THE LEAST UPPER BOUND PROPERTY.

**DEF** For $E \in \mathbb{R}$ and $M \in \mathbb{R}$,

- $M$ is an upper bound of $E$ if $x \leq M \quad \forall x \in E$.
- $M$ is a lower bound of $E$ if $x \geq M \quad \forall x \in E$.
- $M$ is the least upper bound of $E$ if it is the smallest upper bound: $M$ is an upper bound & $\forall$ other upper bound $M'$, $M \leq M'$.

- (Likewise for greatest lower bound.)

- $\sup(E)$, the supremum of $E$, is the least upper bound of $E$.
- $\inf(E)$, the infimum of $E$, is the greatest lower bound of $E$.

**Problem** If $A, B \subset \mathbb{R}$ nonempty & bounded above, so is $A+B := \{ a+b \mid a \in A, b \in B \}$ and $\sup(A+B) = \sup A + \sup B$.

**Sol.** Clearly $A+B \neq \emptyset$. Also, $a \leq \sup A$ and $b \leq \sup B \quad \forall a \in A, b \in B$ and

$\sup A + \sup B$ is an upper bound of $A+B$. Thus $\sup(A+B) \leq \sup A + \sup B$.

To prove the converse, $\forall \varepsilon > 0$ choose $a \in A, b \in B$ such that

$\sup A - \frac{\varepsilon}{2} < a$ and $\sup B - \frac{\varepsilon}{2} < b$. (This is possible since $\sup A - \frac{\varepsilon}{2}$ is not an upper bound of $A$ and likewise for $B$). Then

$(\sup A + \sup B) - \varepsilon = (\sup A - \frac{\varepsilon}{2}) + (\sup B - \frac{\varepsilon}{2}) < a + b \leq \sup(A+B)$.

So we have

$0 \leq (\sup A + \sup B) - \sup(A+B) < \varepsilon \quad \forall \varepsilon > 0$. \hspace{1cm} (\star)

To complete the proof, assume $\varepsilon > 0$. By the Archimedean property, there exists $N$ such that $N \varepsilon > 1$, or equivalently, $\varepsilon > \frac{1}{N}$. Now plug $\varepsilon = \frac{1}{N}$ to (\star) to get a contradiction.

**Remark** If $x \in \mathbb{R}$ satisfies $0 < x < \varepsilon \quad \forall \varepsilon > 0$, then $x = 0$. This follows from the Archimedean property.
**Cardinality of a Set**

**Def**

For \( f : X \rightarrow Y \) a function,

- \( f \) is **injective** if \( \forall x, x' \in X, \quad x \neq x' \Rightarrow f(x) \neq f(x') \).
- \( f \) is **surjective** if \( \forall y \in Y, \exists x \in X \text{ s.t. } y = f(x) \).
- \( f \) is **bijective** if both injective and surjective.

**Def**

- \( X, Y \) have **same cardinality** if \( \exists \) bijection \( X \rightarrow Y \).
- \( X \) have **cardinality** \( n \in \mathbb{N} \) if \( \exists \) bijection \( 1, \ldots, n \rightarrow X \).
- \( X \) is **finite** if it has cardinality \( n \) for some \( n \in \mathbb{N} \). Otherwise, **infinite**.
- \( X \) is **countable** if \( \exists \) bijection \( \mathbb{N} \rightarrow X \).
- \( X \) is at **most countable** if either finite or countable.

**HW#3.2**

Show that \( \mathbb{N} \) is infinite by contradiction.

**Sol**

Suppose that \( \mathbb{N} \) is finite: \( \exists \) bijection \( f : 1, \ldots, n \rightarrow \mathbb{N} \). Then put \( m = \max \{ f(1), \ldots, f(n) \} + 1 \). This is also a natural number such that \( f(k) < m \) for \( 1 \leq k \leq n \). So \( f(k) \neq m \ \forall \ 1 \leq k \leq n \). On the other hand, we must have \( m = f(k) \) for some \( k \in 1, \ldots, n \) by the surjectivity, \( \times \).

**Remark**

The only defining structure on \( \mathbb{N} \) is the order property, characterized by its inductive structure. A major consequence of this property is that \( \mathbb{N} \) enjoys the **Well-Ordering Principle**: Any non-empty subset of \( \mathbb{N} \) has a minimum.

**HW#3.3**

Let \( X \subseteq \mathbb{N} \). Show that \( X \) is at most countable.

**Sol**

We prove that any infinite subset \( X \) of \( \mathbb{N} \) is countable.
(IDEA) Arrange the elements of \( X \) in the increasing order, and use it to define a bijection.

To this end, we recursively define \((a_n)_{n=0}^\infty\) as follows:

\[
\begin{align*}
(a_0 &= \min X, \\
(a_{n+1} &= \min (X - \varepsilon a_n, \ldots, a_n^3).
\end{align*}
\]

Indeed, this gives a well-defined sequence \((a_n)\), which can be proved by induction:

CLAIM: \((*)\) defines \(a_0, \ldots, a_n\) and they satisfy \(a_0 < a_1 < \ldots < a_n\).

- **Base case:** Since \(X\) is non-empty, \(a_0 = \min X\) is well-defined.

