EQUIVARIANT INSTANTON HOMOLOGY

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Abstract. We define four versions of equivariant instanton Floer homology \((I^+, I^-, I^\infty)\) and \(\hat{I}\) for a class of 3-manifolds and \(SO(3)\)-bundles over them including all rational homology spheres. These versions are analogous to the four flavors of monopole and Heegaard Floer homology theories. This construction is functorial for a large class of 4-manifold cobordisms, and agrees with Donaldson’s definition of equivariant instanton homology for integer homology spheres. Furthermore, one of our invariants is isomorphic to Floer’s instanton homology for admissible bundles, and we calculate \(I^\infty\) in all cases it is defined, away from characteristic 2.

The appendix, possibly of independent interest, defines an algebraic construction of three equivariant homology theories for dg-modules over a dg-algebra, the equivariant homology \(H_+^+(A, M)\), the coBorel homology \(H^-\), and the Tate homology \(H^\infty\). The constructions of the appendix are used to define our invariants.
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INTRODUCTION

In [Flo88], Andreas Floer introduced the instanton homology groups $I_\bullet(Y)$, Z/8-graded abelian groups associated to integer homology 3-spheres. These form a sort of TQFT in which oriented cobordisms $W : Y_0 \to Y_1$ induce homomorphisms on the corresponding instanton homology groups. Since then, similar TQFT-style invariants have found themselves a powerful tool in 3- and 4-dimensional topology, especially the related monopole Floer homology of [KM07] and the Heegaard Floer homology of [OS04].

In ideal circumstances, the instanton homology groups are defined by a chain complex generated by irreducible flat $SU_p^q$ connections up to isomorphism (equivalently, representations $\pi_1(Y) \to SU(2)$ whose image is non-abelian, modulo conjugacy by elements of $SU(2)$); the component of the differential between two flat connections $\alpha_-^\prime, \alpha_+^\prime$ is given by an algebraic count of solutions on the cylinder $\mathbb{R} \hat{\times} Y$ to the ASD equation

$$F_{A^\prime} = 0,$$

where $A^\prime$ is a connection on the trivial $SU(2)$-bundle over the cylinder which is asymptotically equal to the $\alpha_{\pm}$. We can think of this as the “Morse chain complex” of the Chern-Simons functional on the space of irreducible connections modulo gauge equivalence, $B^+_Y = A^+_Y / G$, whose critical points are the flat connections and gradient flow equation is (formally) the ASD equation. While these equations depend on a choice of metric on the 3-manifold $Y$, the homology groups are an invariant of the $Y$ itself.

Floer’s theory is constrained to homology 3-spheres because of the presence of reducible connections. While the instanton chain complex above is still a chain complex for rational homology spheres, the proof that its homology is independent of the choice of metric fails in the presence of $U(1)$-reducible connections (corresponding to representations $\pi_1(Y) \to SU(2)$ with image lying inside a circle subgroup). One would need to take these reducible connections into account in the definition of the chain complex, but this cannot be done naively: while the instanton chain complex is a Morse complex for the space of irreducible connections, which is an infinite-dimensional manifold, the gauge group $G$ does not act freely on the entire space of connections, and so the configuration space of all connections modulo gauge, $B_Y$, is not a manifold.

Austin and Braam in [AB96] resolve this difficulty for a class of 3-manifolds (including all rational homology spheres) by defining an invariant called the equivariant instanton homology of $Y$, a Z/8-graded $\mathbb{R}$-vector space $I^G_\bullet(Y)$ with an action of $\mathbb{R}[U] = H^*(BSO(3); \mathbb{R}$), to be a form of $SO(3)$-equivariant Morse theory on an infinite-dimensional $SO(3)$-manifold $\bar{B}_Y$ with

$$\bar{B}_Y / SO(3) = B_Y.$$
The manifold $\tilde{B}_Y$ might be called the configuration space of framed connections on (the trivial $SO(3)$-bundle over) $Y$. Their invariant is defined using the equivariant de Rham complex as a model form the equivariant (co)homology of a smooth $G$-manifold, and thus inherently uses real coefficients.

Floer also defined, in [Flo91], instanton homology groups for $SO(3)$-bundles $E$ over 3-manifolds $Y$ satisfying the admissibility criterion that $\omega_3(E) \in H^2(Y;\mathbb{Z}/2)$ lifts to a non-torsion class in $H^2(Y;\mathbb{Z})$; in particular, $b_1(Y) > 0$. In this case, there are no reducible connections, and the homology of the Floer complex is a well-defined invariant of the pair $(Y,E)$. This case is important for his work on surgery triangles in instanton homology.

Using this, Kronheimer and Mrowka introduce framed instanton homology groups $I^\#(Y,E)$ for an arbitrary $SO(3)$-bundle over a 3-manifold in [KM11b] by studying the instanton homology of $(Y\#T^3,E\#Q)$ for a certain admissible bundle $Q$ over $T^3$. This is meant to be a version of the (non-equivariant) Morse homology of the space of framed connections $\tilde{B}_E$.

In this paper, we jointly generalize Floer’s work on admissible bundles and Austin-Braam’s work for rational homology spheres; to speak of both in the same breath, we say that an $SO(3)$-bundle $E$ over a 3-manifold $Y$ is weakly admissible if either $\omega_3(E)$ has no lifts to a torsion class in $H^2(Y;\mathbb{Z})$, or if $b_1(Y) = 0$.

We take an alternate approach to Kronheimer and Mrowka’s to the framed instanton homology groups: instead of taking a connected sum with $T^3$, we work on the space of framed connections $\tilde{B}_E$ itself. We do this with a sort of Morse-Bott complex for a smooth $G$-manifold equipped with an equivariant Morse function; our definition is partly inspired by the Morse-Bott complex introduced for monopole Floer homology in [Lin18]. This uses Lipyanskiy’s notion of the geometric chain complex $C^{gm}_\bullet(X;R)$ of a smooth manifold $X$, introduced in [Lip14], whose homology gives the usual singular homology of $X$.

While there are technical obstructions to carrying this out for all 3-manifolds, this has the advantage of providing more structure: for $(Y,E)$ a weakly admissible bundle and $R$ a commutative ground ring, we can define a $\mathbb{Z}/8$-graded chain complex of $R$-modules, $\hat{C}I(Y,E,\pi;R)$, which carries the action of the differential graded algebra $C_\bullet(SO(3);R)$. (This is what we find to be the cleanest notion of a chain complex with an action of the Lie group $SO(3)$.) This chain complex depends on further data $\pi$, including a metric on the 3-manifold itself and a perturbation of the functional defining the Morse complex, but this turns out to be mostly inessential: associated to a perturbation $\pi$ on a pair $(Y,E)$ is an element of a finite set $\sigma(Y,E)$ of signature data; this will be defined in section 3.5. For concreteness, we remark that if $Y$ is a rational homology sphere whose universal abelian cover $\tilde{Y}$ has $H^1(\tilde{Y};\mathbb{C}) = 0$, then $\sigma(Y,E)$ consists of a single element for all $E$. The set $\sigma(Y,E)$ corresponds precisely to the “natural classes of perturbations” stated in the main theorem of [AB96].

In fact, the TQFT structure of the usual instanton Floer homology groups can be lifted to the level of the homology groups of $\hat{C}I$. This is the main theorem of this paper.

**Theorem 1.** There is a category $\text{Cob}_{3,b}^{U(2),w}$ of based 3-manifolds $(Y,E,\sigma,b)$ equipped with weakly admissible $U(2)$-bundles and signature data, whose morphisms are certain ‘weakly admissible, $\rho$-monotonic’ oriented cobordisms $(W,\tilde{E})$ equipped with a
path between the basepoints on the ends. There is also a category of relatively \( \mathbb{Z}/8 \)-graded \( R \)-modules with an action of \( H_\bullet (SO(3); R) \), and a functor
\[
\tilde{I} : \text{Cob}_{A,b}^{U(2),w} \to \text{Mod}_{H_\bullet (SO(3); R)}^\mathbb{Z}/8.
\]

The relatively graded group \( \tilde{I}(Y, E, \sigma; R) \) is called the framed instanton homology of the triple \( (Y, E, \sigma) \). When \( Y \) is a rational homology sphere equipped with the trivial bundle, the relative grading may be lifted to an absolute \( \mathbb{Z}/8 \)-grading.

Note the sudden shift to \( U(2) \) bundles in this statement. This is to pin down signs in the definition of the differential and cobordism maps, and nothing else: the majority of this text is written in the context of \( SO(p,q) \)-bundles, and we only make the passage to \( U(2) \)-bundles in Section 4.9, in our discussion of orientations on the moduli spaces. In particular, if we had chosen to work over a coefficient ring where \( 1 = -1 \), we may omit discussion of \( U(2) \)-bundles entirely.

The \( \rho \)-monotonicity condition is defined in terms of the Atiyah-Patodi-Singer \( \rho \) invariant for flat connections and the signature data \( \sigma \): roughly, their sum should always increase across the cobordism. This condition has to do with achieving transversality normal to the reducible locus; when it fails, it is not clear how to try to define the cobordism maps.

For a more precise statement, see Section 3.5 for Definition 3.3 of signature data; the \( \rho \)-invariant and \( \rho \)-monotonicity condition are introduced at the end of Section 4.5, and Definition 4.14 gives the definition of weakly (and fully) admissible bundles. Finally, Section 5.2 contains Definition 5.10 of the weakly admissible cobordism category (and below it, two relatives) as well as Theorem 5.17 defining the framed instanton functor.

It is our expectation that the notion of signature data above is inconsequential:

**Conjecture 1.** If \( Y \) is a rational homology sphere, \( \tilde{C}_I^\bullet (Y, E, \sigma; R) \) is independent of the choice of signature data \( \sigma \) up to graded \( C_\bullet (SO(3); R) \)-equivariant quasi-isomorphism.

From here, if \( R \) is a principal ideal domain, we may construct a complex \( CI^\bullet (Y, E, \sigma; R) \) using the bar construction (topologically, the Borel construction) on \( \tilde{C}_I^\bullet \). Its homology groups are denoted \( I^\bullet_+(Y, E, \sigma; R) \) and form a relatively graded module under the action of \( H^\bullet (BSO(3); R) \). This construction is standard in algebraic topology, and reviewed in the appendix.

One appealing feature of the monopole and Heegaard Floer theories (which are in some sense \( S^1 \)-equivariant homology theories) is the existence of two other variants of the Floer homology groups that fit into an exact triangle. (Largely, we choose our notation to fit with that of Heegaard Floer theory.)

In the appendix, we describe algebraic constructions \( C^\bullet (A, M) \), where \( \bullet = +, -, \infty \), where \( A \) is a dg-algebra and \( M \) a dg-module. \( C^{-}(A, M) \) is a cobar construction, dual to the above bar construction, and \( C^\infty(A, M) \) is the Tate complex, constructed as a comparison between a variant \( C^+ \) and \( C^- \). Thus up to a sort of twist (often only a grading shift), we have an exact sequence
\[
0 \to C^+(A, M) \to C^-(A, M) \to C^\infty(A, M) \to 0,
\]
leading to an exact triangle of homology groups. The Tate homology groups \( H^\infty(A, M) \) satisfy a short list of axioms, including that \( H^\infty(A, A) = 0 \).
Applying a version of this for $\mathbb{Z}/8$-graded complexes to the framed instanton complex $\tilde{C}I$ as a module over $C_\bullet(SO(3); R)$, and taking homology, we arrive at the equivariant instanton homology groups

$$I^+(Y, E, \sigma; R), I^-(Y, E, \sigma; R), I^\infty(Y, E, \sigma; R).$$

As desired, these are all modules over $H^*(BSO(3); R)$, and fit into an exact triangle.

**Theorem 2.** There are functors from the cobordism category of pointed 3-manifolds equipped with weakly admissible $U(2)$ bundles and signature data to the category of relatively $\mathbb{Z}/8$-graded $H^*-*(BSO(3); R)$-modules,

$$I_\bullet^\bullet : \text{Cob}^{U(2), w}_{3,b} \to \text{Mod}^{E_{\mathbb{Z}/8}}_{H^*-*(BSO(3); R)},$$

$\bullet = +, -, \infty$. When $Y$ is a rational homology sphere, $I^*(Y, E, \sigma; R)$'s relative grading lifts to an absolute $\mathbb{Z}/8$-grading. Furthermore, there is a long exact sequence

$$\ldots \to I^+[3], I^- \to I^\infty \xrightarrow{-[4]}, I^+ \to \ldots$$

where $[n]$ denotes that the map increases grading by $n$.\(^1\)

The main tool we use to establish invariance properties and perform calculations and compare with existing theories is a collection of spectral sequences that calculate these equivariant homology groups.

**Theorem 3.** Let $(Y, E, \sigma)$ be a closed oriented 3-manifold equipped with weakly admissible $U(2)$-bundle and signature datum. Supposing $\pi$ is a regular perturbation for $(Y, E, \sigma)$, we denote by $\mathcal{C}_\pi$ the set of critical $SO(3)$-orbits.

Then there are $\mathbb{Z}/8 \times \mathbb{Z}$-graded spectral sequences

$$\bigoplus_{\alpha \in \mathcal{C}_\pi} H^*_{SO(3)}(\alpha; R) \to I^*(Y, E, \sigma; R).$$

These spectral sequences converge strongly for $\tilde{I}, I^+, \text{ and } I^-$, meaning they can be used to calculate the homology groups, and converge weakly for $I^\infty$, meaning that we can detect quasi-isomorphisms as isomorphisms on some finite page $E^r$.

This is proved in two parts, Theorems 5.18 and 5.21, first for nonequivariant homology and then for equivariant homology.

Using the fact that Tate homology of a free $\mathbb{A}$-module vanishes, the groups $I^\infty(Y, E)$ are especially computable. For an admissible bundle, they vanish, and we can compare $I^+$ to Floer’s invariant $I_\bullet(Y, E; R)$ for admissible bundles.

**Theorem 4.** If $E$ is an admissible bundle over a 3-manifold $Y$, then

$$I^\infty(Y, E; R) = 0, \quad \text{and } I^+(Y, E; R) \cong I_\bullet(Y, E; R).$$

If we also have $\frac{1}{2} \in R$, the action of $H^*(BSO(3); R) = R[U]$ on $I^+(Y, E; R)$ is taken by the isomorphism to the standard Floer-theroetic $U$-map.

The isomorphisms are given in Theorem 6.1 and Corollary 6.9.

Let $\frac{1}{2} \in R$. In [Don02], Donaldson introduced three chain complexes for $Y$ an integer homology sphere equipped with the trivial $SU(2)$-bundle: the framed complex $\tilde{C}F(Y; R)$, the equivariant homology complex $\tilde{C\mathcal{E}}F(Y; R)$, and the equivariant cohomology complex $\tilde{C\mathcal{E}}F(Y; R)$. The first complex has an action of the exterior

\(^1\)Note that the grading shift is only meaningful when $Y$ is a rational homology sphere, for which the grading is absolute.
algebra $\Lambda(u)$, with $|u| = 3$, and the second two complexes have an action of $R[U]$, where $|U| = -4$. In Section 2, we prove the following.

**Theorem 5.** For $(Y, E)$ a 3-manifold equipped with a weakly admissible bundle, there is a finite dimensional $\Lambda(u)$-module $\overline{DCI}(Y, E; R)$ and a finite type $R[U]$-modules $\overline{DCI}^\pm(Y, E; R)$, so that there are equivariant quasi-isomorphisms

$$\overline{DCI}(Y, E; R) \cong \overline{CI}(Y, E; R)$$

$$\overline{DCI}^+(Y, E; R) \cong CI^+(Y, E; R)$$

$$\overline{DCI}^-(Y, E; R) \cong CI^-(Y, E; R)$$

Furthermore, when $Y$ is an integer homology sphere, we have equalities

$$\overline{DCI}(Y, E; R) = \overline{CI}(Y, E; R)$$

$$\overline{DCI}^+ = \overline{CI}^+$$

$$\overline{DCI}^- = \overline{CI}^-$$

up to a rescaling of basis.

Here finite type means that it is a direct sum of finitely many simple pieces: for $\overline{DCI}^+$, they are $R$ with trivial $U$-action and $R[U^+]$, where $U$ contracts against $U^+$; for $\overline{DCI}^-$, they are $R$ with trivial $U$-action and $R[U]$ with canonical $U$-action.

It is by passing through this isomorphism that we show that the isomorphisms Theorem 4 above preserve the $U$-action; this is Corollary 6.7.

