

**Problem**

If \( f : [a, b] \to \mathbb{R} \) be s.t. \( |f(x) - f(y)| \leq (x - y)^2 \) \( \forall x, y \in [a, b] \), show that \( f \) is constant.

**Solution**

\[
\frac{|f(x) - f(y)|}{|x - y|} \leq |x - y| \quad \forall x, y \in [a, b], 
\]

so if we take limit as \( y \to x \), we have \( \lim_{y \to x} l y - x l = 0 \) and by squeezing on

\[
-|y - x| \leq \frac{f(y) - f(x)}{y - x} \leq |y - x|,
\]

we get \( \lim_{y \to x} \frac{f(y) - f(x)}{y - x} = 0 \), which is \( f'(x) \). So \( f'(x) = 0 \) \( \forall x \in [a, b] \) and hence \( f \) is constant.

**Problem**

Let \( f(x) = \begin{cases} 
    x^2 \sin \frac{1}{x}, & x \neq 0 \\
    0, & x = 0.
\end{cases} \) on \( \mathbb{R} \).

1. Show that \( f \) is differentiable.
2. Show that \( f' \) is not continuous at \( x = 0 \).

**Solution**

1. On \( \mathbb{R} \setminus \{0\} \), differentiation law applies and

\[
f'(x) = 2x \sin \frac{1}{x} - \cos \frac{1}{x}.
\]

At \( x = 0 \), we have

\[
\left| \frac{f(x) - f(0)}{x - 0} \right| = \frac{|x^2 \sin \frac{1}{x}|}{|x|} = |x| \sin \frac{1}{x} \leq |x|
\]

and by squeezing,

\[
f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = 0.
\]

2. Since \( \cos \frac{1}{x} \) does not converge as \( x \to 0 \) and \( 2x \sin \frac{1}{x} \to 0 \) as \( x \to 0 \), \( f'(x) \) does not converge as \( x \to 0 \). (You may also consider \( a_n = \frac{1}{n\pi} \) to check that \( f(a_n) = (-1)^m \) does not converge as \( n \to \infty \).)
**Thm (Darboux)** If \( f : [a, b] \to \mathbb{R} \) is differentiable, then for any \( y \) between \( f(a) \) and \( f'(b) \) there exists \( c \in (a, b) \) such that \( f'(c) = y \).

**Sol**

Consider first the case \( f(a) < y < f'(b) \). Let \( \varphi(x) = f(x) - yx \). Then \( \varphi(x) \) is also differentiable and \( \varphi'(a) = f'(a) - y < 0 \). Since \( \varphi \) is continuous on \([a, b]\), there exists \( c \in [a, b] \) st. \( \varphi \) achieves minimum.

- But since \( \varphi'(a) < 0 \), for \( \epsilon = -\frac{1}{2} \varphi'(a) > 0 \), \( \exists \delta > 0 \) st. whenever \( x \in [a, b] \) and \( |x - a| < \delta \), we have

\[
|\varphi(x) - \varphi(a) - \varphi'(a)(x-a)| \leq \epsilon |x-a|
\]

\[
\varphi(x) \leq \varphi(a) + \varphi'(a)(x-a) + \epsilon(x-a)
\]

\[
= \varphi(a) + \frac{1}{2} \varphi'(a)(x-a) < \varphi(a)
\]

\( \Rightarrow \) \( x = a \) is not a minimum of \( \varphi \).

- Likewise, \( x = b \) is not a minimum of \( \varphi \).

- So we find that \( c \in (a, b) \), and thus \( c \) is a local minimum of \( \varphi \) and \( \varphi'(c) = 0 \). \( \Rightarrow \) \( f'(c) = y \).

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**Cor** If \( f : [a, b] \to \mathbb{R} \) is differentiable, then \( f' \) has NO jump discontinuity.

**Ex** \( f(x) = \begin{cases} x^2 \sin \frac{1}{x} & , x \neq 0 \\ 0 & , x = 0 \end{cases} \). Then \( y = f'(x) \) shows...
**Problem**

If \( a_0 + a_1 x + \ldots + a_n x^n = 0 \), show that \( f(x) = a_0 + a_1 x + \ldots + a_n x^n \) has one zero on \([0,1]\).

**Sol.**

Let \( F(x) = a_0 x + \frac{a_1}{2} x^2 + \ldots + \frac{a_n}{n+1} x^{n+1} \). Then \( F(0) = 0 \) and \( F(1) = 0 \), and also \( F'(x) = f(x) \). By the Rolle's Thm, \( \exists c \in [0,1] \) s.t. \( F'(c) = 0 \). So \( f(c) = 0 \) for that \( c \in [a,b] \) and we are done. \( \Box \)

(Banach fixed point thm, rudimentary ver.)

