\[ x \in (A \cap B)^c \iff \neg (x \in A \cap B) \iff \neg (x \in A \text{ and } x \in B) \iff (\neg x \in A) \text{ or } (\neg x \in B) \iff (x \in A^c) \text{ or } (x \in B^c) \iff x \in A^c \cup B^c \] (\because \text{ def. of complement})

\[ \text{III/III} \]

HW\#1.3

We prove that \( n+m \) is a natural number using mathematical induction on \( n \).

1. When \( n = 0 \), \( 0 + m = m \in \mathbb{N} \) by definition.

2. Suppose \( n+m \in \mathbb{N} \) for all \( m \in \mathbb{N} \). What we want to prove is that \( (n++)+m \in \mathbb{N} \) for all \( m \). But since \( n+m \in \mathbb{N} \),

\[ (n++)+m = (n+m)++ \in \mathbb{N} \text{ by (PA2)} \]

and \( (n++)+m \) is again an integer for any \( m \in \mathbb{N} \).

Therefore by (PA5), \( n+m \) is a natural number for any \( n,m \in \mathbb{N} \). \( \text{III/III} \)

HW\#1.4

Choose arbitrary \( x, y \in \mathbb{N} \). We show that \( x+(y+z) = (x+y)+z \) for all \( z \in \mathbb{N} \) by mathematical induction on \( z \).

1. When \( z = 0 \),
\[ x+(y+0) = x+y \quad \text{(by Lemma 2.1)} \]
\[ (x+y)+0 = x+y \quad \text{(by Lemma 2.1)} \]

2. Assume that \( x+(y+z) = (x+y)+z \) for \( z \in \mathbb{N} \). Then what we want to prove is that \( x+(y+(z++)) = (x+y)+(z++) \). Indeed,

\[ x+(y+(z++)) = x+(y+(z++)) \quad \text{(by Lemma 2.9)} \]
\[ = (x+(y+z))++ \quad \text{(by Lemma 2.9)} \]

and

\[ (x+y)+z++ = ((x+y)+z)++ \quad \text{(by Lemma 2.9)} \]
\[ = (x+(y+z))++ \quad \text{(by induction hypothesis)}. \]

Therefore they are the same and the claim follows by induction. \( \text{III/III} \).
(1) \( a = a + 0 \) implies \( a \geq a \).

(2) \( a \geq b \Rightarrow \exists m \in \mathbb{N}, a = b + m \)
\[ \{b \geq c \Rightarrow \exists n \in \mathbb{N}, b = c + n\} \Rightarrow a = c + (m+n), \ m+n \in \mathbb{N}. \]

This proves \( a \geq c \).

(3) \( a \geq b \Rightarrow \exists m \in \mathbb{N}, \ a = b + m \)
\[ \{b \geq a \Rightarrow \exists n \in \mathbb{N}, b = a + n\} \Rightarrow \]
\[ a = a + (m+n), \ \Rightarrow \]
\[ o = m+n \] (\(*: \text{cancellation law}\)
\[ \Rightarrow m = n = 0 \] (\(*: \text{Corollary 2.18}\)
\[ \Rightarrow b = a. \]

(4) \( a \geq b \iff \exists m \in \mathbb{N}, a = b + m \)
\[ \iff \exists m \in \mathbb{N}, a + c = b + c + m \]
\[ \iff a + c \geq b + c. \]

(5) \( a < b \iff a \leq b \text{ and } a \neq b \)
\[ \iff a + c \leq b + c \text{ and } a + c \neq b + c \]
\[ \iff a + c < b + c. \]

HW#1.6

Let \( \delta > 0 \). We prove \( \exists m \in \mathbb{N}, \exists r \in \mathbb{N} \) st. \( n = m \delta + r, \ 0 \leq r < \delta \) by induction.

1. \( 0 = 0 \cdot \delta + 0 \) implies that \( m = 0, \ r = 0 \) satisfies the condition.

2. Assume \( n = m \delta + r \). We want to prove that \( \exists m' \in \mathbb{N}, \exists r' \in \mathbb{N} \) such that
\[ n+1 = m' \delta + r' \].
By considering that
\[ n+1 = (m \delta + r) + 1 = m \delta + (r+1) \]
by induction hypothesis, we divide into two cases:

- Case 1: If \( 0 \leq r < \delta - 1 \), then choose \( m' = m \) and \( r' = r + 1 \). Then
\[ 0 \leq 1 \leq r' < \delta \]
\[ n+1 = m \delta + (r+1) = m' \delta + r'. \]

- Case 2: If \( r = \delta - 1 \), then choose \( m' = m + 1 \) and \( r' = 0 \). Then
\[ n+1 = m \delta + (r+1) = m \delta + \delta = (m+1) \delta + 0 = m' \delta + r'. \]
Thus \( n \) also satisfies the condition.