While more complicated than the Tate calculations for admissible bundles, we are able to exploit the isomorphisms above to calculate instanton Tate homology for an arbitrary rational homology 3-sphere in Section 3.

**Theorem 6.** Let $(Y, E, \sigma)$ be a rational homology 3-sphere equipped with $U(2)$-bundle and signature datum, and suppose $R$ is a PID in which 2 is invertible.

We write $R[H^2(Y)]$ to mean the group algebra of the finite group $H^2(Y; \mathbb{Z}) \cong H_1(Y; \mathbb{Z})$. If $c = c_1 E$ is $H^2(Y)$, there is an action of $\mathbb{Z}/2$ on $H^2(Y)$ given by the involution $x \mapsto c - x$. Further define an action of $\mathbb{Z}/2$ on the ring $R[U^{1/2}, U^{-1/2}]$, acting on the basis by $U^{n/2} \mapsto (-1)^n U^{n/2}$. Here $|U^{1/2}| = -2$.

Then there is a canonical isomorphism of $\mathbb{Z}/8$-graded modules over $R[U]$,

$$I^Z(Y, E, \sigma; R) \cong R[U^{1/2}, U^{-1/2}] \otimes_{R[\mathbb{Z}/2]} R[H^2(Y)].$$

**Remark 1.** In [AB96], an equivariant instanton Floer homology is associated to any 3-manifold with $H_1(Y; \mathbb{Z}) \cong \mathbb{Z}^a \oplus (\mathbb{Z}/2)^b$. Their invariant is defined by careful (non-generic) choices of equivariant perturbation. We expect that the instanton complex defined in this paper can be extended to that level of generality, as well as the proof of its invariance up to signature data. However, the analysis of this case is somewhat more delicate, and we do not undertake it here.

**Survey of the homology theories.** As a number of different chain complexes are introduced in this paper and exist in the literature, we survey here the definitions and relationships.

Here $(Y, E)$ is a closed oriented 3-manifold with weakly admissible $U(2)$-bundle, $\pi$ is a regular perturbation and $R$ is a PID. This means, in particular, that there are
The finite set whose elements are connected components of the critical set is written \( \mathcal{C}_\pi \); an element \( \alpha \in \mathcal{C}_\pi \) is an \( SO(3) \)-space, either a point, \( S^2 \), or \( SO(3) \) itself. If \( E \) is equipped with a trivialization, each \( \alpha \in \mathcal{C}_\pi \) has an associated grading \( i(\alpha) \in \mathbb{Z}/8 \); otherwise, we instead have relative gradings \( i(\alpha, \beta) \in \mathbb{Z}/8 \). For uniformity of notation we often write the relative grading as if it were given by an absolute grading.

The first chain complex introduced is \( \hat{\mathcal{C}}I(Y, E, \pi; R) \). As an \( R \)-module, this is given by

\[
\bigoplus_{\alpha \in \mathcal{C}_\pi} C^\text{gm}_* (\alpha; R)[i(\alpha)].
\]

The individual terms \( C^\text{gm}_* (\alpha; R) \) are the geometric chain complexes of the orbits \( \alpha \) described in Section 5.1. A generic basis element of \( C^\text{gm}_* (\alpha; R) \) is given by a “strong \( \delta \)-chain”, but it is helpful to imagine that a generator of \( C^\text{gm}_* (\alpha; R) \) is given by a smooth map \( \sigma : P \to \alpha \), where \( P \) is a compact smooth manifold with corners, and \( \sigma \) is considered up to diffeomorphism of the domain. (This is essentially a special case of the more general notion of \( \delta \)-chain.)

The boundary operator on \( \hat{\mathcal{C}}I(Y, E, \pi; R) \) is given as the sum of the geometric boundary operator (sending \( \sigma : P \to \alpha \) to \( \partial P \to \alpha \)) and a fiber product map with the moduli spaces whose properties are discussed in the first half of this paper, taking a map \( \sigma : P \to \alpha \) to a map

\[
P \times_\alpha \overrightarrow{\mathcal{M}}(\alpha, \beta) \to \beta.
\]

This chain complex carries the action of a \( C^\text{gm}_* (SO(3); R) \)-module.

After this, one applies the results of the appendix to construct chain complexes

\[
\hat{CI}^+(Y, E, \pi; R), \quad \hat{CI}^-(Y, E, \pi; R), \quad \hat{CI}^\mu(Y, E, \pi; R).
\]

These are constructed algebraically, and we will not give detailed descriptions here of a generic element of these chain complexes. Writing \( \bullet \) for one of \( \{+, -, \infty\} \), suffice it to say that if \( G \) is a connected Lie group and \( H \) a connected subgroup, then for \( \alpha = G/H \), there are chain complexes \( \hat{C}^\bullet_G(\alpha; R) \) (the completed group homology complexes) so that \( H^\mu_G(\alpha) \) computes group homology of \( H \), while \( H^\mu_G(\alpha) \) computes group cohomology of \( H \) in negative degrees and \( H^\mu_G(\alpha) \) computes Tate cohomology of \( H \). These all having some degree shift and an action of \( H^{-\bullet}(BG; R) \) induced by restriction to cohomology of \( BH \). Further,

\[
\hat{CI}^+(Y, E, \pi; R) = \bigoplus_{\alpha \in \mathcal{C}_\pi} \hat{C}^\bullet_{SO(3)}(\alpha; R).
\]

Next when \( \frac{1}{2} \in R \) we define a finite-dimensional chain complex \( DCI(Y, E, \pi; R) \), given as an \( R \)-module by

\[
DCI(Y, E, \pi; R) = \bigoplus_{\alpha \in \mathcal{C}_\pi} H_* (SO(3); R)[i(\alpha)],
\]

with an action of \( H_* (SO(3); R) \); this is now an exterior algebra on a single degree-3 generator, the fundamental class of \( SO(3) \). One may think of this as contributing, for each irreducible orbit \( \alpha \), a copy of \( R \) in degree \( i(\alpha) \) and \( i(\alpha) + 3 \) with the action of \( H_* (SO(3); R) \) taking the first to the second, for each \( SO(2) \)-reducible a copy of \( R \) in degrees \( i(\alpha) \) and \( i(\alpha) + 2 \), and for each full reducible a copy of \( R \) in degree \( i(\alpha) \). The differential counts 0-dimensional moduli spaces between irreducibles, as well as the degrees of maps of moduli spaces between different orbits. One of these maps
is often called the $U$-map, which we write $U_{F1}$ to distinguish from later algebraic terms, also written $U$.

There is a $C_\#(SO(3); R)$-equivariant quasi-isomorphism

$$DCI(Y, E, \pi; R) \simeq \widetilde{CI}(Y, E, \pi; R).$$

When $Y$ is an integer homology sphere and $E$ is the trivial bundle, $DCI(Y, \pi; R)$ is isomorphic to the complex Donaldson writes as $HF(Y, \pi; R)$ in [Don02, Section 7.3.3]; though Donaldson only writes his for rational coefficients, there is no difficulty extending the definition to the broader case $\frac{1}{2} \in R$.

One may immediately apply the constructions of the appendix to $DCI(Y, E, \pi; R)$. Applying the plus-homology construction we arrive at $DCI^+(Y, E, \pi; R)$, and applying the minus-homology construction we arrive at $DCI^-(Y, E, \pi; R)$. If $U$ is a degree $-4$ element and $U^*$ is a degree 4 element, the underlying $R$-modules are given by

$$DCI^+(Y, E, \pi; R) = DCI(Y, E, \pi; R)[U^*]$$

and

$$DCI^-(Y, E, \pi; R) = DCI(Y, E, \pi; R)[U].$$

One thinks of each irreducible here as contributing what looks like a copy of $\overset{\ast}{ESO}(3)$.

Applying a trick of Seidel and Smith from [SS10], we pass from these to complexes $\overline{DCI}^+$ and $\overline{DCI}^-$. The first has underlying $R$-module given by

$$\bigoplus_{\alpha \in \mathcal{E}_s} H^S_O(\alpha; R)[i(\alpha)]$$

and the second by

$$\bigoplus_{\alpha \in \mathcal{E}_s} H^\text{dim}^{-\ast}(\alpha; R)[i(\alpha)].$$

That is, each irreducible contributes a copy of $R$ (in degree $i(\alpha)$ or $i(\alpha) + 3$), while each $SO(2)$-reducible contributes a tower $R[|U^*|^2]$ or $R[U^{1/2}][2]$ respectively, where $|U^*|^2 = 2$ and $|U^{1/2}| = -2$. Lastly, each full reducible contributes a copy of $R[[U^*]]$ or $R[U]$, respectively. The differentials involve large powers of $U_{F1}$.

We have $\overset{-\ast}{C}(BSO(3); R)$-equivariant quasi-isomorphisms

$$CI^+(Y, E, \pi; R) \simeq DCI^+(Y, E, \pi; R) \simeq \overline{DCI}^+(Y, E, \pi; R)$$

and

$$CI^-(Y, E, \pi; R) \simeq DCI^-(Y, E, \pi; R) \simeq \overline{DCI}^-(Y, E, \pi; R).$$

For an integer homology sphere $Y$ equipped with the trivial bundle $E$, the complex $\overline{DCI}^-(Y, E, \pi; R)$ is isomorphic as a $U$-module to the complex Donaldson writes as $\overline{CF}(Y; R)$, and similarly $\overline{DCI}^+(Y, E, \pi; R) \simeq \overline{CF}(Y; R)$.

Donaldson also defines complexes $\overline{CF}(Y; R)$ and $\overline{CF}(Y; R)$. These are best understood as the quotient and fixed points of the chain complex $DCI(Y, \pi; R) \simeq \overline{CF}(Y, \pi; R)$ under the $H_\#(SO(3); R)$-action. We do not use these, but do notice them as appearing in certain spectral sequences for integer homology spheres. In that context, we write the resulting homology groups as $\overline{T}$ and $\overline{L}$.

There are also Froyshov’s reduced instanton Floer homology groups, which he writes $HF$ and Donaldson writes $\overline{HF}$. We identify these in Section 6.4 as being the image of $I^+(Y; R)$ inside $I^-(Y; R)$ (or more precisely, we see this at the level of an $E^\infty$ page of a spectral sequence calculating those), but otherwise do not use...
them. If we were to give these groups a notation, we could call them \( \hat{I} \). There is also Floer’s original version of instanton homology, \( I_k(Y) \), which is defined only for integer homology spheres and does not use the reducibles. This, too, may be seen in terms of the spectral sequence for \( I^+ \) (ignoring the reducible piece).

Finally, Kronheimer and Mrowka define instanton homology groups \( I^#(Y, E; R) \) for all pairs of 3-manifolds and \( SO(3) \)-bundles, by taking the connected sum with a pair \((T^3, E)\) where \( w_2(E) \) is Poincaré dual to \( T^2 \times \{*\} \), also called ‘framed instanton homology’. These are not the same as the groups \( \hat{I}(Y, E; R) \). A calculation of [Sca15] using Fukaya’s connected sum theorem shows that when \( Y \) is an integer homology sphere, the group \( I^#(Y; R) \) may be calculated using a chain complex very much like that defining \( DCI(Y, \pi; R) \), but they differ in one component of the matrix defining the differential: the term \( U_{FI} \) in \( \hat{c}_{DCI} \) is instead given by \( U_{FI} - \delta \) in the complex defining \( I^#(Y; R) \). So one finds instead that \( I^#(Y, E; R) \) is a sort of deformation of \( \hat{I}(Y, E; R) \). We do not discuss this relationship further here.

In summary, there are nine versions of instanton homology for integer homology spheres,

\[
\hat{I}, I^+, I^-, I^\infty, \hat{I}, \hat{I}, I^#, I;
\]

all but the last of these are also defined for pairs of a rational homology sphere and \( U(2) \)-bundle. In this paper we will work most extensively with the first four of these. We will briefly mention the next three ‘reduced’ homology theories \( \hat{I}, \hat{I} \), and \( I \) when discussing spectral sequence calculations for integer homology spheres. We will not mention \( I^# \) and \( I \) any further.

In addition to the complexes defining each of these homology theories, we will also use five important complexes for calculation, the Donaldson models

\[
DCI, \ DCI^+, \ \overline{DCI}^+, \ DCI^-, \ \overline{DCI}^-.
\]

The first of these has homology naturally isomorphic to \( \hat{I} \), the second two of these have homology naturally isomorphic to \( I^+ \), and the last two of these have homology naturally isomorphic to \( I^- \).

**Organization.** Section 1 introduces the framed configuration spaces on which we attempt to do Morse theory, and discusses the action of \( SO(3) \) on them. Section 2 explains how to complete these topological spaces using Sobolev spaces and obtain the structure of a Banach manifold.

The technical heart of the paper is in Sections 3 and 4, where we use the standard holonomy perturbations in instanton Floer theory to show that we can achieve equivariant transversality: a generic perturbation gives rise to a finite set of critical \( SO(3) \)-orbits, and the space of trajectories between them forms a smooth manifold. As long as the dimension is sufficiently small, this has a compactification to a topological manifold with corners, and we use these compactifications to define the complex \( \hat{CI} \). This includes a calculation of the reducible perturbed instantons and their indices in Section 4.4, which is then used to give a condition which guarantees we can achieve transversality normal to the reducible locus. This condition uses the notion of signature data introduced in Section 3.5.

Sections 1-4 provide us with the technical machine (the moduli spaces and their properties) we need. Sections 5 and 6 are devoted to defining and calculating equivariant instanton homology given this machine, and may be read independently of the first part of the paper.
In Section 5, we define the invariants $CI^\bullet$. First we review Lipyanskiy’s geometric homology, which is a crucial technical tool in our definition of $\tilde{CI}$: it allows us to define the chain complex without triangulating the moduli spaces appearing in (*), and only requires the use of moduli spaces of small dimension; in particular, small enough that the Uhlenbeck bubbling phenomenon does not arise. The chain complex $\tilde{CI}(Y, E, \pi; R)$, which depends on a choice of metric and perturbation, is defined in Section 5.2. This construction comes with cobordism maps, and these provide us with the usual invariance properties, giving Theorem 1.

In Section 5.3 we explain the notion of index filtration for $\tilde{CI}$, which gives rise to a $\mathbb{Z}/8 \times \mathbb{Z}$-graded spectral sequence. This is not a filtration on $\tilde{CI}$ in the standard sense, but rather what we call a periodic filtration. In Section 5.4 we use the idea of periodic filtrations to define the equivariant instanton homology complexes $CI^\bullet$, following the construction of Section A.8.

In Section 6, we use the index spectral sequences of sections 5.4 and 5.5 to carry out some calculations and comparisons.

In Section 6.1 we warm up with a calculation of $I^\bullet(Y, E)$ for admissible bundles, giving Theorem 4 above.

After this, in Section 6.2 we compare $\tilde{CI}$ to Donaldson’s complex $DCI$ (written in [Don02] as $\tilde{CF}$), and in particular show that their homologies are isomorphic, justifying the notation $H(\tilde{CI}) = \tilde{I}$. We are then able to show $I^+(Y; R) \cong \overline{HF}(Y; R)$ for rings containing $1/2$, as well as $I^-(Y; R) \cong \underline{HF}(Y; R)$. In the same section, we extend the definition of Donaldson’s chain complexes to rational homology spheres.

Section 6.3 uses the Donaldson model, and a localization/periodicity theorem for Tate homology, to give the explicit formula for $I^\kappa$ mentioned as Theorem 6 above. Section 6.4 contains a calculation of $I^\bullet$ for the 3-sphere, lens spaces, and the Poincaré homology sphere, as well as a description of the spectral sequences for integer homology spheres and a brief comment on the connection to Frøyshov’s reduced groups. We conclude in Section 6.5 with a discussion of orientation-reversal and duality in $CI^\bullet(Y)$.

The appendix describes the algebraic constructions $C^\bullet(A, M)$ and their invariance properties, as well as providing calculational tools. In particular, if $A$ is the group algebra $C_*(G; R)$, where $G$ is a compact Lie group, then $C^\bullet(G, G/H)$ is calculated for all $\bullet$ and any closed subgroup $H \subset G$. Section A.8 concludes with an extension of this machinery to the $\mathbb{Z}/8$-graded case, when our complexes come equipped with well-behaved periodic filtrations.