- **Induction step:** Suppose the claim is true for \(n \geq 0\). Since \(a_n \neq a_{n+1}\), if 
  
  \[\varepsilon a_n, \ldots, a_n^3, a_{n+1}\] 

  is injective. Since \(X\) is infinite, this map cannot be surjective and 

  \[a_{n+1} = \min (X - \varepsilon a_n, \ldots, a_n^3)\]

  is well-defined. (we used the fact that \(\emptyset \neq S \subset \mathbb{N}\), then \(\min S \geq \min S\))

To prove that \(a_n < a_{n+1}\), we first notice that 

\[\varepsilon a_n, \ldots, a_n^3, a_{n+1} \implies X - \varepsilon a_n, a_n^3 \subset X - \varepsilon a_n, a_{n+1}\]

\[\implies a_{n+1} = \min (X - \varepsilon a_n, a_n^3) \geq \min (X - \varepsilon a_n, a_n^3) = a_n.\]

If it happens that \(a_{n+1} = a_n\), then 

\[a_n = a_{n+1} \in X - \varepsilon a_n, a_n^3\]

and we get a contradiction! So \(a_{n+1} > a_n\), and \((*)\) holds with \(n+1\)

Instead of \(n\).

Using this, we constructed \((a_n)_{n=0}^\infty\) satisfying \((*)\). Then we define \(f: \mathbb{N} \to X\) by \(f(k) = a_k\). Clearly \(f\) is injective. (Indeed, this is strictly increasing.) Moreover, \(f\) is not surjective then pick \(m = \min (X - f(\mathbb{N}))\). Since 

- \(m \neq a_0 \implies m > a_0 > 0,\)

- If \(m > a_n > n\) then \(m \neq \min (X - \varepsilon a_n, \ldots, a_n^3) = a_{n+1}\) shows 
  
  \[m > a_n, a_{n+1} > n \implies m > a_{n+1} \geq n+1.\]
This gives \( m \geq n \ \forall n \in \mathbb{N} \), a contradiction by letting \( n = m+1 \). So \( f \) is also surjective and hence bijective.

**HW #3.4**

\( f : \mathbb{N} \to Y \) a function. Then \( f(\mathbb{N}) \) is at most countable.

**Sol.** Define \( A = \{ n \in \mathbb{N} : f(n) \neq f(m) \text{ for all } 0 \leq m < n \} \). We can define \( f_A : A \to f(\mathbb{N}) \) by \( f_A(n) = f(n) \) for \( n \in \mathbb{N} \).

(Since \( f(m) \in f(\mathbb{N}) \ \forall n \in A \), \( f_A \) is well-defined.) We claim that \( f_A \) is bijective:

1. **(Injectivity)** Suppose \( m, n \in A \) be distinct. We may assume \( m < n \).

   Then \( f_A(m) \neq f_A(n) \) by the construction of \( A \).

2. **(Surjectivity)** Let \( y \in f(\mathbb{N}) \). Then the set \( X = \{ n \in \mathbb{N} : f(n) = y \} \) is non-empty and we can pick \( n = \min X \in \mathbb{N} \). We claim \( n \in A \).

   Indeed, if \( m < n \) then \( f(m) \neq y = f(n) \) by our choice of \( n \). So \( n \in A \) indeed and \( f_A(n) = y \).

Therefore 1 + 2 shows that \( A \) and \( f(\mathbb{N}) \) have the same cardinality. Since \( A \subset \mathbb{N} \) is at most countable by #3.3, so is \( f(\mathbb{N}) \).

**Remark.** There are at least 2 ways of proving that a map is bijective:

- Check if it is both injective & surjective, or
- Check if it has an inverse.

The latter is useful when we can easily construct a function which should serve as an inverse function.
**Lemma**

If \((a_n)_{n=0}^{\infty}\) seq. of rationals and \(x = \lim_{n \to \infty} (a_n)\) as formal limit, then
\[ x = \lim_{n \to \infty} a_n \text{ as limit in } \mathbb{R}. \]

**Proof**

\((\forall \varepsilon > 0)\) using the Archimedes principle choose \(\varepsilon' \in \mathbb{Q}\) with \(0 < \varepsilon' < \varepsilon\).

(Pick \(N \in \mathbb{N}\) with \(N\varepsilon > 1\) and let \(\varepsilon' = \frac{1}{N}\).) Then \(\exists N\) such that
\[ \forall j, k \geq N, \quad |a_j - a_k| < \varepsilon'. \]

Then \(\forall n \geq N\), we have
\[ \forall k \geq N \quad |a_k - a_n| < \varepsilon' \]
\[ \implies \forall k \geq N \quad a_n - \varepsilon' \leq a_k \leq a_n + \varepsilon' \]
\[ \implies \lim_{k \to \infty} (a_n - \varepsilon') \leq \lim_{k \to \infty} a_k \leq \lim_{k \to \infty} (a_n + \varepsilon') \quad \text{by Prop 6.30} \]
\[ \implies a_n - \varepsilon' \leq x \leq a_n + \varepsilon' \]
\[ \implies |a_n - x| \leq \varepsilon' < \varepsilon. \]

Summarizing, we have proved that
\(\forall \varepsilon > 0, \exists N \text{ s.t. } n \geq N \text{ then } |a_n - x| < \varepsilon\)
\[ \implies (a_n)_{n=0}^{\infty} \text{ converges to } x. \]