\textbf{Limsup, Liminf, Limit Point}

**Keywords** - Limit point, limsup, liminf, comparison & squeezing, exponentiation

**Ex 5.1**

Let \((a_n), (b_n)\) be a sequence of reals, \(\limsup a_n, \limsup b_n < \infty\). Prove that

\[
\limsup (a_n + b_n) \leq \limsup a_n + \limsup b_n.
\]

Show by example that a strict inequality is possible.

**Sol.**

\begin{enumerate}
\item We claim that \(\sup_{k \geq n} (a_k + b_k) \leq \sup_{k \geq n} a_k + \sup_{k \geq n} b_k\). Indeed, for any \(n \in \mathbb{N}\),
\[
k \geq n \Rightarrow a_k \leq \sup_{j \geq n} a_j, \quad b_k \leq \sup_{j \geq n} b_j
\]
\[
\Rightarrow (a_k + b_k) \leq \sup_{j \geq n} a_j + \sup_{j \geq n} b_j; \text{ upper bound of } \limsup (a_n + b_n) = k \geq n
\]
and the claim follows.
\item Proof: take \(\sup\) and apply the comparison principle (Lemma 6.10).
\item For a strict ineq, let \(a_n = (-1)^n\), \(b_n = (-1)^n\). Then
  \begin{itemize}
  \item \(a_n + b_n = 0 \Rightarrow \limsup (a_n + b_n) = 0\).
  \item \(\limsup a_n = 1, \limsup b_n = 1\).
  \end{itemize}
\end{enumerate}

**Example**

Let \((r_n)_{n=1}^\infty\) be an enumeration of \(\mathbb{Q} \cap (0, 1)\). Calculate \(\liminf r_n\).

**Sol.**

We claim that \(\liminf r_n = 0\) by showing \(\inf_{k \geq n} r_k = 0\). Indeed, write

\[r_k = r_k / \delta_k \text{ with } r_k, \delta_k \in \mathbb{Z} \text{ and } r_k \geq 1, \delta_k > 1.\]

Then for any \(n\), \(Q = \max \{\delta_1, \ldots, \delta_n\}\) is a positive integer, and thus for any \(\delta > Q\), \(1 \leq k \leq n\), we have \(\frac{1}{\delta} \leq \frac{1}{\delta_k} \leq \frac{r_k}{\delta_k} = r_k\). So

\[
\frac{1}{\delta} \leq Q \left(0, 1\right) \text{ and } \frac{1}{\delta} \in \mathbb{Q} \Rightarrow \frac{1}{\delta} = r_k \text{ for some } k \geq n
\]

and we have

\[
\inf_{k \geq n} (r_k) \leq \frac{1}{\delta} \quad \forall \delta > Q.
\]

Taking \(\delta \to \infty\), we get \(\sup_{k \geq n} (r_k) \leq 0\). The reverse inequality is trivial, so

**Sol.**

\(0 < r_n \quad \forall n \Rightarrow \lim_{n \to \infty} r_n = 0\). If \(\varepsilon > 0\), pick \(N \in \mathbb{N}\) st. \(N > \frac{1}{\varepsilon}\). Then
Exercise 3.1
Show that \( \limsup (-a_n) = -\liminf a_n \).

Solution:
This follows from \( \sup (-A) = -\inf A \).

Exercise 4.3
If \( |a_n| \leq 2^n \), prove that \( b_n = a_1 + \ldots + a_n \) converges.

Solution:
(Idea) It is very unlikely that we can guess the limit. So we make an indirect approach. Prove that \((b_n)\) is Cauchy.