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1. Configuration spaces and their reducibles

Let $Y$ be an oriented closed 3-manifold equipped with a Riemannian metric, and $E 	o Y$ an $SO(3)$-bundle over $Y$; it is worth mentioning that $SO(3)$-bundles over a 3-complex are determined up to isomorphism by their second Stiefel-Whitney class $w_2(E) \in H^2(Y; \mathbb{Z}/2)$. Associated to $E$ by the adjoint representation $SO(3) \subset \mathfrak{so}(3)$ is the adjoint bundle $\mathfrak{g}_E \subset \text{End}(E)$, the subbundle given by skew-adjoint endomorphisms. The space $\mathcal{A}_E$ of orthogonal connections on $E$ is affine over $\Omega^1(\mathfrak{g}_E)$. There is a gauge group, $\mathcal{G}_E$, the set of smooth bundle automorphisms of $E$ that cover the identity; equivalently, this is the set of smooth sections of the non-principal $SO(3)$ bundle $\text{Aut}(E)$, the associated bundle to $E$ under the conjugation action $SO(3) \curvearrowright SO(3)$. Using the isomorphism $\text{Inn}(SU(2)) \cong SO(3)$, we may form the bundle

$$\widetilde{\text{Aut}}(E) = \text{Aut}(E) \times_{SO(3)} SU(2).$$

We say that $\sigma \in \mathcal{G}_E$ is an even gauge transformation if $\sigma$ lifts to a section of $\widetilde{\text{Aut}}(E)$, and denote the group of even gauge transformations by $\mathcal{G}_E^e$. Obstruction theory applied to sections of $\text{Aut}(E)$ provides a short exact sequence of groups

$$1 \to \mathcal{G}_E^e \to \mathcal{G}_E \to H^1(Y; \mathbb{Z}/2) \to 0.$$

There is a map, the Chern-Simons functional,

$$\text{cs} : \mathcal{A}_E \to \mathbb{R},$$

defined as follows. Pick a compact oriented 4-manifold $X$, equipped with an $SO(3)$ bundle $E_X$, such that $\partial(X, E_X) = (Y, E)$. Given a connection $A$ on $E$, extend it to a connection $A_X$ on $E_X$. Then define

$$\text{cs}(A) = \int_X \text{Tr}(F^2_{A_X}).$$

This gives a function that depends only on the choice $(X, E_X)$, not on the extension $A_X$. If $(X', E_{X'}, A_{X'})$ is another extension, we may define $M = X \cup_Y X'$ with the obvious choice of $SO(3)$-bundle and connection over it and invoke the Chern-Weil formula

$$-2\pi^2 p_1(E') = \int_M \text{Tr}(F^2_{A_X}).$$

Thus cs is defined in general up to an $8\pi^2 \mathbb{Z}$ ambiguity (using that $4 \mid p_1(E')$). If $(X', E_{X'}) = (X, E_X)$, then $p_1(E_X \cup E_{X'}) = 0$, giving us a well-defined functional cs on $\mathcal{A}_E$ conditional on that choice of bounding 4-manifold $(X, E_X)$.

The gauge group $\mathcal{G}_E$ acts on $\mathcal{A}_E$ by $\sigma(A) = A - (\nabla_A \sigma)\sigma^{-1}$, where we take the covariant derivative of $\sigma$ by considering it as a section of $\text{End}(E) = E \otimes E^*$, where $E$ is considered now as an oriented vector bundle equipped with metric. We denote the quotient by this group action

$$\mathcal{B}_E = \mathcal{A}_E / \mathcal{G}_E,$$

the configuration space of connections on $E$, and also write the even configuration space as the quotient

$$\mathcal{B}_E^e := \mathcal{A}_E / \mathcal{G}_E^e.$$

Immediately from the definition we see that the stabilizer at $A$ by the action of $\mathcal{G}_E$ or $\mathcal{G}_E^e$ is precisely the subset of (even) $A$-parallel gauge transformations; thus elements of the stabilizer are determined by their value at a single point, and evaluating at a point $b \in Y$ gives an isomorphism to a subgroup of $SO(3) = \text{Aut}(E_b)$. 

The Chern-Simons functional \( cs : A_E \to \mathbb{R} \) does not descend naively to \( B_E \) — its value may change after applying an element of gauge group \( G_E \). However, if \( u \) is a gauge transformation,
\[
    cs(u(A)) = cs(A) + 8\pi^2 k
\]
for some integer \( k \), so \( cs \) descends to a continuous map \( B_E \to \mathbb{R}/8\pi^2 \mathbb{Z} \). This statement is little more than saying that there is always a 4-manifold with \( SO(3) \)-bundle \((X, E)\) over which the gauge transformation \( u \) extends, for then one sees that
\[
    cs(u(A)) = \int_X \text{tr} \left( F_{u(A)}^2 \right) = \int_X \text{tr} \left( F_A^2 \right);
\]
that the integrals are equal is Stokes’ theorem. It is this circle-valued functional, on a slightly modified space, that we hope to do Morse theory with.

1.1. Framed connections and the framed configuration space. The configuration space \( B_E \) is in no sense a manifold, because \( G_E \) does not act freely on the space of connections. To free up the action of the gauge group on the space of connections so that the quotient by gauge is a manifold (at least heuristically, at this point), we pick a basepoint \( b \in Y \) and consider the space of framed connections
\[
    \hat{A}_E = A_E \times E_b.
\]
We call a point \( p \in E_b \) a framing because it determines an isomorphism \( SO(3) \cong E_b \) sending the identity to the point \( p \); here we are thinking of \( E \) as a principal bundle, not a vector bundle. A gauge transformation \( \sigma \) evaluates to \( \sigma(b) \in \text{Aut}(E_b) \), which acts on \( E_b \) on the left by applying the automorphism. (By definition, to say \( f \in \text{Aut}(E_b) \) means \( f(pg) = f(p)g \).) This gives us an action of \( G_E \) on the space of framed connections \( \hat{A}_E \). There is further a natural right \( SO(3) \)-action on \( \hat{A}_E \) by acting on \( E_b \) by translation. If \( \sigma \in G_E^c, \ (A,p) \in \hat{A}_E \), and \( g \in SO(3) \), we have that
\[
    \sigma \cdot ((A,p) \cdot g) = (\sigma \cdot (A,p)) \cdot g,
\]
because the gauge group acts by automorphisms of the right \( G \)-set \( E_b \).

Because any gauge transformation in the stabilizer of \( A \) is \( A \)-parallel, it is trivial if its value at any point is the identity. As a consequence, \( G_E^c \) acts freely on \( \hat{A}_E \). Its quotient under the gauge group action, denoted \( \hat{B}_E \), retains the right \( SO(3) \)-action. We call this the framed configuration space of connections on \( E \). The stabilizer of \([A,p] \in \hat{B}_E^c \) in \( SO(3) \) is the isomorphic image of the stabilizer of \( A \in A_E \) under the action of the even gauge group following evaluation \( G_E^c \to \text{Aut}(E_b) \) and the natural isomorphism \( \text{Aut}(E_b) \cong SO(3) \). There is a map
\[
    \hat{B}_E^c \to B_E^c,
\]
given by quotienting by the leftover \( SO(3) \) action or equivalently by forgetting the framing. Said another way, orbits of the \( SO(3) \)-action on \( \hat{B}_E^c \) or \( B_E^c \) correspond to gauge equivalence classes of connections, and a point on an orbit \( \alpha \subset \hat{B}_E^c \) corresponding to \([A]\) is an equivalence class of framings, with \( \alpha \cong E_b/\text{Stab}([A]) \).

We will soon define Hilbert manifold completions of these spaces, in the context of which we will find that \( \hat{B}_E^c \) is a smooth Hilbert manifold with smooth \( SO(3) \)-action. These remarks also apply to the full gauge group, with quotient \( \hat{B}_E = \hat{A}_E/G_E \).
Remark 1.1. One could also define $\widetilde{B}_E = \mathcal{A}_E / \mathcal{G}^{c,b}_E$, quotienting by the group of based gauge transformations (gauge transformations with $\sigma(b) = \text{Id}$). This works just as well here, and the right $SO(3)$ action corresponds to the inverse of the left $SO(3)$ action given by $\mathcal{G}^{e,b}_E$. However, we find later discussions of the 4-manifold configuration spaces and the restriction maps to their ends clearer in the language of framed connections, so this is our preferred model.

1.2. The equivalent $U(2)$ model. We would like to understand the reducible subspaces of $\widetilde{B}_E$ under the $SO(3)$ action. The $SO(2)$ fixed points are more easily understood if we introduce an auxiliary construction. Pick a principal $U(2)$-bundle $\tilde{E}$ and an isomorphism

$$\tilde{E} \times_{U(2)} SO(3) \simeq E.$$  

This is possible because $U(2)$-bundles on 3-manifolds are classified by their first Chern class (and $SO(3)$ bundles by their second Steifel-Whitney class), so we only need to know that we can pick a lift of $w_2$ to an integral cohomology class; that this is possible follows from the Bockstein long exact sequence and the fact that $H^3(Y;\mathbb{Z}) \simeq \mathbb{Z}$ has no 2-torsion. Thinking of $\tilde{E}$ instead as a complex vector bundle and explicitly identifying the quotient homomorphism $U(2) \to SO(3)$, the construction $\tilde{E} \times_{U(2)} SO(3) \simeq E$ produces the oriented 3-plane bundle $\text{su}(\tilde{E}) \subset \text{End}(\tilde{E})$ of skew-Hermitian endomorphisms of $\tilde{E}$ as the associated $SO(3)$-bundle.

Fix a connection $A_0$ on the determinant complex line bundle $\det(\tilde{E}) := \lambda$ (if $c_1(\tilde{E})$ is finite order, we can choose this to be the flat connection, unique up to gauge transformation). We consider the space of connections with fixed determinant connection

$$\mathcal{A}_E^{\text{det}} = \{ A \in \mathcal{A}_E \mid \text{tr}(A) = A_0 \in \Omega^1(Y; i\mathbb{R}) \};$$

this is also affine over $\Omega^1(\mathfrak{g}_E)$. To every connection on $\tilde{E}$ there is an associated connection on $E$, giving us a map $\mathcal{A}_{\tilde{E}} \to \mathcal{A}_E$ which is a bijection when restricted to $\mathcal{A}_E^{\text{det}}$. To enhance this to a bijection of spaces of framed connections, let

$$\tilde{\mathcal{A}}_E^{\text{det}} = \mathcal{A}_E^{\text{det}} \times (\tilde{E}b \times_{U(2)} SO(3));$$

this carries a natural right action by $SO(3)$. Furthermore, the natural gauge group acting on $\mathcal{A}_E^{\text{det}}$ is the set of gauge transformations whose (pointwise) determinant is 1, denoted

$$\mathcal{G}^{\text{det}}_E = \Gamma(\text{Aut}(\tilde{E}) \mid \det \gamma = 1).$$

We can further identify this latter bundle of groups as isomorphic to $\tilde{\text{Aut}}(E)$ and thus there is a surjective homomorphism

$$\mathcal{G}_{E}^{\text{det}} \twoheadrightarrow \mathcal{G}_E^c.$$  

It is not a bijection: recall that the latter group is defined as a subset of $\mathcal{G}_E^c = \Gamma(\text{Aut}(E))$. Its kernel is the 2-element set of gauge transformations whose pointwise values are $\pm 1 \in SU(2)$. This subgroup acts trivially on $\tilde{\mathcal{A}}_E^{\text{det}}$.

Now it is easy to verify that the bijection $\tilde{\mathcal{A}}_E^{\text{det}} \to \mathcal{A}_E$ is equivariant under the actions of the gauge groups, and thus after quotienting we have an equivariant diffeomorphism

$$\tilde{B}_E^{\text{det}} \cong \overline{B}_E^c.$$
The group acting on the former space is \( SU(2) \), but \( \pm 1 \) act trivially, and so passing to the quotient \( PSU(2) \cong SO(3) \) we identify these two configuration spaces as \( SO(3) \)-spaces. Furthermore, the natural definition of Chern-Simons functional on this first space is sent to our Chern-Simons functional on the second space, and there is no difference in the resulting gauge theory. We only consider this construction auxiliary because it depends on the unnecessary input data of \( \hat{E} \) and \( A_0 \). When working on the \( U(2) \)-bundle, we usually prefer to speak of the \( SU(2) \) action, as \( SU(2) \) naturally sits inside the \( U(2) \) gauge group, even though \( \pm 1 \) act trivially on the framed configuration space.

In the simple case that \( \hat{E} = \eta_1 \oplus \eta_2 \), the associated \( SO(3) \)-bundle is \( i\mathbb{R} \oplus (\eta_1 \oplus \eta_2^{-1}) \), and a connection on the former respecting the splitting is taken under the bijection \( \mathcal{A}^\text{det}_E \to \mathcal{A}_E \) to a connection which respects the latter splitting.

### 1.3. Reducibles on 3-manifolds.

We can explicitly describe the stabilizers that arise under the action of the even gauge group on \( \mathcal{A}_E \), and hence the orbit types in \( \mathcal{B}^e_E \). In understanding these, it’s also convenient to carry out the same analysis for the full gauge group \( \mathcal{G}_E \) and its quotient \( \hat{\mathcal{B}}_E \). Recall that

\[
\mathcal{G}_E/\mathcal{G}^e_E \cong H^1(Y; \mathbb{Z}/2),
\]

identifying the latter with the group of obstructions to lifting a section of \( \text{Aut}(E) \) to a section of \( \text{Aut}(E) \). This leaves an \( H^1(Y; \mathbb{Z}/2) \) action on \( \hat{\mathcal{B}}_E \), whose quotient is \( \mathcal{B}^e_E \), the quotient by all gauge transformations. This is a useful gadget to keep track of, especially in light of the following lemma (a version of [DK90, Lemma 4.28]), calculating the stabilizers of connections under both \( \hat{\mathcal{G}}^e_E \) and \( \mathcal{G}_E \).

**Lemma 1.1.** Let \( A \) be an \( SO(3) \)-connection on \( E \). Then the stabilizer \( \Gamma_A \) under the action of the full gauge group \( \mathcal{G}_E \) is

\[
\mathcal{C}(\text{holonomy of } A) \subset SO(3),
\]

the centralizer of the holonomy group of \( A \) at some choice of basepoint \( b \). Let \( \pi: SU(2) \to SO(3) \) be the projection. \( \Gamma_A \cap \mathcal{G}^e_E \) is

\[
\mathcal{C}_{SU(2)}(\pi^{-1}H_A) \subset SO(3),
\]

the set of elements of \( SO(3) \) that fix \( \pi^{-1}H_A \) under conjugation, considering \( SO(3) = \text{Inn}(SU(2)) \).

**Proof.** If \( SO(3) \) acts smoothly on some manifold \( M \), consider the \( M \)-bundle \( E \times_{SO(3)} M \). If \( A \) is an \( SO(3) \)-connection on \( E \), there is a natural connection induced on the associated \( M \)-bundle. Then \( m \in M \) extends to a parallel section of this \( M \)-bundle if and only if it is fixed under the action of \( H_A \) on \( M \). Taking \( M \) to be \( SO(3) \) and \( SU(2) \) equipped with the conjugation action gives the desired result. □

A group-theoretic calculation shows that the only subgroups of \( SU(2) \) that arise as centralizers are the center \( \mathbb{Z}/2 \), the circle subgroups \( U(1) \), and the full group \( SU(2) \). So the only possible stabilizers of the action of \( \mathcal{G}^e_E \) on \( \mathcal{A}_E \) are

\[
\{ I \}, SO(2), SO(3).
\]

A similar calculation shows that the subgroups of \( SO(3) \) that arise as centralizers are additionally \( O(2) \), the diagonal subgroup \( V_4 \), and \( \mathbb{Z}/2 \). (Calculate centralizers of elements first, then calculate the possible intersections of these.) Comparing stabilizers, we see that the action of \( H^1(Y; \mathbb{Z}/2) \) on \( \mathcal{B}^e_E \) giving rise to the quotient
by the full gauge group \( B_E \) is free except at reducibles with full stabilizer \( O(2), V_4 \), or \( \mathbb{Z}/2 \), where the action of \( H^1(Y;\mathbb{Z}/2) \) has stabilizer isomorphic to \( \mathbb{Z}/2, V_4 \), and \( \mathbb{Z}/2 \), respectively.

When the stabilizer of a connection \( A \) on \( E \) is \( SO(3) \subset \mathcal{G}_E \), the bundle must be trivial and the connection gauge equivalent to the trivial connection; however, it needn’t be equivalent by an even gauge transformation. In fact, because \( H^1(Y;\mathbb{Z}/2) \) acts freely on points in \( B_E \) with full stabilizer and there is only one such in the full quotient, there are \( H^1(Y;\mathbb{Z}/2) \) different elements of \( \mathcal{B}_E \) with stabilizer \( SO(3) \).