**Problem**

If \( f: \mathbb{R} \to \mathbb{R} \) is differentiable, and \( \exists \epsilon \in [0,1] \) s.t. \( |f'(x)| \leq r \) \( \forall x \in \mathbb{R} \), then \( f \) has a unique fixed point.

**Sol.**

• (Uniqueness) If \( \exists a < b \) : fixed points, then \( \exists c \in [a,b] \) s.t.

\[
1 = \frac{f(b) - f(a)}{b - a} = f'(c) \leq r < 1, \quad \Rightarrow \quad \text{Contradiction}.
\]

• (Existence) Let \( x_0 \in \mathbb{R} \) be arbitrary. Define \( (x_n)_{n=0}^\infty \) by

\[
x_{n+1} = f(x_n).
\]

Then for \( n \geq 0 \), \( \exists \epsilon \) between \( x_n \) and \( x_{n+1} \) s.t.

\[
|\epsilon| = |f(x_{n+1}) - f(x_n)|
= |f'(c)| |x_{n+1} - x_n|
\leq r |x_{n+1} - x_n|.
\]

Now \( \forall \epsilon > 0 \), let \( N \in \mathbb{N} \) be chosen so that \( \forall n \geq N \), \( |x_n - x_N| < \frac{\epsilon}{2r} \). (This is possible since \( r^n \to 0 \) as \( n \to \infty \).) Then \( \forall j > k > N \),

\[
|x_j - x_k| \leq |x_j - x_{j-1}| + \ldots + |x_{k+1} - x_k|
\leq r^{j-1} |x_1 - x_0| + \ldots + r^k |x_1 - x_0|
= r^k \cdot \frac{1 - r^{j-k}}{1-r} |x_1 - x_0|
\leq \frac{rN}{1-r} |x_1 - x_0|
< \epsilon,
\]

and hence \( (x_n)_{n=0}^\infty \) is Cauchy. Hence \( \lim_{n \to \infty} x_n = a \) exists. Then
\[ a = \lim_{n \to \infty} x_n = \lim_{n \to \infty} x_{n+} = \lim_{n \to \infty} f(x_n) = f(a), \]
and hence \( a \) is a fixed point of \( f \).

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**Problem**

\( f: [a,b] \to \mathbb{R} \) function and \( c \in [a,b] \). Then TFAE:

1. \( f \) is differentiable at \( c \).

2. \( \exists m \in \mathbb{R} \) and a function \( \varepsilon: [a,b] \to \mathbb{R} \) such that

\[ \lim_{x \to c} \varepsilon(x) = 0 \quad \text{and} \quad f(x) = f(c) + m(x-c) + \varepsilon(x)(x-c). \]

- In other words, \( f'(c) \) exists iff \( f(x) \) can be approximated by a line near \((c,f(c))\).

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**Solution**

Note that \( \varepsilon(x) \) should be given by

\[ \varepsilon(x) = \begin{cases} \frac{f(x)-f(c)}{x-c} - m, & x \neq c \\ 0, & x = c \end{cases} \]

So

\[ \lim_{x \to c} \varepsilon(x) = 0 \iff \lim_{x \to c} \frac{f(x)-f(c)}{x-c} = m. \]
Integration

HW 9.1  
\text{If } f, g : [a, b] \to \mathbb{R} \text{ Riemann integrable,}

(i) \quad f + g : \text{Riemann integrable and } \int_a^b (f + g) = \int_a^b f + \int_a^b g.

(iv) \quad \text{If } f \geq 0, \text{ then } \int_a^b f \geq 0.

(vii) \quad \text{If } a < c < b, \text{ then } f |_{[a, c]} , f |_{[c, b]} \text{ are Riemann integrable and}

\quad \int_a^b f = \int_a^c f |_{[a, c]} + \int_c^b f |_{[c, b]}.

Remark) \quad \text{If } E \subset \mathbb{R} \text{ non-empty, then } \exists x_0 \in E \text{ s.t. } \lim_{n \to \infty} x_n = \sup E \quad \text{or} \quad \exists x_0 \in E \text{ s.t. } \lim_{n \to \infty} x_n = \inf E.

Solution)  
(ii) \quad \text{Let } P, Q \text{ be any partition of } [a, b]. \text{ Then } P \cup Q \text{ is also a partition of } [a, b]. \text{ Now write } P \cup Q = \{x_0 < x_1 < \cdots < x_n = b\}.

\text{Then}

\quad U(f + g, P \cup Q) = \sum_{i=1}^n \left( \sup_{x_i \leq \xi \leq x_{i+1}} (f(\xi) + g(\xi)) \right) (x_i - x_{i-1})

\quad \leq \sum_{i=1}^n \left( \sup_{x_i \leq \xi \leq x_{i+1}} f(\xi) + \sup_{x_i \leq \xi \leq x_{i+1}} g(\xi) \right) (x_i - x_{i-1})

\quad = U(f, P) + U(g, Q)

\quad \leq U(f, P) + U(g, Q).