Pick \( \varepsilon > 0 \) and choose \( N \) s.t. \( 2^N \varepsilon > 1 \). (Indeed, choose \( N \) s.t. \( N \varepsilon > 1 \) by Archimedean principle and use \( N \leq 2^N \).) Then \( 2^N \varepsilon < 1 \) and for any \( N < m \leq n \),

\[
|b_n - b_m| = |a_m + \ldots + a_n|
\leq |a_m| + \ldots + |a_n|
\leq 2^m + \ldots + 2^n
= 2^m \left( \frac{1}{2} + \ldots + \frac{1}{2^n} \right)
\leq 2^m \cdot 2^N < \varepsilon.
\]

This proves that \((b_n)\) is Cauchy, hence convergent.

Exercise 5.6
If \( |x| < 1 \), \( \lim_{n \to \infty} x^n = 0 \).

Remark:
The following are equivalent:

1. \( \lim_{n \to \infty} a_n = 0 \).
2. \( \lim_{n \to \infty} |a_n| = 0 \).
3. \( \limsup_{n \to \infty} |a_n| = 0 \).

Proof:
1. \( \implies \) 2: \( |a_n - 0| = |a_n - 0| \) applied to the definition of convergence immediately gives the proof.
2. \( \implies \) 3: Property of \( \limsup \).
3. \( \implies \) 1: \( \forall \varepsilon > 0, \exists N \) s.t. \( \forall n \geq N \), \( \sup_{k \geq n} |a_k| < \varepsilon \implies |a_n| < \varepsilon \). \( \checkmark \)
It suffices to prove $\lim_{n \to \infty} |x|^n = 0$. Since $|x|^n$: decreasing & bounded, it converges to some $L$. Then

$$L = \lim_{n \to \infty} |x|^n = \lim_{n \to \infty} |x|^{n+1} = |x| \cdot L$$

and we have $(-1|x|) L = 0$. Since $|x| \neq 1$, we get $L = 0$. 

\[ \text{Ex 5.6} \]

If $x > 0$, \( \lim_{n \to \infty} x^{\frac{1}{n}} = 1 \).

**Sol.**

- Choose any $0 < r < 1$. Then $\lim_{n \to \infty} r^n = 0$ shows that $\exists N = N(r)$ s.t.
  $$n \geq N \Rightarrow r^n = |r^n - 0| < x.$$ Then we have $r \leq x^{\frac{1}{n}}$ for $n \geq N$ and
  $$r \leq \liminf_{n \to \infty} x^{\frac{1}{n}}.$$ But since $0 < r < 1$ is arbitrary, we have $1 \leq \liminf_{n \to \infty} x^{\frac{1}{n}}$. (Indeed, $\liminf_{n \to \infty} x^{\frac{1}{n}}$ is an upper bound of $(0, 1)$ and $\sup(0, 1) = 1$.)

- Similarly, choose any $r > 1$. Then $\lim_{n \to \infty} \frac{1}{r^n} = 0$ shows that $\exists N$ s.t.
  $$\frac{1}{r^n} < \frac{1}{x} \text{ for any } n \geq N.$$ Then $x^n \leq r$ for $n \geq N$ and
  $$\limsup_{n \to \infty} x^n \leq r.$$ Since $r > 1$ is arbitrary, this gives $\limsup_{n \to \infty} x^{\frac{1}{n}} \leq 1$.

- Therefore $\limsup_{n \to \infty} x^{\frac{1}{n}} = \liminf_{n \to \infty} x^{\frac{1}{n}} = 1$ and $\exists n \lim_{n \to \infty} x^{\frac{1}{n}} = 1$. 

**Example**

**Sol.**

Let $\varepsilon_n$ be defined by $N^{\varepsilon_n} = 1 + \varepsilon_n$. Then

$$\varepsilon_n = \left(\frac{1 + \varepsilon_n}{N}\right)^n \quad \Rightarrow \quad \varepsilon_n^2 \leq \frac{2}{n^2},$$

$$\Rightarrow \quad \varepsilon_n \leq \sqrt{\frac{2}{n}}.$$ Take $n \to \infty$ and apply squeezing to conclude: $\varepsilon_n \to 0 \Rightarrow N^{\varepsilon_n} \to 1$. 

\[ 3 \]
Cor 7.11 (Fabini's Thm for finite sets) \( X, Y \): finite sets, \( f : X \times Y \to \mathbb{R} \) a function.