Inside the \( U(2) \)-model, [SS17, Section 3] identifies the action of \( H^1(Y;\mathbb{Z}/2) \) as sending a connection on \( \tilde{E} \) to the corresponding connection on \( \tilde{E} \otimes \xi_C \), where \( \xi_C \) is the complexification of a real line bundle (equivalently, a complex line bundle equipped with a unitary connection with holonomy in \( \pm 1 \)).

We will ultimately be interested in reducible orbits of the \( SO(3) \)-action. To describe the reducible orbits in a \( G \)-space \( X \) with orbit type isomorphic to \( G/H \), it suffices to describe the set of \( H \)-fixed points \( X^H \), and the action of the Weyl group

\[
W(H) = N_G(H)/H
\]
on \( X^H \); every orbit \( Gx \) of a point \( x \) with stabilizer conjugate to \( H \) intersects \( X^H \) nontrivially; and in fact, if \( x \in X^H \), then

\[
Gx \cap X^H = W(H)x.
\]

Thus, for instance, a \( G \)-invariant function on the subspace of points whose stabilizer contains a conjugate of \( H \) is determined uniquely by a \( W(H) \)-invariant function on \( X^H \). The description in terms of fixed subspaces and Weyl groups tends to be easier to state and prove, so we largely prefer that language as long as possible. The only case we actually use is that of \( SO(2) \subset SO(3) \), which has Weyl group isomorphic to \( \mathbb{Z}/2 \).

**Proposition 1.2.** Suppose \( E \) is an \( SO(3) \)-bundle over a closed 3-manifold \( Y \). Denote \( \tilde{E} \) a \( U(2) \) bundle with \( c_1(\tilde{E}) \mod 2 = \omega_2(\tilde{E}) \) and write \( \lambda = \det \tilde{E} \), as in Section 1.2.

1. The set of points of \( \mathcal{B}_E \) fixed by \( SO(2) \subset SO(3) \) is identified with

\[
\bigsqcup_{\eta} \mathcal{B}_\eta := \bigsqcup_{\eta} \mathcal{A}_\eta / \mathcal{G}_\eta
\]

where \( \eta \) varies over isomorphism classes of complex line bundles, \( \mathcal{A}_\eta \) is the configuration space of unitary connections on \( \eta \), affine over \( \Omega^1(\tilde{Y};i\mathbb{R}) \), and the gauge group \( \mathcal{G}_\eta \) is the space of sections of \( \text{Aut}(\eta) \), which is the same as the space of maps \( \text{Map}(Y,S^1) \). The \( SO(3) \) orbit of the \( SO(2) \)-fixed point space is the set of all reducible connections. The action of the Weyl group sends a class of connection \( [A] \) on \( \eta \) to the class of \( [A_0 - A] \) on \( \lambda \otimes \eta^{-1} \).

2. If \( E \) is trivial, \( \mathcal{B}_E \) has \( SO(3) \) fixed point set in bijection with \( H^1(Y;\mathbb{Z}/2) \); if \( E \) is nontrivial, the even configuration space has no \( SO(3) \) fixed points.

**Proof.** Following equation (1), it suffices to find the fixed subspaces in \( \mathcal{B}_E^{\text{det}} \). Consider the subset of framed connections \( (\tilde{A}, p) \) in \( \mathcal{A}_E^{\text{det}} \) whose connection term \( \tilde{A} \) has stabilizer consisting of gauge transformations with \( \sigma(b) \) in the diagonal subgroup

\[
SU(1) \times U(1) \subset SU(2),
\]
using the framing $p$ to specify the isomorphism
\[ \text{Aut}(\tilde{E}_b) \cong U(2). \]
All other circle stabilizers in $G^{\text{det}}_E$ are conjugate to this subgroup, and this fixed subspace of the space of framed connections projects to the subset of $SO(2)$-fixed connections in $\tilde{B}^{\text{det}}_E$. Our assumption implies that $\tilde{A}$ has holonomy contained in the diagonal subgroup
\[ U(1) \times U(1) \subset U(2). \]
This leaves invariant a splitting
\[ \tilde{E}_b \cong \mathbb{C}^2 \cong \mathbb{C} \oplus \mathbb{C}, \]
the first isomorphism given by the framing $p$, and hence gives a parallel splitting $\tilde{E} \cong \eta \oplus \eta'$. Sending $(\tilde{A}, p)$ to the corresponding connection $A'$ on $\eta$ gives us the map from this subset to $A_{\eta'}$, and because any gauge transformation of $\eta$ extends to a determinant-1 gauge transformation of $\tilde{E}$ (act by the inverse in the $\eta'$ coordinate), this descends to a well-defined map
\[ \left( \tilde{B}^{\text{det}}_E \right)^{U(1)} \rightarrow \bigcup_{\eta} B_{\eta} \]
modulo gauge. Conversely, any connection on $\eta$ induces a connection on $\tilde{E}$ of the specified form, and any gauge transformation of $\eta$ is induced by a unique determinant-1 gauge transformation of $\tilde{E}$. Any framing on $\eta$ then induces a framing on $\tilde{E}$, but there is a determinant-1 gauge transformation changing any framing on $\eta$ to another, so after modding out by gauge the choice of framing on $\eta$ didn’t matter. This gives the stated bijection for those connections whose induced splitting gives a line bundle of topological type $\eta$ in the first coordinate; but note that on a 3-manifold, a $U(2)$-bundle is determined by its first Chern class, and so we have an isomorphism
\[ \tilde{E} \cong \eta \oplus (\lambda \otimes \eta^{-1}). \]
Thus every complex line bundle $\eta$ arises in such a splitting.

The action of the Weyl group is to swap the coordinates of the framing; this then swaps the components $[A' \oplus (A_0 - A')]$ of the connection, as stated in the lemma. In particular, it only fixes $A'$ if there is an isomorphism
\[ \eta \cong \lambda \otimes \eta^{-1} \]
and this isomorphism sends $A'$ to $A_0 - A'$. This is only possible if $\lambda$ is twice an integral class, and so $w_2(E) = 0$; but if one such choice is made, all others are affine over $H^1(Y; \mathbb{Z}/2)$, tensoring the whole bundle (and hence each component) with $\xi_E$, the complexification of a real line bundle, or equivalently a complex line bundle with holonomy in $\pm 1$.

**Corollary 1.3.** Let $E$ and $\tilde{E}$ be as in the previous proposition; write $c = c_1(\lambda) = c_1(\tilde{E})$. Then the reducible subspace of $\tilde{B}^{\text{det}}_E$, consisting of framed connections with nontrivial stabilizer, is a disjoint union over connected components labeled by pairs $\{z_1, z_2\} \subset H^2(Y; \mathbb{Z})$ with $z_1 + z_2 = c$, where the $z_i$ are cohomology classes corresponding to complex line bundles $\eta_i$.

If $z_1 \neq z_2$, this connected component is a fiber bundle over $B_{\eta_1} \cong E_{\eta_2}$ with fiber $S^2$, where $SO(3)$ acts trivially on the base and via the standard action on the fiber. If $z_1 = z_2$, this connected component contains a unique fully reducible connection.
Proof. Consider the subspace of framed reducibles corresponding to the splitting \( \tilde{E} \cong \eta_1 \oplus \eta_2 \); we write this space as \( \tilde{B}^\text{red}_{\eta_1, \eta_2} \). The quotient of this space by the \( SO(3) \)-action is the same as the quotient of its \( SO(2) \)-fixed subspace by the action of \( \mathbb{Z}/2 \).

In the case that \( z_1 \neq z_2 \), the \( SO(2) \)-fixed subspace is \( \mathcal{B}_{\eta_1} \cup \mathcal{B}_{\eta_2} \), and the action of the Weyl group identifies these. Thus the desired fiber bundle is the quotient map

\[
\tilde{B}^\text{red}_{\eta_1, \eta_2} \to \mathcal{B}_{\eta_1} \cong \mathcal{B}_{\eta_2}.
\]

When \( z_1 = z_2 = z \), corresponding to a complex line bundle \( \eta \), the \( SO(2) \)-fixed subspace is \( \mathcal{B}_\eta \). Its quotient by the action of the Weyl group is connected, so \( \tilde{B}^\text{red}_{\eta_1, \eta_2} \) is connected. In this case, \( \eta \) is trivial, so we may choose \( \tilde{E} \) trivial for convenience of discussion; then the action of the Weyl group may be described as complex conjugation, using the isomorphism \( \eta \cong \xi \mathbb{C} \) for a unique real line bundle \( \xi \); the induced connection is the fully reducible connection.

\[\blacksquare\]

Remark 1.2. The full group of gauge transformations preserves the reducible set, so \( \mathcal{G}_E/\mathcal{G}_E^\text{red} \cong H^1(Y; \mathbb{Z}/2) \) acts on \( \text{Red}(Y, E) \). If \( \beta : H^1(Y; \mathbb{Z}/2) \to H^2(Y; \mathbb{Z}) \) is the Bockstein homomorphism, then in the notation of the above corollary, the action on \( \text{Red}(Y, E) \) is given by sending

\[x \cdot \{z_1, z_2\} \mapsto \{z_1 + \beta x, z_2 + \beta x\}.
\]

In particular, this action is free and transitive on \( \text{Red}_\alpha(Y, E) \).

If \( x \) stabilizes \( \{z_1, z_2\} \), then \( z_2 = z_1 + \beta x \), and we see that \( c = 2z_1 + \beta x \), and we see that we may rechoose \( c \) to be \( \beta x \); then the above corresponds to the reduction

\[E \cong \xi \mathbb{C} \oplus \mathbb{R} \cong \xi \oplus \xi \oplus \mathbb{R},
\]

where \( \xi \) is the real line bundle corresponding to \( x \).

The discussion after Lemma 1.1 shows that the \( H^1(Y; \mathbb{Z}/2) \) action has stabilizer equal to \( \mathbb{Z}/2 \) at an \( O(2) \) connection, and otherwise acts freely on the reducible set. So in fact we see that \( H^1(Y; \mathbb{Z}/2) \) acts freely on the set of reducible components that do not contain an \( O(2) \)-connection, and has stabilizer \( \mathbb{Z}/2 \) on those that do.

We summarize the content of this section as notation:

**Definition 1.1.** Let \( Y \) be a closed oriented 3-manifold and \( E \) an \( SO(3) \)-bundle over \( Y \). We write \( \text{Red}(Y, E) \) for the set of connected components of the reducible subspace of \( \tilde{B}^\text{red}_E \). This may be written as

\[\text{Red}(Y, E) = \text{Red}_\alpha(Y, E) \cup \text{Red}_{SO(2)}(Y, E),\]

where the first term refers to those components containing a fully reducible orbit, and the second refers to those components entirely consisting of framed connections whose stabilizers are conjugate to \( SO(2) \).

If we fix a choice of \( c \in H^2(Y; \mathbb{Z}) \) that reduces mod 2 to \( w_2(E) \), we are furnished with a bijection between \( \text{Red}(Y, E) \) and the set of unordered pairs \( \{z_1, z_2\} \subset H^2(Y; \mathbb{Z}) \) with \( z_1 + z_2 = c \). The set \( \text{Red}_\alpha(Y, E) \) is sent to the 1-element sets\(^2\), and the set \( \text{Red}_{SO(2)}(Y, E) \) is sent to the 2-element sets \( \{z_1, z_2\} \) with \( z_1 \neq z_2 \).

We write elements of \( \text{Red}(Y, E) \) as \( \{z_1, z_2\} \), making the choice of \( c \) implicit.

In the case that \( b_1(Y) = 0 \), we determine the cardinality of these sets as a simple exercise in algebraic topology.

\[\text{that is, those pairs with } z_1 = z_2; \text{ equivalently the set of } z \text{ with } 2z = c\]
Proposition 1.4. Let $Y$ be a closed oriented 3-manifold with $b_1(Y) = 0$, and $E$ an $SO(3)$-bundle over it. We may determine the number of reducible components as follows.

- If $E$ is trivial, then
  \[ |\text{Red}_*(Y, E)| = |H^1(Y; \mathbb{Z}/2)| \]
  \[ |\text{Red}_{SO(2)}(Y, E)| = (|H^2(Y; \mathbb{Z})| - |H^1(Y; \mathbb{Z}/2)|)/2. \]

- If $E$ is nontrivial, $\text{Red}_*(Y, E)$ is empty, and
  \[ |\text{Red}_{SO(2)}(Y, E)| = |H^2(Y; \mathbb{Z})|/2. \]

Proof. If $E$ is trivial, we choose the integral lift $\lambda$ of $w_2(E)$ to be $0$. Then $\text{Red}_*(Y, E)$ is in bijection with 2-torsion elements of $H^2(Y; \mathbb{Z})$. The Bockstein exact sequence

\[ 0 = H^1(Y; \mathbb{Z}) \xrightarrow{\text{mod } 2} H^1(Y; \mathbb{Z}/2) \xrightarrow{\beta} H^2(Y; \mathbb{Z}) \xrightarrow{\times 2} H^2(Y; \mathbb{Z}) \]

shows that the set of 2-torsion elements is in bijection with $H^1(Y; \mathbb{Z}/2)$, the bijection sending $x \in H^1(Y; \mathbb{Z}/2)$ to $\beta x \in H^2(Y; \mathbb{Z})$. The set $\text{Red}_{SO(2)}(Y, E)$ is in bijection with the set of pairs

\[ \{z, -z\} \in H^2(Y; \mathbb{Z}) \]

for $z$ not 2-torsion. This proves both equalities.

If $E$ is nontrivial, and $c$ is an integral lift of $w_2 E$, then a set $\{z_1, z_2\}$ with $z_1 + z_2 = c$ cannot be a singleton: if $c = 2z_1$, then $c$ reduces to $0 \mod 2$, and thus $w_2 E = 0$, so $\text{Red}_*(Y, E)$ is empty. Thus every pair $(z_1, z_2)$ with $z_1 + z_2 = c$ consists of distinct elements, and the $\mathbb{Z}/2$ action swapping $z_1$ and $z_2$ has no fixed points. Because a pair $(z_1, z_2)$ is determined completely by $z_1 \in H^2(Y; \mathbb{Z})$, the desired equality is proven.

As a sanity check, observe that $|H^2(Y; \mathbb{Z})|$ is divisible by $|H^1(Y; \mathbb{Z}/2)|$, which is a power of 2, and we assumed $H^1(Y; \mathbb{Z}/2)$ to be nontrivial in assuming that $w_2(E)$ is nontrivial, so $|H^2(Y; \mathbb{Z})|/2$ is indeed an integer. ■

1.4. Configurations on the cylinder. There are two more kinds of configuration spaces we should consider. First, we should have a configuration space of connections on the bundle $\pi^*E$ over $\mathbb{R} \times Y$. We write $A^{(4)}_{\pi^*E}$ as the space of connections which, when restricted sufficiently far on each end, are pullbacks of connections on $Y$. (We will denote any gauge group or space of connections on a 4-manifold with a (4) unless there is no risk of confusion.) When we later give a Hilbert manifold modification of this construction, it will instead be replaced by an appropriate space of connections which have exponential decay to certain constant connections on the end. The space of framed connections on $\pi^*E$ is

\[ \tilde{A}^{(4)}_{\pi^*E} := A^{(4)}_{\pi^*E} \times (\pi^*E)(0, b), \]

with the framing chosen at $(0, b) \in \mathbb{R} \times Y$. The gauge group acting on this is $G^{(4), c}_{\pi^*E}$ — even gauge transformations which are constant on the ends. As usual, the framed configuration space is the space

\[ \tilde{B}^{(4), c}_{\pi^*E} := \tilde{A}^{(4)}_{\pi^*E}/G^{(4), c}_{\pi^*E}. \]
with the right \( SO(3) \) action on \( \pi^*E_{(0,b)} \). The ‘restriction to the ends’ map is slightly more complicated than what one might do naively: there is a map

\[
\text{ev}_- : \tilde{A}_{\pi^*E}^{(4)} \to \tilde{A}_E
\]
given by sending \((A, p)\) to \((A_{-\infty}, \gamma_{-\infty}(A, p))\), where in the latter term we send \( p \) to its parallel transport under \( A \) along \( \gamma \) traversed backwards from \((0, b)\). When the constant framing of the path (assigning the same \( p \) to \( \pi^*E_{(t,b)} \) for all \( t \)) is \( A \)-parallel — if \( A \) is in temporal gauge, for instance — this map is particularly simple: \( \gamma_{-\infty}(A, p) = p \). This map is equivariant on the left under the restriction-to-\( \gamma_{-\infty} \) map

\[
\mathcal{G}_{\pi^*E}^{(4), e} \to \mathcal{G}_E^c,
\]
and on the right under the \( SO(3) \)-action, because parallel transport is an isomorphism

\[
\pi^*E_{(0,b)} \to \pi^*E_{(-\infty,b)}
\]
of right \( SO(3) \)-sets. There is also a corresponding map \( \text{ev}_+ \).