Therefore, by induction on \( n \), every \( n \in \mathbb{N} \) admits such \( m \) and \( r \).

\[ \text{HW#1.7} \]

We prove this by contradiction. Assume such a sequence \( P_0 > P_1 > \ldots \) exists.

We claim the following:

- **CLAIM** \( \forall n \in \mathbb{N} \) and \( \forall N \in \mathbb{N}, \ P_n \geq N \).

**Proof of Claim**

We use induction on \( N \).

1. Since \( P_n \in \mathbb{N}, \ P_n \geq 0 \) is obvious.

2. Assume \( P_n \geq N \) for any \( n \in \mathbb{N} \). We want to prove that

\[ P_n \geq N+1 \text{ for any } n \in \mathbb{N}. \]

Note that by the condition on \( (P_n) \) and the induction hypothesis,

\[ P_n > P_{n+1} \geq N. \]

Since \( P_n \geq N \), \( P_n = N+m \) for some \( m \in \mathbb{N} \). But if \( m=0 \), then \( P_n = N \).

Then \( P_m \geq N \) (induction hypo.) and \( N = P_n \geq P_m \) implies \( P_m = N \) as well,

which implies \( P_{m+1} = P_n \), contradicting \( P_0 > P_n \) by trichotomy. So \( m \neq 0 \). This implies \( m \geq 1 \) by the following lemma:

**Lemma** Every natural number other than 0 is the successor of some natural number. Consequently, \( n \neq 0 \) implies \( n \geq 1 \).

**Proof** Define \( S \) by \( S = \{ n \in \mathbb{N} : n \text{ is the successor of some natural number} \} \).

1. \( 0 \in S \) \( \cup \) \( \varepsilon_0 \).

2. If \( n \in S \), \( m \in S \cup \varepsilon_0 \) by the construction of \( S \).

So by induction on \( n \), \( S \cup \varepsilon_0 = \mathbb{N} \). Now since \( 0 \notin S \) by (PA3),

\[ S = \mathbb{N} - \varepsilon_0. \]

That means, if \( n \neq 0 \) then \( n \in S \) and so \( n = m + 1 \) for some \( m \in \mathbb{N} \). Since \( n = m + 1 \), this implies \( n \geq 1 \).

Thus \( P_n = N+m \geq N+1 \) as desired.

Now for any \( n \), pick \( N \geq P_n + 1 \). Then the claim shows \( P_n \geq P_n + 1 \),
which implies $0 > 1$, a contradiction to the trichotomy.

**HW#1.8**

Choose $(a_j)$ as follows:

$$a_j = \begin{cases} 1 & \text{if } j = 1 \\ -1 & \text{if } j = 2+1 \\ 0 & \text{otherwise} \end{cases} \Rightarrow (a_j) = \begin{pmatrix} 1 & -1 & 0 & 0 & \cdots \\ 0 & 1 & -1 & 0 & \cdots \\ 0 & 0 & 1 & -1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \end{pmatrix}.$$

Then on the one hand,

$$\sum_{j=1}^{\infty} a_j = 0 + \cdots + 0 + a_{2+1} + a_{3+1} + 0 + \cdots = 1 - 1 = 0.$$

$$\Rightarrow \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij} = 0.$$

On the other hand, if $j \geq 2$ then

$$\sum_{i=1}^{\infty} a_{ij} = 0 + \cdots + 0 + a_{i+1} + a_{i+2} + 0 + \cdots = -1 + 1 = 0.$$

but if $j = 1$, then

$$\sum_{i=1}^{\infty} a_{ij} = a_{i1} + 0 + \cdots = 1.$$

So we have

$$\sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij} = 1.$$

**HW#2.1**

1. We first prove that $-|x| \leq x \leq |x|$.
   
   1) If $x \geq 0$, then $|x| = x$ and hence

   $$-|x| = -x \leq 0 \leq x = |x| \leq |x|.$$

   2) If $x < 0$, then $|x| = -x > 0$ and hence

   $$-|x| = -(x) = x < 0 < -x = |x|.$$

2. Now we prove that $|x+y| \leq |x| + |y|$.
   
   1) $x+y \geq 0$, then $|x+y| = x+y \leq |x| + |y|.$

   2) $x+y < 0$, then $|x+y| = -(x+y) = -x-y \leq |x| + |y|.$