Then
\[
\sum_{x \in X} \sum_{y \in Y} f(x,y) = \sum_{(x,y) \in X \times Y} f(x,y) = \sum_{(y,x) \in Y \times X} f(x,y) = \sum_{y \in Y} \sum_{x \in X} f(x,y).
\]

Remark) If \( P(x,y) \) is a statement of \( x \) and \( y \), then we also have

\[
\sum_{(x,y) \in X \times Y} f(x,y) = \sum_{\substack{(x,y) \in X \times Y \\ P(x,y) \text{ is true}}} f(x,y).
\]

Indeed, let \( S = \{ (x,y) \in X \times Y : P(x,y) \} \) and \( S' = \{ (y,x) \in Y \times X : P(x,y) \} \). Then \( \exists \) bijection \( S \to S' \) defined by \( (x,y) \mapsto (y,x) \) and hence the claim follows. For example,

\[
\sum_{i=1}^{n} \sum_{j=1}^{\tau} f(i,j) = \sum_{1 \leq j \leq \tau \leq \eta} f(i,j) = \sum_{j=1}^{n} \sum_{i=1}^{\tau} f(i,j).
\]

This kind of techniques are called double summation.

Ex 5.4

\( x, y \in \mathbb{R}, x, y > 0, m, n \in \mathbb{Z}, m, n \geq 1 \). Show (3) \( x^{kn} > 0 \).

1. \( (x^{mn})^n = x \).

2. \( (x^n)^{1/n} = x \).

3. \( (xy)^{1/m} = x^{1/m}y^{1/m} \)

4. \( (x^{1/m})^{1/m} = x^{1/(m^2)} \).

5. \( (x^{1/m})^{1/n} = \frac{x}{(mn)} \).

6. \( (x^{1/m})^{1/n} = \frac{x}{(mn)} \).

7. If \( 0 < x < 1 \), then \( 0 < x^n < x \) and by def. of \( x^m \), \( 0 < x^n < x^{1/m} \).

8. If \( x > 1 \), then \( x^n \leq x \) and \( 1 \leq x^{1/m} \).
(1): Let \( y = x^{1/m} \). Then \( y > 0 \) by (1). Now assume otherwise. Then either \( y^n < x \) or \( y^n > x \).

- Case \( y^n < x \): We claim that \((y+\varepsilon)^n < x\) for some \( \varepsilon > 0 \). Otherwise \((y+\varepsilon)^n \geq x\) for all \( \varepsilon > 0 \) and by taking \( \varepsilon = \frac{1}{k}, \ k \to \infty \), we get \( y^n \geq x \), a contradiction.

Now \((y+\varepsilon)^n < x\) for some \( \varepsilon > 0 \) shows that \( x^{1/n} \geq y + \varepsilon \), a contradiction since \( y = x^{1/n} \).

- Case \( y^n > x \) follows similarly.

In either case we get a contradiction and hence \((x^{1/m})^n = y^n = x\) !!!

(2): If \( (x^n)^{1/m} \neq x \), then either \( (x^n)^{1/m} < x \) or \( (x^n)^{1/m} > x \). Now since both are positive, taking power to the \( n \) gives

\[ x^n = (x^{1/m})^n < x^n \quad \text{or} \quad x^n = (x^{1/m})^n > x^n, \]

a contradiction!

(3): \( x^{1/m} y^{1/m} = (x^{1/m} y^{1/m})^{1/m} \) by (2).

(4): \( (x^{1/m})^{1/m} = \left( \frac{1}{m} (x^{1/m})^{1/m} \right)^{1/m} \) by (2).

(5): \( (x^{1/m})^{1/m} = \left( \frac{1}{m} (x^{1/m})^{1/m} \right)^{1/m} \) by (2).