Thus the evaluation maps descend to a map of right \( SO(3) \)-spaces

\[
[\text{ev}_-] \times [\text{ev}_+] : \tilde{B}_{\pi^*E}^{(4), e} \to \tilde{B}_E^c \times \tilde{B}_E^c.
\]

We are most frequently interested in the spaces of connections on \( \mathbb{R} \times Y \) with specified limits \([ev_+] \in \alpha_+ \) inside previously specified \( SO(3) \)-orbits. Furthermore, we should pick a relative homotopy class \( z \in \pi_0(B_{\pi^*E}^c, [\alpha_-], [\alpha_+]) \). Because \( A_E \) is contractible and \( \mathcal{G}_E^c \) acts with connected stabilizers, these are in bijection with

\[
\pi_0 \mathcal{G}_E^c \cong H^3(Y; \pi_3 \text{Aut}(E_b)) \cong \mathbb{Z}
\]
by an obstruction theory argument. Given limiting framed connections \([A_\pm, p_\pm] \in \alpha_\pm \), we choose lifts (denoted \( A_- \) and \( A_+ \)) so that a path from \( A_- \) to \( A_+ \) in \( A_E \) projects to the relative homotopy class \( z \) in \( \tilde{B}_E^c \). We then define

\[
\tilde{A}_{\pi^*E, z}(A_, A_+) = \{(A, p) \in \tilde{A}_{\pi^*E}^{(4)} \mid A_{\pm \infty} = A_{\pm \infty} \}.
\]

(Note that the endpoint conditions only include the connection and nothing about the framing.) The natural gauge group here is the group of even gauge transformations that are harmonic on the ends:

\[
\mathcal{G}_{\pi^*E}^{c, h}(A_, A_+) = \{ \sigma \in \mathcal{G}_{\pi^*E}^{(4), e} \mid d\pi_* A_{\pm} \sigma = 0 \text{ on the corresponding end} \}.
\]

The following is straightforward, and justifies this simplification.

\textbf{Lemma 1.5.} The natural map

\[
\tilde{A}_{\pi^*E, z}(A_, A_+)/\mathcal{G}_{\pi^*E}^{c, h}(A_, A_+) \to [\text{ev}]^{-1}(\alpha_- \times \alpha_+)
\]
is an isomorphism of \( SO(3) \)-spaces, where the final space denotes the corresponding subset of \( \tilde{B}_{\pi^*E}^{(4), e} \).

While the latter space is clearly the object of interest, the former is easier to describe, so we prefer to use it as our working definition of the configuration space of framed connections between two orbits. We record this as a definition:
Definition 1.2. If \( \alpha_{\pm} \) are \( SO(3) \)-orbits in \( \tilde{\mathcal{B}}_E \), the naive configuration space\(^3\) of smooth framed connections from \( \alpha_- \) to \( \alpha_+ \) is
\[
\tilde{\mathcal{B}}_{\pi*E,z}(\alpha_-, \alpha_+) := \tilde{A}_{\pi*E,z}(A_-, A_+)/G^E_{\pi*E}(A_-, A_+)
\]
where \( A_- \) and \( A_+ \) are representatives for the connections in the \( \alpha_{\pm} \), and a path from \( A_- \) to \( A_+ \) projects to the homotopy class \( z \). This carries a continuous \( SO(3) \) action, acting by translation on \( \pi^*E(0,b) \) on the right. Identifying
\[
\alpha_{\pm} = [A_{\pm}, p] \cong E_b/G_{\pm}
\]
with \( p \) varying over all possible framings at the basepoint, the endpoint maps
\[
[ev_{\pm}] : \tilde{\mathcal{B}}_{\pi*E,z}(\alpha_-, \alpha_+) \to \alpha_{\pm}
\]
are induced by \( \tilde{A}_{\pi*E}(A_-, A_+) \to E_b \), sending \((A, p)\) to \( \gamma_{\pm}(A, p) \), respectively.
There is a left translation action of \( \mathbb{R} \), induced by the family of diffeomorphisms \( \tau_t : \mathbb{R} \to \mathbb{R} \) with \( \tau_t(s) = s - t \), given by sending
\[
\tau_t^*(A, p) = (\tau_t^*A, \gamma_t(\tau_t^*A, \tau_t^*p)).
\]
In the framing coordinate, we are parallel transporting \( \tau_t^*p \) forward by \( t \) along \( \gamma \), so that it is sent back to the basepoint.

We have a corresponding version of Proposition 1.2 for the configuration spaces on cylinders. We do not repeat the setup of the determinant-1 \( U(2) \)-model, but refer to it freely in the proof below. Lemma 1.1 still gives that the only possible stabilizers of elements in the 4-dimensional configuration space are the trivial group, the full group \( SO(3) \), and circle subgroups conjugate to \( SO(2) \).

Proposition 1.6. Let \( E \) be an \( SO(3) \)-bundle over a 3-manifold \( Y \), and \( \alpha_{\pm} \) are \( SO(3) \)-orbits in \( \tilde{\mathcal{B}}_E \). We have the following descriptions of the reducible subspaces in the configuration space \( \tilde{\mathcal{B}}_{\pi*E,z}(\alpha_-, \alpha_+) \) under the \( SO(3) \) action:

1. If the \( \alpha_{\pm} \) are not both reducible orbits belonging to the same component \( \text{Red}(Y, E) \), or after concatenating a path from \( \alpha_+ \) to \( \alpha_- \) in the (simply connected) reducible component they lie in, the homotopy class \( z \) is non-trivial, then \( SO(3) \) acts freely on the configuration space.
2. If \( \alpha_{\pm} \in \text{Red}_{SO(2)}(Y, E) \) are \( SO(2) \)-reducible orbits lying in the same connected component of reducibles which contains no fully reducible point, labelled by \( \{z_1, z_2\} \subset H^2(Y) \), and the homotopy class \( z \) is trivial, then the reducible subspace is a fiber bundle with base \( \mathcal{B}_{\eta}(\alpha_-, \alpha_+) \) and fiber \( S^2 \), and in particular consists of one connected component. There are no fully reducible points.
3. If \( \alpha_- = \alpha_+ \) are the same unique fully reducible orbit in a fixed component \( \{z_1, z_2\} \in \text{Red}_{\eta}(Y, E) \), and \( z \) is the trivial homotopy class, there is a unique fully reducible point in the configuration space on the cylinder, lying in the unique connected component of reducible orbits.

Proof. It’s easier to find the reducibles inside the larger space \( \tilde{\mathcal{B}}_{\pi*E}^{(4)} \) and afterwards take the intersection with \([ev]^{-1}(\alpha_- \times \alpha_+)\). As in Proposition 1.2, working with connections on \( \pi^*\tilde{E} \) with fixed trace, a connection representing an \( SO(2) \)-fixed point has holonomy inside \( U(1) \times U(1) \) and hence induces a global splitting \( \pi^*\tilde{E} \cong \eta_1 \oplus \eta_2 \).

\(^3\)We use the term naive to contrast this space of smooth connections which are constant near \( \infty \) to the later Hilbert spaces of connections which decay exponentially at \( \infty \).
As before, this gives a natural correspondence between the $SO(2)$-fixed points and \( \bigcup \mathcal{B}_\eta \) when \( \eta_1 \neq \eta_2 \) and with the single space \( \mathcal{B}_\eta \) when \( \eta_1 = \eta_2 \). In the first case the involution given by the Weyl group action swaps the two spaces and in the second case it acts with a single fixed point.

Write \( \mathcal{B}_\eta^{(4),\text{red}} \) for the subspace of \( \mathcal{B}_\pi \), consisting of reducible connections for which the corresponding line bundle is of topological type \( \eta \); in particular, we do not specify the stabilizer. If \( \eta \) is an $SO(2)$-reducible component, the above shows that the map

\[
\mathcal{B}_\eta^{(4),\text{red}} \to \mathcal{B}_\eta^{(4),\text{red}} / SO(3)
\]

is a constant rank submersion, with fiber \( S^2 \). The base may be identified with \( \mathcal{B}_\eta^{(4)} \) : it is identified as the $SO(2)$-fixed point space modulo the Weyl group action.

To calculate the intersection with \( [\text{ev}]^{-1}(\alpha_- \times \alpha_+) \), observe that the line bundles \( \eta \) restrict to isomorphic line bundles on both ends (because the inclusion \( Y \to \mathbb{R} \times Y \) is a homotopy equivalence), and so the pair of line bundles associated to \( \alpha_- \) and \( \alpha_+ \) must be the same. Furthermore, after being put in temporal gauge, the relative homotopy class \( z \) traced out by one of these induced connections lies inside the simply connected space \( \mathcal{B}_\eta^{(4)}(\alpha_-, \alpha_+) \). This gives us the restrictions in (2)-(3).

Remark 1.3. If the $SO(3)$-bundle \( E \) is nontrivial, there are no full reducibles, and so only cases (1) and (2) arise.

When the limiting connections are flat, we can distinguish the relative homotopy classes \( z \) by the values of the integral calculating the “relative Pontryagin number” of the connection \( A \) by

\[
p_1(A) := \frac{1}{8\pi^2} \int_{\mathbb{R} \times Y} F_A^2;
\]

this takes on a well-defined value modulo \( \mathbb{Z} \), and the different discrete values it can take in \( \mathbb{R} \) parameterize the set of connected components \( \pi_0 \mathcal{B}_\pi(\alpha_- \times \alpha_+) \). When the limits are not flat, one could pick a base connection \( A_0 \) on the cylinder interpolating between them and compute \( \int (F^2_\eta - F^2_{A_0}) \); alternatively, later the limits will be critical points of a perturbed Chern-Simons functional and we will define the action of a connection in terms of this perturbed functional.

1.5. Configurations on a cobordism. Finally, suppose we are given a complete Riemannian manifold \( W \) with two cylindrical ends (i.e., specified isometries to \( (-\infty, 0] \times Y_1 \) and \( [0, \infty) \times Y_2 \)) and an $SO(3)$ bundle \( E \) with specified isomorphisms over the ends to bundles \( E_i \to Y_i \). We think of this as a cobordism \( Y_1 \to Y_2 \). Here \( E_i \to Y_i \) is an $SO(3)$ bundle, framed over a basepoint \( b_i \in Y_i \). Furthermore, we pick a smooth embedded path \( \gamma : \mathbb{R} \to W \) with \( \gamma(t) = (b_1, t) \) for \( t < 0 \) sufficiently small and \( \gamma(t) = (b_2, t) \) for \( t \) sufficiently large, and a trivialization of \( \gamma^* E \) restricting to the given trivializations on the ends. This path \( \gamma \) will serve the role \( \mathbb{R} \times \{ b \} \) did for the cylinder. Given orbits \( \alpha_i \subset \mathcal{B}_{E_i} \), we construct the configuration space \( \mathcal{B}_{E, \gamma}(\alpha_1, \alpha_2) \) much as we did before. The basepoint is instead \( \gamma(0) \), and the framing portion of the endpoint maps given by parallel transport to \( \pm x \) along \( \gamma \). There is still a decomposition of the configuration space into disconnected components, which we still label by \( z \), but this no longer has a description in terms of relative homotopy classes. Our above discussion on relative Pontryagin numbers, however, still does, as will the definition of an action using the perturbed Chern-Simons functional.
Proposition 1.7. Suppose \( B \) with an \( SO(\alpha) \) where the former denotes components that include some fully reducible orbit. If \( \text{Red} \), connected components they lie in.

Proof. If \( \beta w_2 E = 0 \), then \( E \) cannot be written as the direct sum of a trivial line bundle and an oriented 2-plane bundle: \( \beta w_2(E) = e(E) \), the Euler class, which is the obstruction to finding a nonvanishing section; but an \( SO(2) \)-reducible point induces such a splitting (there is a parallel section). Note that this is also the obstruction to finding an integral lift of \( w_2(E) \). So suppose now that \( c \in H^2(W; \mathbb{Z}) \) is an integral lift of \( w_2(E) \).

Almost identically to Proposition 1.6, the \( SO(2) \)-fixed point space of \( B^\text{ev}_E \) is identified with the disjoint union over configuration spaces of connections on line
bundles over $W$: $\sqcup_{\eta} B^W_\eta$. If $\eta_1 \oplus \eta_2$ has first Chern class $c$, then the Weyl group acts on $B_\eta$, by sending a connection $[A]$ on one to the connection $[A - A_0]$ on the other, where $A_0$ is a fixed connection on the complex line bundle $\lambda$. Quotienting by the Weyl group we are only left with components labelled by pairs $\{z_1, z_2\}$ with $z_1 + z_2 = c$, and if $z_1 = z_2$, the Weyl group fixes a unique point in $B^W_\eta$, the corresponding fully reducible point. ■
2. ANALYSIS OF CONFIGURATION SPACES

2.1. Tangent spaces. Before we introduce the Hilbert manifold versions of these constructions, we investigate the tangent spaces and differentials involved in the naive constructions (where all connections and gauge transformations are smooth).

The space $\mathcal{A}_E$ of all smooth connections on $E$ is an affine space over $\Omega^1(\mathfrak{g}_E)$; the natural Riemannian metric on $T\mathcal{A}_E = \Omega^1(\mathfrak{g}_E)$ is the (incomplete!) $L^2$ inner product on forms, using a bi-invariant inner product on the Lie algebra bundle $\mathfrak{g}_E$ to define the Hodge star. With this, we can define $\nabla cs$ as a vector field on $\mathcal{A}_E$; we may identify $(\nabla cs)(A) = *F_A$. The action of the gauge group $\mathcal{G}_E^b$ at $A \in \mathcal{A}_E$ has differential

$$d_A : \Omega^0(\mathfrak{g}_E) \cong T_{e}\mathcal{G}_E^b(E) \to T_A\mathcal{A}_E \cong \Omega^1(\mathfrak{g}_E).$$

The kernel of this map is the space of $A$-harmonic sections of $\mathfrak{g}_E$. The differential of the action of $\mathcal{G}_E^b$ on the fiber $E_b$ (whose tangent spaces are all canonically identified with $\mathfrak{g}_b$, the fiber of $\mathfrak{g}_E$ at $b$) is given by evaluation at the basepoint

$$\text{ev}_b : \Omega^0(\mathfrak{g}_E) \to \mathfrak{g}_b. \quad \text{We write}$$

$$\tilde{d}_A : \Omega^0(\mathfrak{g}_E) \to \Omega^1(\mathfrak{g}_E) \oplus \mathfrak{g}_b$$

for the differential of the action of $\mathcal{G}_E^b$ on $\tilde{\mathcal{A}}_E$ at $(A, p)$. Unfortunately, there is no natural identification of the cokernel of $\tilde{d}_A$ — which we hope to eventually identify with $T_{(A, p)}\tilde{\mathcal{B}}_E^c$ — with the $L^2$ orthogonal complement of its image, since the image is not $L^2$-closed. Its closure is $\text{Im}(d_A) \oplus \mathfrak{g}_b$, the essential difficulty here being that point-evaluation is not $L^2$-continuous. Rather, this orthogonal complement is precisely the same as $\ker(d_A^*)$. However, we can identify

$$(\text{Im}(d_A) \oplus \mathfrak{g}_b)/\text{Im}(d_A) \cong \mathfrak{g}_A^\perp,$$

the orthogonal complement inside $\mathfrak{g}_b$ of the subspace $\mathfrak{g}_A$ of elements that extend to $A$-parallel sections. Equivalently, $\mathfrak{g}_A$ is the tangent space to the image of $\Gamma_A$ in $\text{Aut}(E_b)$.

