous, is $f$ zero?

**Sol**

(a) For any partition $P = \{ a = x_0 < \ldots < x_n = b \}$, $\exists x^* \in [x_0, x_1]$ s.t.

\[f(x^*) = 0.\]

\[\inf_{x \in [x_0, x_1]} f(x) \leq f(x^*) = 0 \leq \sup_{x \in [x_0, x_1]} f(x).\]
Implies that
\[ L(f; P) \leq 0 \leq U(f; P) \]
\[ \Rightarrow \int_a^b f \leq 0 \leq \int_a^b f. \]
Since \( \int_a^b f = \int_a^b f = \int_a^b f \), we have \( \int_a^b f = 0 \).

(b) No. Consider \( f : [0, 1] \to \mathbb{R} \) defined by
\[ f(x) = \begin{cases} 1, & x = \frac{1}{n} \text{ for some } n = 1, 2, \ldots \\ 0, & \text{otherwise.} \end{cases} \]
Then \( \forall \delta > 0 \), \( f \) is piecewise continuous on \([0, 1]\) and hence integrable by Exercise 2.7 (or Proposition 2.8). Also \( f \) satisfies the condition, so \( \int_0^1 f = 0 \). On the other hand, \( f \) is not zero.

(c) Yes. Let \( x_0 \in [a, b] \). Then \( x_n = x_0 + \frac{1}{n} \to x_0 \) and for each \( n \), (\( n \) large so that \( x_n \leq b \)), \( \exists x_n^* \in [x_0, x_n] \) s.t. \( f(x_n^*) = 0 \). Now \( x_n^* \to x_0 \) and by continuity,
\[ f(x_0) = \lim_{n \to \infty} f(x_n^*) = 0. \]
The case \( x_0 = b \) is also easily checked. So \( f \equiv 0 \).

\[ \square \]

**Exercise**

Let \( f : [a, b] \to \mathbb{R} \) be continuous.

(a) (Mean Value Thm) \( \exists c \in [a, b] \) s.t. \( \int_a^b f(x) \, dx = f(c) \cdot (b-a) \).

(b) (2nd Mean Value Thm) Let \( g : [a, b] \to [0, \infty) \) be Riemann integrable.
\[ \exists c \in [a, b] \text{ s.t. } \int_a^b f(x)g(x) \, dx = f(c) \int_a^b g(x) \, dx. \]

**Sol.** (a) is a special case of (b). So it suffices to prove (b). But if we write \( m = \min_{x \in [a,b]} f(x) \), \( M = \max_{x \in [a,b]} f(x) \), then by Maximum Principle, \( \exists x^*, y^* \) s.t. \( m = f(x^*) \) and \( M = f(y^*) \). Then
\[ \int_a^b mg(x) \, dx \leq \int_a^b f(x)g(x) \, dx \leq \int_a^b Mg(x) \, dx. \]
So if \( \int_a^b g(x) \, dx = 0 \), then we can pick any \( c \in [a,b] \) and we have

\[
\int_a^b f(x) g(x) \, dx = 0 = f(c) \int_a^b g(x) \, dx.
\]

If \( \int_a^b g(x) \, dx \neq 0 \), then by \( g \geq 0 \) we have \( \int_a^b g(x) \, dx > 0 \). So

\[
f(x^*) = m \leq \frac{\int_a^b f(x) g(x) \, dx}{\int_a^b g(x) \, dx} \leq M = f(y^*)
\]

\( \Rightarrow \) By Intermediate Value Thm., \( \exists c \in [x^*, y^*] \subset [a,b] \) s.t.

\[
f(c) = \frac{\int_a^b f(x) g(x) \, dx}{\int_a^b g(x) \, dx},
\]

\[\text{Counterexamples}\]

(1) If \( f_n : [0,1] \to \mathbb{R} \) is integrable and \( f_n = \lim_{n \to \infty} f_n(x) \) exists, still it is not guaranteed that \( f \) is Riemann integrable.

(2) If \( f_n : [0,1] \to \mathbb{R} \) is Riemann integrable and \( f(x) = \lim_{n \to \infty} f_n(x) \) exists and is Riemann integrable, still we can have

\[
\lim_{n \to \infty} \int_0^1 f_n(x) \, dx \neq \int_0^1 \lim_{n \to \infty} f_n(x) \, dx.
\]

(3) But if we add the assumption that \( f_n \) are uniformly bounded: \( \exists M > 0 \) s.t. \( |f_n(x)| \leq M \) for all \( n \) and \( x \), then if \( f(x) = \lim_{n \to \infty} f_n(x) \) is Riemann integrable, we have

\[
\lim_{n \to \infty} \int_0^1 f_n(x) \, dx = \int_0^1 \lim_{n \to \infty} f_n(x) \, dx.
\]

(Bounded convergence thm.) Proof of this fact is not easy, and if we learn Lebesgue integral this follows much easily.
HW9.5 Let $f : [0,1] \rightarrow \mathbb{R}$ be continuous and $f(x) \geq 0 \ \forall x \in [0,1]$, and $\int_0^1 f = 0$. Show that $f(x) = 0 \ \forall x \in [0,1]$. 

S1 1) Assume otherwise. $\exists x^* \in [0,1]$ s.t. $f(x^*) \neq 0$ ($\Rightarrow f(x^*) > 0$).

Then with $\varepsilon = \frac{1}{2} f(x^*)$, $\exists \delta > 0$ s.t.

$$|x-x^*| < \delta, \ x \in [0,1] \Rightarrow |f(x) - f(x^*)| < \varepsilon$$

$$\Rightarrow f(x) > f(x^*) - \varepsilon = \varepsilon.$$ 

In particular, if we write $[0,1] \cap [x^*-\frac{\delta}{2}, x^*+\frac{\delta}{2}] = [c,d]$, then

$f(x) \geq \varepsilon$ on $[c,d]$. So

$$0 = \int_0^1 f \geq \int_c^d \varepsilon = \varepsilon(d-c) > 0, \ \neq 0.$$ 

S1 2) Let $F(x) = \int_0^x f$ for $0 \leq x \leq 1$. Then $f \geq 0$ shows that

$$0 \leq \int_0^x f \leq \int_0^1 f = 0$$

$$\Rightarrow F(x) = 0. \ \text{So it follows from FTC that}$$

$$f(x) = F'(x) = 0.$$ 

Example (Fundamental Theorem of Calculus of Variations) If $f : [a,b] \rightarrow \mathbb{R}$ is continuous and $\int_a^b fg = 0$ for every continuous function $g$ on $[a,b]$ with $g(a) = g(b) = 0$, then $f = 0$. on $[a,b]$.

Sol 1) Put $g(x) = (x-a)(b-x)f(x)$. Then $f(x)g(x) = (x-a)(b-x)f(x)^2 \geq 0$ on $[a,b]$ and $\int_a^b fg = 0$ implies that $f(x)g(x) = 0$ on $[a,b]$. This implies that $f(x) = 0$ on $(a,b)$, and hence on $[a,b]$ by continuity. \(\square\)