(6): \( (x^{1/m})^{1/m} = \left( \frac{1}{m} (x^{1/m})^{1/m} \right)^{1/m} = \frac{1}{m} (x^{1/m})^{1/m} \) by (2).

(7): \( (x^{1/m})^{1/m} = \left( \frac{1}{m} (x^{1/m})^{1/m} \right)^{1/m} = \frac{1}{m} (x^{1/m})^{1/m} \) by (2).

(The idea is that \( f_n : \mathbb{R}^+ \to \mathbb{R}^+ : x \mapsto x^n \), then \( x \mapsto x^{1/n} \) is the inverse of \( f_n \), by (1) and (2). Then we can rephrase the exponentiation rules in terms of \( f_n^{-1} \). That is what we did in (6) and (7). )
**Example**

Let \( a_1 = 1 \), \( a_{nm} = 1 + \frac{1}{1 + a_n} \). Show that \((a_n)\) converges and find its limit.

**Sol.**

1. Anticipate what the limit will likely to be.

2. Show the convergence by estimating \(|a_n - L|\), where \( L \) is the anticipated limit.

   Indeed, if \( a_n \to L \) then \( a_{nm} \to L \) and we have
   
   \[
   L = 1 + \frac{1}{1+L} \quad \Rightarrow \quad L^2 = 2 \quad \Rightarrow \quad L = \sqrt{2}, \text{ since } L \geq 1.
   \]

   This heuristic observation shows that we may consider
   
   \[
   \varepsilon_n = |a_n - \sqrt{2}|.
   \]

   Indeed,
   
   \[
   \varepsilon_{nm} = |a_{nm} - \sqrt{2}| = \left| 1 + \frac{1}{1 + a_n} - \sqrt{2} \right| = \frac{|a_n + 2 - (\sqrt{2} - 1)|}{1 + a_n} = \frac{\sqrt{2} - 1}{1 + a_n} \varepsilon_n \leq (\sqrt{2} - 1) \varepsilon_n.
   \]

   Since \( 0 < \sqrt{2} - 1 < 1 \) and \( \varepsilon_1 \leq (\sqrt{2} - 1) \varepsilon_2 \leq (\sqrt{2} - 1)^2 \varepsilon_3 \leq \cdots \leq (\sqrt{2} - 1)^m \varepsilon_1 \),

   by squeezing theorem, \( \varepsilon_n \to 0 \). Then \( |a_n - L| \to 0 \implies a_n \to L \).

**Example**

Let \( a_1 = 2 \), \( a_{nm} = \frac{1}{2} \left( a_n + \frac{2}{a_n} \right) \). Show that \((a_n)\) converges and find its limit.

**Sol.**

We claim that 1. \( a_n \geq \sqrt{2} \), 2. \( a_{nm} \leq a_n \) for any \( n \). Indeed, \( a_1 \geq \sqrt{2} \) is clear. Also, \( a_3 = \frac{3}{2} \leq 2 \).

To prove induction step, assume 1 and 2. Then

\[
\begin{align*}
   a_{nm} &= \frac{1}{2} \left( a_n + \frac{2}{a_n} \right) = \frac{a_n^2 + 2}{2a_n} \\
   \Rightarrow a_{nm} - \sqrt{2} &= \frac{a_n^2 - 2\sqrt{2}a_n + 2}{2a_n} = \frac{(a_n - \sqrt{2})^2}{2a_n} \geq 0. \\
   a_{nm} - a_{n+1} &= a_{nm} - \frac{1}{2} \left( a_{n+1} + \frac{2}{a_{n+1}} \right) = \frac{a_{nm} - 2}{2a_{nm}} \geq 0.
\end{align*}
\]

Therefore \((a_n)\) bounded, monotone \( \Rightarrow \) convergent \( \Rightarrow L = \lim a_n \) satisfies

\[
L = \frac{1}{2} \left( L + \frac{2}{L} \right), \quad L \geq \sqrt{2} \implies L = \sqrt{2}.
\]