Picking a choice of (necessarily non-orthogonal) complement of $\text{Im}(\tilde{d}_A)$ inside $\text{Im}(d_A) \oplus \mathfrak{g}_b$ to identify with $\mathfrak{g}_A^\perp$, we may still decompose

$$T_{(A, p)}\tilde{\mathcal{A}}_E = \ker(d_A^*) \oplus \text{Im}(\tilde{d}_A) \oplus \mathfrak{g}_A^\perp.$$

We will henceforth write this as

$$T_{A,p} = \mathcal{K}_A \oplus \mathcal{G}_A \oplus \mathfrak{g}_A^\perp,$$

giving us the isomorphism

$$T_{[A,p]}\tilde{\mathcal{B}}_E^c \cong \mathcal{K}_A \oplus \mathfrak{g}_A^\perp.$$

This can not be a decomposition into a direct sum of locally trivial bundles (the last factor has dimension varying between 0, 2, and 3), but it is when restricted to the submanifold of $\tilde{\mathcal{B}}_E^c$ consisting of framed connections with stabilizer of fixed conjugacy class (irreducible, conjugate to $SO(2)$, or fully reducible), where $\mathfrak{g}_A^\perp$ actually defines a locally trivial vector bundle.

Because $cs$ is, up to a constant, fixed under the action of $\mathcal{G}_E^b$ and not just $\mathcal{G}_E^{c,b}$, its $L^2$ gradient on $\mathcal{A}_E$ must take values in $\mathcal{K}_A$. It is clear from

$$\nabla cs(A) = *F_A$$

that the critical points of $cs$ are precisely the flat connections.
If \( A_- \) and \( A_+ \) are connections on the ends of \( \pi^*E \) over \( \mathbb{R} \times Y \) (and \( \alpha_\pm \) the corresponding orbits in the configuration space), we constructed the 4-dimensional configuration space in Definition 1.2 as a quotient of \( \tilde{\mathcal{A}}_{\pi^*E,2}(A_-, A_+) \). If we pick a base connection \( A_0 \) in this space, we get a description

\[
\tilde{\mathcal{A}}_{\pi^*E,2}(A_-, A_+) = \{ A_0 + \Omega^1_\mathcal{A}(\pi^*g_E) \} \times (\pi^*E)_{0,b},
\]

and in particular of the space of unframed connections as an affine space. The Lie algebra of the gauge group \( \mathcal{G}_{\pi^*E}^c(A_-, A_+) \) is \( \Omega^0_\mathcal{A}(g_E) \), the space of sections of \( g_E \) that are \( A_\pm \)-harmonic on the corresponding ends. The differential of the gauge group action at some \((A, p)\) is given by \( \sigma \mapsto (d_A \sigma, \sigma(b)) \). Picking a complement \( g_A^1 \) of \( \text{Im}(d_A) \), we again have our decomposition

\[
T_{(A,p)}\tilde{\mathcal{A}}_E = \ker(d_A^h) \oplus \text{Im}(d_A) \oplus g_A^1.
\]

We also have the endpoint maps to \((A_+, E_b) \cong SO(3)\). Before modding out by gauge, the positive endpoint map sends \( ev_+(A, p) = \gamma_\infty(A, p) \), the parallel transport of \( p \) to \( \infty \) using \( A \). The differential of this map at \((A, p)\) is a linear map

\[
\Omega^1_\mathcal{A}(g_E) \oplus g_{0,b} \to g_b,
\]

equal to the integral

\[
\int_0^\infty \gamma^*_A \sigma(t, b) dt.
\]

Here we use parallel transport backwards along \( \gamma \) using \( A \) so that \( \sigma(t, b) \) is taken to the single vector space \( g_{0,b} \), where it makes sense to integrate. Observe that if \( A \) is in temporal gauge, this is just \( \int_0^\infty \sigma(t, b) dt \).

Nothing changes in the above description when passing to a general Riemannian 4-manifold \( W \) with cylindrical ends.

### 2.2. Hilbert manifold completions.

To have a useful transversality theory, we must replace the naive “infinite-dimensional manifolds” \( \mathcal{A}, \mathcal{G} \), and so on with certain completions which are manifolds modeled on Hilbert spaces. If \( E \) is an \( SO(3) \)-bundle over a 3-manifold, define \( A_{E,k} \) to be the set of \( L^2_k \) connections on \( E \); more precisely, if \( \Omega^1_k \) denotes the Hilbert space of \( L^2_k \) 1-forms, then

\[
A_{E,k} = A_0 + \Omega^1_k(g_E)
\]

for any choice of smooth base connection \( A_0 \). The resulting set does not depend on the choice of smooth connection. Similarly, we may define

\[
\mathcal{G}_{E,k+1} = \{ \sigma \in L^2_{k+1}(\text{End}(E)) \mid \sigma(x) \text{ is an isomorphism for all } x \},
\]
given the constraint that \( k + 1 \geq 2 \) so that gauge transformations are automatically continuous and we may take point evaluations as in the definition. Then \( \tilde{\mathcal{G}}_{E,k+1}^{c} \) is the subset of sections which lift continuously to \( \tilde{\text{Aut}}(E) \). As in [DK90, Section 4.2], the space \( \tilde{\mathcal{G}}_{E,k+1}^{c} \) becomes a Banach Lie group, and we have a smooth action of \( \tilde{\mathcal{G}}_{E,k+1}^{c} \) on \( \tilde{\mathcal{A}}_{E,k} = A_{E,k} \times E_k \); the same argument as the case of smooth gauge transformations shows that this action is free. The following is [Bou08, Proposition 3.1.5.10]. To facilitate comparison to the version stated there, note that Bourbaki’s definition of immersion includes the assumption that the differential has closed image with a closed complement.
Proposition 2.1. Suppose $G$ is a Banach Lie group and $X$ a Banach manifold on which $G$ acts. Suppose that $G$ acts freely and properly on $X$, and such that if $ho(x) : T_xG \to T_xX$ is the differential of the action at the point $x$, the image of $\rho(x)$ is closed and has some closed complement for all $x$. Then the quotient topology on $X/G$ is Hausdorff, and there is a unique smooth structure on $X/G$ such that $\pi : X \to X/G$ is a submersion. Furthermore, $X \to X/G$ is a principal $G$-bundle.

We should see that all these assumptions apply to the action of $\mathcal{G}_{E,k+1}^b$ on $\tilde{A}_{E,k}$, giving us the quotient manifold $\tilde{\mathcal{B}}_{E,k}$. We record this as a lemma.

Lemma 2.2. $\tilde{\mathcal{B}}_{E,k}^c$ is a Hausdorff Hilbert manifold if $k \geq 1$.

Proof. Properness of this action would follow from properness of the action on the unframed space of connections, because the map $\tilde{A}_{E,k} \to A_{E,k}$ is proper (because $E_b$ is compact). That the action of $\mathcal{G}_{E,k+1}$ on $\mathcal{A}_{E,k}$ is proper is proved in [DK90, Section 2.3.7], by a bootstrapping argument.

Because $\tilde{A}_{E,k}$ is a Hilbert manifold, we only need to verify that the image of the differential is closed; write this explicitly as the sum

$$(d_A, \text{ev}_b) : \Omega_{k+1}^0(\mathfrak{g}_E) \to \Omega_k^1(\mathfrak{g}_E) \oplus \mathfrak{g}_b.$$

First note that $\text{Im}(d_A)$ is closed: there is an $L^2$-orthogonal decomposition

$$\Omega_k^1(\mathfrak{g}_E) = \ker(d_A^b) \oplus \text{Im}(d_A);$$

that this is true in $\Omega_k^1$ and not just $\Omega_k^1$, follows from elliptic regularity (if $d_A \sigma = \eta$ where $\eta \in L^2_k$, then $\sigma \in L^2_{k+1}$).

Then suppose $(\eta_n, g_n) \to (\eta, g)$ is a convergent sequence, for which there is a sequence of $\sigma_n \in \Omega_{k+1}^0$ with $d_A \sigma_n = \eta_n$ and $\sigma_n(b) = g_n$. Because $\text{Im}(d_A)$ is closed, there is some $\sigma$ with $\eta = d_A \sigma$. Thus there is some sequence $\psi_n \in \ker(d_A)$ so that $\sigma_n + \psi_n \to \sigma$. As long as $k + 1 \geq 2$, point-evaluation is continuous, and so $g_n + \psi_n(b) \to g$; because $g_n \to g$, we see that $\psi_n(b) \to 0$. Thus $\sigma(b) = g$ and $d_A \sigma = \eta$ as desired. 

This provides somewhat less inspiring charts than the case of irreducible connections and unbased gauge group, where one can explicitly present slices for the action of the gauge group as those connections with

$$d_{A_0}^*(A - A_0) = 0.$$

One can at least use the decomposition

$$T_{A,k} = K_{A,k} \oplus G_{A,k} \oplus \mathfrak{g}_A^+$$

to provide slices for the based gauge group as exponentials of $K_{A,k} \oplus \mathfrak{g}_A^+$. In addition to the natural tangent bundle $T_{\tilde{A}_{E,k}} = \Omega_k^1(\mathfrak{g}_E)$, we may also define completed tangent bundles $T_{\tilde{J}}\tilde{A}_{E,k} = \Omega_j^1(\mathfrak{g}_E)$ for any $j \leq k$.

After passing to this completion, the analogue of Proposition 1.2 remains true. Whenever we construct a gauge transformation via parallel transport, it has one greater order of differentiability than the connection we used to define parallel transport.
2.3. The 4-dimensional case. In contrast to the case of homology 3-spheres, where to define Floer’s instanton homology we only needed to study instantons on the cylinder with irreducible limits on the ends, the same flavor of Sobolev completion will not suffice to define the configuration space of 4-manifolds with general limits. Instead, we will need to use weighted Sobolev spaces. Recall $\pi^*E$ is an $SO(3)$-bundle on $\mathbb{R} \times Y$, given as the pullback of some bundle $E$ on $Y$ under the projection map. Let $f_\delta$ be a smooth positive function on $\mathbb{R}$ such that $f_\delta(t) = e^{\delta|t|}$ for $|t| \geq 1$.

Suppose we have fixed limiting connections $A_-$ and $A_+$ on $E$. Here we are fixing connections and not gauge equivalence classes; we write the corresponding orbits in $\mathcal{B}_E^\pi$ as $\alpha_{\pm}$. Pick a smooth base connection $A_0$ on $\pi^*E$ over $\mathbb{R} \times Y$ that agrees with the pullback connections of $A_-$ and $A_+$ near $\pm \infty$ respectively; it traces out a relative homotopy class $z \in \pi_1(\mathcal{B}_E^\pi, A_-, A_+)$. Then one may define the space of $L^2_{k,\delta}$ sections of $\pi^*E$ as those sections for which

$$\int_{\mathbb{R} \times Y} f^2 \left( |\sigma|^2 + |\nabla_{A_0} \sigma|^2 + \ldots |\nabla_{A_0} \sigma|^2 \right) < \infty;$$

this is the $L^2_{k,\delta}$ norm with respect to $A_0$. Then we say

$$\mathcal{A}_{E,z,k,\delta}(A_-, A_+) := \{ A_0 + \Omega_{k,\delta}^1 (\theta_E) \}.$$  

This only depends on $A_0$ through the choice of $z$, but it clearly depends on the limits $A_{\pm}$; if two pairs $(A_{\pm}')$ of limits are gauge equivalent by gauge transformations in the same homotopy class, then there is a gauge transformation taking

$$\mathcal{A}_{E,z,k,\delta}(A_-^{1 \pm}, A_+^{1 \pm}) \cong \mathcal{A}_{E,z,k,\delta}(A_-^{2 \pm}, A_+^{2 \pm}).$$

Similarly, this set does not depend on $f$, only $\delta$.

We need a Banach Lie group $\mathcal{G}_{E,k+1,\delta}(A_-, A_+)$. Set

$$\Gamma_{\pm} = \Gamma(\alpha_{\pm}) \subset \mathcal{G}_{E,k+1}^c;$$

this is precisely the set of $\alpha_{\pm}$-harmonic gauge transformations. As before, we make sense of the $L^2_{k+1,\delta}$ norm by considering gauge transformations as certain sections of $\pi^*(\text{End}(E))$. If we only cared about constant limits (as is the case when the $\alpha_{\pm}$ are irreducible), we could take the Lie group

$$\mathcal{G}_{E,k+1,\delta} = \{ \sigma \in \Gamma(\text{Aut}(\pi^*E)) \subset \Gamma(\text{End}(\pi^*E)) \mid (\sigma - 1) \in L^2_{k+1,\delta}(\text{End}(\pi^*E)) \}.$$  

That is, roughly, gauge transformations in this group are sections of $\pi^*(E \otimes E^*)$ that are pointwise automorphisms and decay exponentially quickly to the identity. This is again a Banach Lie group that acts smoothly on $\mathcal{A}_{E,z,k,\delta}(A_-, A_+)$; this is essentially the content of [Don02, Section 4.3]. We write

$$\mathcal{G}_{E,k+1,\delta}^h(\alpha_{\pm}) = \{ \sigma \in \Gamma(\text{Aut}(E)) \mid \exists \psi \text{ with parallel ends and } \sigma - \psi \in L^2_{k+1,\delta}(\text{End}(E)) \};$$

This is again a Banach Lie group, but now we have the surjective endpoint evaluation map

$$\mathcal{G}_{E,k+1,\delta}(A_-, A_+) \to \Gamma_- \times \Gamma_+.$$  

There is also the natural subgroup $\mathcal{G}_{E,k+1,\delta}(A_-, A_+)$, those transformations that lift to sections of $\mathcal{G}_{E,k+1,\delta}(A_-, A_+)$, following Definition 1.2, we use this to define the configuration space as follows.
Definition 2.1. The configuration space of framed connections on $\pi^*E$ between $\alpha_-$ and $\alpha_+$ is defined to be

$$\tilde{B}_{\pi^*E,z,k,\delta}(A_-, A_+) = \tilde{A}_{\pi^*E,z,k,\delta}(A_-, A_+)/G_{\pi^*E,k,\delta}(A_-, A_+).$$

This carries an action of SO(3) by acting on the fiber above the basepoint on the right. The endpoint maps given by parallel transport along $\mathbb{R} \times \{b\}$ are right SO(3) maps to $[\alpha \pm] = E_b/G_\pm$; these maps are smooth and furthermore submersions. Equivariant with respect to these is a (left) $\mathbb{R}$ action by pullback of the connection and parallel transport of the framing.

Again, we should check that this group action is well-behaved.

Lemma 2.3. $\tilde{B}_{\pi^*E,z,k,\delta}(\alpha_-, \alpha_+)$ carries the natural structure of a smooth Hilbert manifold when $k \geq 2$.

Proof. We need to check that the action on $\tilde{A}_{\pi^*E,z,k,\delta}(A_-, A_+)$ is proper, and that the tangent spaces to orbits are closed.

Let $(\sigma_n, A_n)$ be a sequence of $L^2_{k+1,\delta}$ gauge transformations and $L^2_{k,\delta}$ connections so that $\sigma_n^* A_n \to B$ in the $L^2_{k,\delta}$ topology, and so that $A_n \to A$ in the $L^2_{k,\delta}$ topology. Write $B_n = \sigma_n^* A_n$. We want to show that $\sigma_n$ then has a convergent subsequence in the $L^2_{k+1,\delta}$ topology.

On any chart with compact closure, choose an arbitrary trivialization of $\pi^*E$, so that we represent $A_n$ and $B_n$ as matrices of 1-forms and $\sigma_n$ as a map to SO(3). Then as in [DK90, Section 2.3.7] we have the formula

$$d\sigma_n = \sigma_n A_n - B_n \sigma_n;$$

from this it follows that an $L^2_j$ bound on $\sigma_n$ implies an $L^2_{j+1}$ bound as long as $j \leq k$, and an $L^2_j$ bound follows from compactness of SO(3) and of the chart itself. Therefore $\|\sigma_n\|_{L^2_{k+1}}$ is uniformly bounded; by compactness, we may choose a convergent subsequence $\sigma_n \to \sigma$ in $L^2_{k}$. But then we may identify $\sigma^* A = B$, and therefore we have

$$d\sigma - d\sigma_n = \sigma A - \sigma_n A_n - B \sigma + B_n \sigma_n,$$

which may be rewritten as

$$(\sigma - \sigma_n)(A) + \sigma_n(A - A_n) - B(\sigma - \sigma_n) - (B - B_n)(\sigma_n).$$

Then because all of $\sigma - \sigma_n, A - A_n,$ and $B - B_n$ go to 0 in $L^2_k$, we see that $d\sigma - d\sigma_n \to 0$ in $L^2_k$, and therefore $\sigma_n \to \sigma$ in $L^2_{k+1}$.