Here, in the last line, we utilized the following Lemma:

**Lemma** \quad \text{If } f : [a, b] \to \mathbb{R} \text{ is Riemann integrable, } P, P' \text{ partitions.}

\text{Then } U(f, P') \leq U(f, P). \text{ That is, refining partition makes upper sum smaller.}

**Proof of Lemma** \quad \text{Let } P = \{x_0 = x_0 < x_1 < \cdots < x_n = b\}. \text{ For each } i = 1, \cdots, n, \text{ let } P' \cap [x_i, x_{i+1}] = \{x_{i0} = x_{i1} < \cdots < x_{i, m_i} = x_{i+1}\}. 
Then \( P' = \{ a = X_{10} < \cdots < X_{1m_1} = X_{20} < \cdots < \cdots < X_{n,m_n} = b \} \) and
\[
U(f, P') = \sum_{i=1}^{n} \sum_{j=1}^{m_i} \left( \sup_{x \in [X_{ij-1}, X_{ij}]} f(x) \right) (X_{ij} - X_{ij-1})
\]
\[
\leq \sum_{i=1}^{n} \sum_{j=1}^{m_i} \left( \sup_{x \in [X_{ij}, X_{ij+1}]} f(x) \right) (X_{ij} - X_{ij+1})
\]
\[
= \sum_{i=1}^{n} \left( \sup_{x \in [X_{i-1}, X_i]} f(x) \right) (X_i - X_{i-1}) = U(f, P).
\]

Returning to the original problem, we showed that
\[
\int_a^b (f+g) \leq U(f+g, P, Q) \leq U(f, P) + U(g, Q).
\]

Taking infimum for all partition \( P \), we get
\[
\int_a^b (f+g) \leq \int_a^b f + U(g, Q).
\]

Taking infimum for all partition \( Q \), we get
\[
\int_a^b (f+g) \leq \int_a^b f + \int_a^b g.
\]

Similar argument yields
\[
\int_a^b f + \int_a^b g \leq \int_a^b (f+g).
\]

But since \( f, g \) are Riemann integrable, \( \int_a^b f = \int_a^b f = \int_a^b f \) and \( \int_a^b g = \int_a^b g = \int_a^b g \).
and hence \( \int_a^b (f+g) = \int_a^b f + \int_a^b g \).

(iv) If \( f \geq 0 \), then \( L(f, P) \geq 0 \) since if \( P = \{x_0 < \ldots < x_m = b \} \),
then \( \inf_{x_{i-1} \leq x \leq x_i} f(x) \geq 0 \). So \( \int_a^b f \geq 0 \) and the claim follows.

\( \inf_{x_{i-1} \leq x \leq x_i} f(x) \geq 0 \)

(viii) \( P \) : parti. of \([a, c]\), \( Q \) : parti. of \([c, b]\) then
\( P \cup Q \) : parti. of \([a, b]\).

\[
U(f, P \cup Q) = \sum_{i=1}^{m} (\sup_{[x_{i-1}, x_i]} f) (x_i - x_{i-1}) + \sum_{i=1}^{n} (\sup_{[y_{i-1}, y_i]} f) (y_i - y_{i-1})
\]

\[
= U(f|_{[a, c]} \cup P) + U(f|_{[c, b]} \cup Q)
\]

and by the same technique as in (i),
\[
\int_a^c f \leq \int_a^c f|_{[a, c]} + \int_c^b f|_{[c, b]}
\]

and similarly
\[
\int_a^b f \geq \int_a^c f|_{[a, c]} + \int_c^b f|_{[c, b]}
\]

\( f : [a, b] \to [0, \infty) \) continuous and \( \int_a^b f = 0 \), then \( f(x) = 0 \ \forall x \in [a, b] \).

\[\text{Sol.} \]
Assume otherwise, \( \exists c \in [a, b] \) s.t. \( f(c) > 0 \). Then with \( \varepsilon = \frac{1}{2} f(c) \),
\( \exists \delta > 0 \) s.t. \( \forall x \in [a, b] \) with \( |x-c| < \delta \), \( |f(x) - f(c)| < \varepsilon \).

But if we write \( I = [c - \frac{\delta}{2}, c + \frac{\delta}{2}] \cap [a, b] \), then
\[
(x \in I \Rightarrow x \in [a, b] \ \text{and} \ \left|x-c\right| \leq \frac{\delta}{2} < \delta
\]
\[
\Rightarrow f(x) - f(c) > -\varepsilon
\]
\[
\Rightarrow f(x) > \frac{1}{2} f(c) = \varepsilon.
\]
So if we write \( I = [p, q] \),
\[ 0 = \int_a^b f = \int_a^p f + \int_p^b f \]
\[ \geq \int_a^p f \geq \int_p^b \varepsilon = \varepsilon (b-p) \to 0, \]

a contradiction!