This shows that $\sigma_n \to \sigma$ in $L^2_{k+1,\text{loc}}$. We should check convergence on the ends. We may write $A_n$ and $B_n$ as the flat connection $\alpha$ plus an $L^2_{k,\delta}$ 1-form valued in $g_E$. Choose a trivialization of $E$ on a chart $U$ of $Y$, and extend that to a trivialization of $\pi^*E$ over $\mathbb{R} \times U$. Consider this on one end at a time; for convenience we say $(-\infty, 0] \times U$. We write the flat connection $\alpha$ as $d + \omega_\alpha$ in this chart, where $\omega_\alpha$ is a particular matrix of 1-forms, and then the connections in this trivialization take the form $d + \omega_\alpha + A_n$ for an $L^2_{k,\delta}$ 1-form $A_n$.

The above formula now gives us

$$d\sigma_n = \sigma_n (\alpha + A_n) - (\alpha + B_n) \sigma_n = \sigma_n A_n - B_n \sigma_n + (\sigma_n \alpha - \alpha \sigma_n);$$

more simply stated, this is

$$d\alpha \sigma_n = \sigma_n A_n - B_n \sigma_n.$$
For the limit $\sigma \in L^2_{k+1,\text{loc}}$, we then have $(d_\alpha \sigma) = \sigma A - B \sigma$. Because the $L^2_{\delta,\delta}$ norm of $\sigma A$ agrees with that of $A$, we see that $(d_\alpha \sigma)$ has an $L^2_{\delta,\delta}$ bound; because $\sigma$ is the limit of gauge transformations which are asymptotic to 1 on the ends, $\sigma$ is asymptotic to 1 on the ends, and so $\sigma$ is $L^2_{0,\delta}$.

We also have

$$d_\alpha \sigma - d_\alpha \sigma_n = (\sigma(A - A_n) + (\sigma - \sigma_n) A_n - (B - B_n) (\sigma) - B_n (\sigma - \sigma_n)).$$

Therefore

$$\|d_\alpha \sigma - d_\alpha \sigma_n\|_{L^2_{\delta,\delta}} \leq \|A - A_n\| + 2\|A_n\| + \|B - B_n\| + 2\|B_n\|,$$

and in particular $\|\sigma_n\|_{L^2_{\delta,\delta}}$ is uniformly bounded. By the compactness theorem for weighted Sobolev spaces, $\sigma_n \to \sigma$ in $L^2_{1,\delta}$, for $\delta' < \delta$. Inducting with the previous formula, we may see that $\|\sigma_n\|_{L^2_{k+1,\delta}}$ is uniformly bounded. Therefore $\sigma_n \to \sigma$ in $L^2_{k+1,\delta}$ for $\delta' < \delta$. The above formula again shows using Sobolev multiplication that $\|\sigma - \sigma_n\|_{L^2_{k+1,\delta}} \to 0$ on this chart, and by running this argument a few times $\sigma_n \to \sigma$ in $L^2_{k+1,\delta}$ on all of $\mathbb{R} \times Y$, as desired.

The extension to the entire gauge group (with gauge transformations that are exponentially decaying to $\alpha$-harmonic gauge transformations) is formal, as the quotient by the subgroup used above is compact, equal to the space of harmonic gauge transformations on the two ends.

To see that tangent spaces are closed, it suffices to show that $\Omega^1_{k,\delta}(gE)$ admits a closed splitting $\text{Im}(d_A) \oplus \ker(d_A^*)$; we want to show that every element may be written uniquely as a sum of that form.

That these are closed subspaces follows from the closed range theorem for densely defined operators (thinking of this as a densely defined operator in $L^2$): the range of $d_A$ is closed if and only if its image is the $L^2$ orthogonal complement of $\ker(d_A^*) \cap L^2_{k,-\delta}$.

If $\psi \in \ker(d_A^*) \cap L^2_{-k,-\delta}$, then by separation of variables on the end, using that the signature operator on $Y$ has no nonzero eigenvalues of magnitude less than $\delta$, we see that $\psi \in L^2_{-k,-\delta,\text{ext}}$: it is the sum of a section which is constant and $\alpha$- or $\beta$-harmonic on the ends and an $L^2_{k,-\delta}$ section; similarly elliptic regularity and the assumption that $A$ is $L^2_{k,\delta}$ implies that $\psi \in L^2_{k,\delta,\text{ext}}$. Now we may apply the usual integration by parts trick to show that $\langle \sigma d_A \psi, \psi \rangle = \langle \sigma, d_A^* \psi \rangle = 0$ for all $\sigma$.

What is left is to see that the equation $\Delta_A \sigma = -d_A^* \psi$ has a solution for any $\psi \in \Omega^1_{k,\delta}$; to see this, choose $\psi$ so that $d_A^* \psi$ is in the orthogonal complement of $\text{Im}\Delta_A$. But as before we may see that if

$$0 = \langle \Delta_A \sigma, d_A^* \psi \rangle = \langle d_A \sigma, d_A d_A^* \psi \rangle,$$

then in particular $d_A d_A^* \psi = 0$, and as such $d_A^* \psi$ is parallel. Because $\psi$ is asymptotic to 0 on the ends, we see that $d_A^* \psi = 0$, and hence indeed

$$\Delta_A \sigma = -d_A^* \psi$$

is solvable for all $\psi$. Uniqueness follows because if $\Delta_A \sigma = 0$, then because $\sigma$ is asymptotic to 0 on the ends, by the usual integration by parts trick $d_A \sigma = 0$; again because $\sigma$ is asymptotically zero, we see that $\sigma$ is globally zero.

The discussion of tangent bundles and reducible configurations (Proposition 1.6) from the naive case carry over without change.
Finally, suppose we are given a complete Riemannian manifold $W$ with two cylindrical ends (i.e., specified isometries to $(-\infty, 0] \times Y_1$ and $[0, \infty) \times Y_2$), an $SO(3)$ bundle $E$ with specified isomorphisms over the ends to bundles $E_i \to Y_i$, and specified $\alpha_i$ on the $E_i$. Further suppose $W$ is equipped with an embedded path $\gamma$ in $W$, cylindrical and agreeing with specified basepoints $b_i$ on the ends. Then the construction of configuration spaces $\overline{\mathcal{B}}_{E,k,\delta}(\alpha_1, \alpha_2)$ with equivariant endpoint maps to $(E_i)_{b_i}/\Gamma_{\alpha_i}$ carries through with essentially no change from above. When defining cobordism maps in instanton homology, the data of the embedded path will only ultimately matter up to homotopy of such paths, which reduces to the data of a relative homotopy class $(D^1, \{0\}, \{1\}) \to (\overline{W}, Y_1, Y_2)$. 
3. Critical points and perturbations

3.1. Holonomy perturbations. In order to carry out the construction of a Morse-like homology theory on our configuration space \( \mathcal{B}_{E,k} \), the Chern-Simons functional will usually need to be perturbed so that the critical sets are isolated orbits of \( SO(3) \), and the moduli spaces of trajectories are smooth manifolds of the appropriate dimension. Here we recall the standard perturbations used in this situation, as well as their basic properties, from [KM11b, Section 3.2]; they work relative to a knot \( K \), but we may simply take \( K = \emptyset \). These perturbations originate in Floer’s definition of instanton homology and have been used consistently in the development of the subject.

Suppose we are given a collection of immersions

\[ q_i : S^1 \times D^2 \to Y, \quad i = 1, \ldots, N, \]

such that the \( q_i \) all agree in a small neighborhood of \( \{1\} \times D^2 \). Further pick a conjugation-invariant smooth function \( h : G^N \to \mathbb{R} \) and a nonnegative 2-form \( \mu \) on \( D^2 \) that integrates to 1 and vanishes near the boundary. Taking the holonomy of a connection \( A \) along the family of curves \( q_i(e^{2\pi it}, z) \) parameterized by \( i \) and \( z \) gives a map

\[ \text{Hol}_q(A) : D^2 \to SO(3)^N; \]

we are interested in the functions (called \( SO(3) \)-cylinder functions)

\[ f_{q,h,\mu}(A) = \int_{D^2} h(\text{Hol}_q(A))\mu. \]

These functions \( f_{q,h,\mu} \) have smooth extensions to the Hilbert manifolds \( \mathcal{A}_{E,k} \), where (as on \( \mathcal{A}_E \)) they are invariant under the action of the gauge group. By ignoring the framing, we can extend this trivially to a map \( f_{q,h,\mu} : \mathcal{A}_{E,k} \to \mathbb{R} \) invariant under both the left action of the full gauge group and the right action of \( SO(3) \). Thus they descend to \( SO(3) \)-invariant smooth functions \( f_{q,h,\mu} : \mathcal{B}_{E,k} \to \mathbb{R} \). Note that they are furthermore invariant under the remaining \( H^1(Y;\mathbb{Z}/2) \) action.

In the context of the \( U(2) \)-model, we choose an integral cohomology class \( c \in H^2(Y;\mathbb{Z}) \) reducing mod 2 to \( w_2E \) and hence a lift of \( E \) to the \( U(2) \)-bundle \( \lambda \oplus \mathbb{C} \), where \( c_1\lambda = c \). Fixing a base connection \( A_0 \), the configuration space is the set of connections for which \( \text{tr}(A) = A_0 \). Associated to any connection with fixed trace is the holonomy map

\[ \text{Hol}_A : \Omega Y \to U(2), \]

where our loops are based at \( b \); we also have, associated to \( A_0 \), a map

\[ \text{Hol}_{A_0} : \Omega Y \to U(1). \]

Because \( \text{tr}(A) = A_0 \), we have

\[ \det \text{Hol}_A = \text{Hol}_{A_0}; \]

thus we may define a relative holonomy map

\[ \text{Hol}'_A := \text{Hol}_A\text{Hol}_{A_0}^{-1} : \Omega Y \to SU(2). \]

This map is equivariant under the action of the gauge group of determinant-1 automorphisms (acting on \( SU(2) \) by translation by \( \sigma(b) \)). If one further acts on
connections by the operation of tensoring with $\chi_C$, where $\chi$ is a real line bundle, this describes the action of

$$\mathcal{G}_E/\mathcal{G}_E^\chi = H^1(Y; \mathbb{Z}/2)$$

on this space. Abusing notation to write $\chi : \pi_1 W \to \pm 1$ for the holonomy of the unique flat connection on this real line bundle, the corresponding map $\chi_C : \pi_1 W \to U(1)$ has determinant 1; so

$$\text{Hol}_{A\otimes\chi}(\gamma) = \chi(\gamma)\text{Hol}_A.$$ That is, $\text{Hol}$ is not invariant under the $H^1(Y; \mathbb{Z}/2)$ action. Now as above we may define cylinder functions using this relative holonomy map; we call these $SU(2)$-cylinder functions. Note that $SO(3)$-cylinder functions are the special case of $H^1(Y; \mathbb{Z}/2)$-invariant $SU(2)$-cylinder functions.

We’re interested in the gradient flow equation of the Chern-Simons functional on $\tilde{\mathcal{B}}$; the relevant perturbations (called “holonomy perturbations”) will be the formal gradients of the cylinder functions described above, so we will need to know that the formal gradients are well-behaved. The most convenient way to do so is to calculate them explicitly. We write

$$\nabla h : SO(3)^N \to \mathfrak{so}(3)^N$$

for the gradient of $h$ using the Lie group trivialization of the tangent bundle and the standard inner product on $\mathfrak{so}(3)$; write

$$\nabla^j h : SO(3)^N \to \mathfrak{so}(3)$$

for the $j$th component. Kronheimer and Mrowka give a formula for this in [KM11b, Equation (74)] along the base $[-\epsilon, \epsilon] \times D^2 \to Y$ of the embeddings $q_i$ as

$$\nabla_f := (\nabla_{f,q,h,\mu})(A) = * \left( \sum_{i=1}^N (q_i)_\ast((\nabla^i h)(\text{Hol}_q(A))\mu) \right),$$

where $(q_i)_\ast$ is the pushforward of differential forms on each tangent space. This is defined on the rest of $\text{Im}(q)$ via parallel transport, and then extended by 0 to the rest of $Y$. The following properties of the formal gradient $\nabla_f$ are enumerated as in [KM11b, Proposition 3.5] and immediate from the explicit formula above:

**Theorem 3.1.** Let $f : A_{E,k} \to \mathbb{R}$ be a fixed $SU(2)$-cylinder function as above. Then the formal gradient $\nabla_f$ defines a smooth section of the tangent bundle $T_k A_{E,k}$, and its first derivative $D\nabla_f$, considered as a section of the bundle $\text{Hom}(T_k, T_k)$, extends to $\text{Hom}(T_j, T_j)$ for all $j \leq k$. Further, we have the following pointwise norm bounds for constants $C, C_k, C'$:

$$\|\nabla_f(A)\|_{L^p} \leq C$$

$$\|\nabla_f(A)\|_{L^2_k} \leq C_k(1 + \|A - A_0\|_{L^2_k})^k$$

$$\|\nabla_f(A) - \nabla_f(A')\|_{L^p} \leq C'\|A - A'\|_{L^p}$$

In the last line $1 \leq p \leq \infty$ and $C'$ does not depend on $p$ (and thus $\nabla_f$ is continuous in $L^p$). More generally, for any $n$ we may find continuous increasing functions $k_n$ (depending on $f$) such that

$$\|D^n\nabla_f(A)\|_{L^2_k} \leq k_n(\|A - A_0\|_{L^2}),$$
where on the left side we’re taking an operator norm. Furthermore, $f$ (and thus $\nabla_f$) are invariant under the action of the full gauge group.

Suppose now we have a collection of $SU(2)$-cylinder functions $f_i$; we may associate an increasing sequence of constants ($K_i$) so that when $\sum_i K_i |a_i|$ converges, $\sum a_i \nabla f_i$ converges in $C^k$ on compact sets. One may take $K_i$ to be, for instance,

$$K_{i-1} + \sum_{n=1}^i k_n(f_i)(i).$$

Then we may define the $K$-weighted $\ell^1$ Banach space as the space of sequences $(a_i)$ with $\|a\|_P = \sum K_i |a_i| < \infty$. There is a map $\mathcal{P} \to C^\infty(\tilde{B}_{E,k}, \mathbb{R})$ given by sending

$$\pi = \sum_i a_i \mapsto \sum_i a_i f_i =: f_\pi$$

and a map to smooth sections of the tangent bundle by sending $\pi$ to its formal gradient, which we still call $\nabla_\pi$. We call $\mathcal{P}$ a Banach space of perturbations and the induced functions $f_\pi$ (which are now $L^1$ sums of $SU(2)$ cylinder functions, but not necessarily cylinder functions themselves) holonomy perturbations.

The results of the previous theorem extend to smoothness of the $\mathcal{P}$-parameterized section of the tangent bundle, and parameterized inequalities for $\nabla_\pi(A)$ where now $\pi$ may vary within $\mathcal{P}$ (with an extra factor of $\|\pi\|_P$ in the right-hand sides); this is [KM11b, Proposition 3.7].

The following is the relevant geometric fact about $SO(3)$ and $SU(2)$-cylinder functions — and therefore also holonomy perturbations. It is well-known, though usually stated nonequivariantly and for irreducible connections. Note that in the first statement we have quotiented by the entire (as opposed to even) based gauge group.

**Lemma 3.2.** Given any compact $SO(3)$-invariant submanifold $S \subset \tilde{B}_{E,k}$, the restriction of $SO(3)$-cylinder functions to $S$ are dense in the space of smooth invariant functions $C^\infty(S)^{SO(3)}$. This is furthermore true even if we demand the cylinder functions vanish in a small invariant open neighborhood $O$ of a finite set of orbits.

Similarly, given a compact $SO(3)$-invariant submanifold $S \subset \tilde{B}_{E,k}^{\text{det}} \cong \tilde{B}_E$, the restriction of $SU(2)$-cylinder functions to $S$ are dense in the space of smooth invariant functions on $S$.

**Proof.** If $y \in Y$ is the basepoint, consider the maps $A_{E,k} \to SO(3)$ given by $\text{Hol}(\gamma_i)$ where $\gamma_i$ are some family of immersed loops, based at $y$, all with the same germ there. (‘Immersed’ is implicit for all loops in the rest of this argument.) These maps are equivariant with respect to the action of the gauge group on $SO(3)$ (conjugating by the value at the basepoint), and hence descend to $SO(3)$-equivariant maps $\tilde{B}_{E,k} \to SO(3)$. The claim is that these maps separate points and tangent vectors in $\tilde{B}_{E,k}$; then, by compactness of $S$ there is some finite collection $\gamma_i$, $1 \leq i \leq N$, such that $\text{Hol}_A(\gamma_i) : S \to SO(3)^N$ is an embedding.

If two connections have the same holonomy along every immersed loop $\gamma$ based at $y$ with specified germ, they have the same holonomy along every piecewise smooth immersed loop: we may write any piecewise smooth loop $\gamma$ based at $y$ as being the composition of an arbitrarily small loop and a loop with the specified germ, and so $\text{Hol}_A \gamma = \text{Hol}_{A'} \gamma$ for any piecewise smooth based loop $\gamma$. For any point $p \in Y$, pick
a path $\alpha$ from $y$ to $p$, and define

$$\sigma(p) = \text{Hol}_A(\alpha)\text{Hol}_A^{-1}(\alpha);$$

that this is well-defined follows from the assumption that the (based) holonomies always agree along closed loops. Taking derivatives along any smooth curve based at $y$ we see that $A - d_A\sigma = A'$, and clearly $\sigma(y) = 1$. So based holonomy separates connections modulo based gauge. Equivalently, holonomy separates framed connections modulo gauge.

At the level of tangent spaces, we run the same argument with any $\omega$ such that $d\text{Hol}_A(\gamma)(\omega) = 0$ for all $\gamma$ to see that $\omega = d_A\sigma$ for some $\sigma \in \Omega^0(\mathfrak{g}_E)$. Pick a path $\alpha$ from $y$ to an arbitrary point $p$; then we have a natural isomorphism

$$\mathfrak{g}_p \cong T\text{Isom}(E_y, E_p)$$

coming from the framing on $E_y$, and so we may define

$$\sigma(p) = d\text{Hol}_A(\alpha)(\omega);$$

as before this is independent of the choice of $\alpha$ and it is easy to verify from this definition that $d_A\sigma = \omega$. Thus based holonomy separates tangent vectors in $\tilde{B}_{E,k}$.

Now choose some finite set of embedded based curves $\gamma_i$ with the same germ so that $\text{Hol}_A(\gamma)$ embeds $S$ into $SO(3)^N$. Given $h : S \to \mathbb{R}$ an $SO(3)$-invariant function, extend it to $\tilde{h} : SO(3)^N \to \mathbb{R}$ arbitrarily using the equivariant tubular neighborhood theorem. We may then approximate $h$ by holonomy perturbations by picking a sequence of embeddings $q_i$ of solid tori of small radius around the curves so that the radii go to zero as $j \to \infty$; then the cylinder functions

$$f_{q_i},h,\mu : S \to \mathbb{R}$$

approach $h$ in $C^\infty(S)$ for any choice of 2-form $\mu$, as desired.

The final claim about $SO(3)$-cylinder functions follows by applying the previous paragraphs to embed the submanifold $S$ and the disjoint finite set of orbits $C$ into $SO(3)^N$; if $C$ is, for instance, the inverse image of a small neighborhood of the image of $C$ in $SO(3)^N$ (whose closure is disjoint from the image of $S$). Then choose the extended function $\tilde{h}$ to be zero on this small neighborhood.

The corresponding fact for $SU(2)$-cylinder functions is a straightforward modifications of the same proof.

Following the lemma, we choose our Banach space of perturbations $\mathcal{P}_E$ to be generated by a choice of a countable set of holonomy perturbations $\{\pi_i\}$, where $\pi_i$ are generated by $(h, \gamma, \mu)$, where $h$ varies over a countable dense set of $SO(3)$-invariant functions on $SU(2)^N$, $\gamma$ varies over a countable dense subset of the space of $N$ immersions of $S^1 \times D^2$ that share a germ around $\{1\} \times D^2$, $\mu$ is specified, and $N$ varies over all positive integers. We say that a Banach space of perturbations $\mathcal{P}_E$ arising in this way is sufficiently large. The particular countable family of holonomy perturbations is inessential (other than these denseness conditions). While the particular choice of countable dense set to choose is noncanonical, the union of any two gives another; so any theorem on independence of perturbation in $\mathcal{P}_E$ extends to show that the particular choice of $\mathcal{P}_E$ is irrelevant.

In practice we will need to make further perturbations, but we want to do so without changing the existing functional at a finite set $C$ of ‘acceptable’ critical orbits. We say that holonomy perturbations vanishing on a small open neighborhood
\( O \) of this set \( C \) adapted to \( O \); the lemma shows these are in large supply. If \( P_E \) is a Banach space of perturbations and \( \pi_0 \) is some fixed perturbation, we denote
\[
\{ \pi \in P_E \mid f_\pi \big|_O = f_{\pi_0} \big|_O \} =: P_{E, O}.
\]

It is well-known that the space of flat connections is compact. The set of critical points of the unperturbed Chern-Simons functional framed flat connections, modulo even gauge; it’s equivalently described as a sort of framed projective representation variety of \( E \). We recall the following compactness principle for critical points from [KM11b, Lemma 3.8]; their argument is unchanged for the framed moduli space, modulo even gauge.

**Lemma 3.3.** Let \( P \) be a Banach space of perturbations. The map
\[
F : P \times \hat{\mathcal{B}}_{E,k} \to P \times \mathcal{T}_{k-1} \hat{\mathcal{B}}_{E,k},
\]
\((\pi, [A]) \mapsto (\pi, \nabla(\text{cs} + f_\pi)([A])), \) is proper. In particular, if \( \mathcal{C}_k \subset P \times \hat{\mathcal{B}}_{E,k} \) denotes the subset whose fiber over \( \pi \in P \) is the set of critical points of \( \text{cs} + f_\pi \) — that is, \( \mathcal{C}_k = F^{-1}(P \times \{0\}) \) — then the projection \( \mathcal{C}_k \to P \) is proper.

3.2. Linear analysis. It follows immediately from [Don02, Equation 2.18], modified to fit our normalization of \( \text{cs} \), that
\[
\text{cs}(A + \omega) = \text{cs}(A) + \int_Y \text{tr}(d_A \omega \wedge \omega + \frac{2}{3} \omega \wedge \omega \wedge \omega)
\]
that the Hessian of the Chern-Simons functional on \( A_E \) is precisely
\[
* d_A : \Omega^1_k(g_E) \to \Omega^1_{k-1}(g_E).
\]
(When working with Sobolev completions, the Hessian is a smooth map between tangent bundles of different regularity.) Because the tangent space at a framed connection \([A,p] \in \hat{\mathcal{B}}_{E,k}\) can be decomposed as \( K_{A,k} \oplus g_A^k \) (recall that \( K_{A,k} \) is defined to be the Coulomb slice \( \text{ker}(d_A^* \big|_{\Omega^1_k(g_E)})) \), this identifies the Hessian of the Chern-Simons functional at a flat connection as
\[
\text{Hess}_A = *d_A \oplus 0 : K_{A,k} \oplus g_A^k \to K_{A,k-1} \oplus g_A^{k-1}.
\]

We call the summand \( *d_A : K_{A,k} \to K_{A,k-1} \) the normal Hessian, \( \text{Hess}_A^N \), which is the Hessian restricted to the normal space to the orbit. The Hessian is a summand of a larger elliptic operator, the extended Hessian
\[
\text{Hess}_A : \Omega^0_k(g_E) \oplus \Omega^1_k(g_E) \to \Omega^0_{k-1}(g_E) \oplus \Omega^1_{k-1}(g_E)
\]
by the matrix
\[
\begin{pmatrix}
0 & d_A^*
\end{pmatrix}.
\]
It is clear that \( \text{Hess}_A^N = \Delta_A \) on \( \Omega^0 \oplus \Omega^1 \) and thus \( \text{Hess}_A \) is a self-adjoint elliptic operator. At a flat connection, where \((d_A)^2 = 0\), the extended Hessian has the further decomposition under
\[
\Omega^1_k(g_E) = K_{A,k} \oplus G_{A,k}
\]
as the orthogonal direct sum of \( \text{Hess}_A^N \) on \( K_{A,k} \) and the signature operator \( S_A \) on \( \Omega^0 \oplus G_{A,k} \). The kernel of the Hessian is identified with the kernel of the Laplacian on 0- and 1-forms (and is thus invertible if the Laplacian is).

Adding a perturbing term, the Hessian of the perturbed Chern-Simons functional \( \text{Hess}_{A,e} \) differs only by the gradient of the vector field \( \nabla_x \) at \([A]\), written
We define the perturbed extended Hessian the same way. The perturbation of $\text{Hess}_A$ is a compact self-adjoint operator, and so the resulting perturbed extended Hessian operator is still self-adjoint first order elliptic. At critical points of $\text{cs}_f$, using the decomposition $\Omega^0 = \mathcal{H}^0_A \oplus \text{Im}(d^*_{\chi})$, we see that

$$\text{Hess}_{A, \pi} = 0 \oplus \text{Hess}_{\nu} \oplus S_A$$
onumber

on

$$\mathcal{H}^0_A \oplus (\text{Im}(d^*_A) \oplus \mathcal{G}_{A,k}) \oplus \mathcal{K}_{A,k}.$$

Thus at critical points $A$ of the perturbed Chern-Simons functional, $\text{Hess}_{\nu}$ is a summand of a self-adjoint elliptic operator, and so it has discrete eigenvalues and the space it acts on enjoys a direct sum decomposition over these (finite-dimensional) eigenspaces.

**Definition 3.1.** A critical orbit $[A, p] \in \tilde{\mathcal{B}}_{E,k}$ of the perturbed function $\text{cs}_f$ is called nondegenerate if $\text{Hess}_{[A], \pi}$ is invertible, or, equivalently, the kernel of $\text{Hess}_{A, \pi}$ on $\Omega^1_k(q) \oplus \Omega^1_{-1}(q)$ consists only of harmonic 0-forms.

This terminology agrees with [Don02]; if the connection $A$ is furthermore irreducible, and so its extended Hessian operator is invertible, Donaldson calls $A$ acyclic.

### 3.3. Reducible critical points.

Because of the linear nature of the $SO(2)$-fixed subspace of $\tilde{\mathcal{B}}_E$,

$$(\tilde{\mathcal{B}}_E)^{SO(2)} = \bigcup_{\eta} \mathcal{B}_\eta,$$

it’s especially easy to determine the critical points of the Chern-Simons functional. When $b_1(Y) = 0$, there is a unique flat connection in each component (corresponding to the calculation $\text{Hom}(\pi_1(Y), \mathbb{Z}) = 0$), at which the Hessian inside the $SO(2)$-fixed subspace is simply the restriction of $\ast d : \Omega^1_k(Y; i\mathbb{R}) \rightarrow \Omega^1_{k-1}(Y; i\mathbb{R})$ to

$$\text{Hess}_0 = \ast d : \ker(d^*)_k \rightarrow \ker(d^*)_{k-1}.$$  

This depends only on the underlying metric on $Y$, not the connection $A$. Because $b_1(Y) = 0$, Hodge theory guarantees that this is an isomorphism, and so these critical points are nondegenerate in the $SO(2)$-fixed locus.

**Proposition 3.4.** Suppose $Y$ is a Riemannian 3-manifold with $b_1(Y) = 0$, equipped with an $SO(3)$-bundle $E$. There is a positive constant $\epsilon$ so that for any perturbation $\pi \in \mathcal{P}$ with $\|\pi\| \leq \epsilon$, there is precisely one critical point of $\text{cs}_f$ in each component $\mathcal{B}_{\eta,k}$ of the $SO(2)$-fixed point space, and the Hessian of each inside $\mathcal{B}_{\eta,k}$ is nondegenerate.

**Proof.** The compactness principle Lemma 3.3 also applies to $\mathcal{B}_{\eta,k}$. The set of nondegenerate critical points in $\mathcal{C}_*$ (which we call $\mathcal{C}_*^\circ$) is open, and hence the subset of degenerate points is closed; because proper maps are closed, the set of perturbations $\pi$ for which some perturbed critical point $[A]$ is cut out non-transversely is closed. Conversely, regular perturbations (written $\mathcal{P}^*$) form an open set. Thus, because every reducible is cut out transversely in the fixed locus for $\pi = 0$, the same is true for $\|\pi\| \leq \epsilon$ for some small $\epsilon$. Finally, if $p : \mathcal{C}_* \rightarrow \mathcal{P}$ is the projection, the map

$$p : p^{-1}(\mathcal{P}^*) \rightarrow \mathcal{P}^*$$


is still a proper local diffeomorphism and hence the inverse function theorem guarantees that (for sufficiently small $|\pi|$) there is precisely one critical point of $cs + f_\pi$ in each component $B_{y,k}$ of the $SO(2)$-fixed point set.

There is a class of $SO(3)$-bundles over 3-manifolds with $b_1(Y) > 0$ for which we can avoid reducibles entirely, called admissible bundles (sometimes ‘non-trivial admissible’ bundles). These were introduced in Floer’s work on Dehn surgery in [Flo95], and were extensively used in Kronheimer and Mrowka’s study of instanton knot homology, beginning with [KM11b]. In defining instanton Floer homology we will restrict to these two somewhat orthogonal cases: $b_1 = 0$ or $E$ admissible.

**Definition 3.2.** An $SO(3)$-bundle over a 3-manifold $Y$ is admissible if every lift of $w_2(E)$ to a class in $H^2(Y;\mathbb{Z})$ is non-torsion. We say that an $SO(3)$-bundle $E$ over $Y$ is weakly admissible if either $E$ is admissible or $b_1(Y) = 0$.

Ultimately, the instanton chain complex is described in terms of critical orbits, not fixed points, so we give a description of these. The class of weakly admissible bundles allows for a succinct description of the class of reducible critical orbits.

**Proposition 3.5.** Let $Y$ be a Riemannian 3-manifold equipped with weakly admissible $SO(3)$-bundle $E$, equipped with a perturbation $\pi$ small enough that Proposition 3.4 applies. Then we have the following description of the reducible critical $SO(3)$-orbits of $cs + f_\pi$ in $\widetilde{B}_{E,k}$.

- If $b_1(Y) = 0$, then there is a unique critical orbit lying in each reducible component $\{z_1, z_2\} \in \text{Red}(Y, E)$ of the configuration space. In the components $\text{Red}_s(Y, E)$ that contain a fully reducible orbit, the full reducible is the critical orbit.
- If $E$ is admissible, there are no reducible critical orbits.

**Proof.** First suppose $b_1(Y) = 0$. By Proposition 3.4, there is a unique critical point in each connected component of the $SO(2)$-fixed subspace. The components of the $SO(2)$-fixed subspaces were enumerated in Corollary 1.3.

In general, a critical orbit of $cs$ in the component $\{z_1, z_2\} \in \text{Red}(Y, E)$ is a gauge equivalence class of flat connection respecting the isomorphism

$$E \cong \mathfrak{g} \oplus \eta_1 \otimes \eta_2^{-1}.$$ 

In particular, $\eta_1 \otimes \eta_2^{-1}$ would support a flat connection, and thus have torsion first Chern class. Because $c = z_1 + z_2$, this implies that $c - 2z_2$ is a torsion cohomology class; as this also represents $w_2(E)$, this implies that $E$ is not admissible. Because $E$ supports no reducible critical points for the unperturbed Chern-Simons functional, the same is true of all sufficiently small perturbations.

**3.4. Transversality for critical points.** Before discussing the 4-dimensional case, where we will assume that the limiting connections are nondegenerate, we should verify that this situation is achievable! This leads us to the first transversality theorem of this paper; its proof is a model for the rest of the transversality results we will need. It essentially follows the corresponding proof in [AB96].

**Theorem 3.6.** Let $Y$ be a Riemannian 3-manifold, equipped with an $SO(3)$-bundle $E$. Suppose either that $b_1(Y) = 0$ or that $E$ is admissible. Given a sufficiently large Banach space of perturbations $P$, there is a dense open set of $\pi \in P$ for which there are finitely many critical orbits of the perturbed Chern-Simons functional $\widetilde{B}_{E,k}$ on each of which the normal Hessian is invertible.
The proof proceeds by achieving transversality in each locus inductively. In the course of achieving transversality over the reducible locus, we need a simple functional analysis lemma.

**Lemma 3.7.** Let $H$ be a Hilbert space, equipped with a densely defined self-adjoint Fredholm operator $T : H \to H$; write $H_0 = \ker(T)$ and $H_1 = \im(T)$ for the associated orthogonal splitting of $H$. Let $K : H \to H$ be a bounded self-adjoint operator with $K_{00} : H_0 \to H_0$ injective. Then for sufficiently small positive $\varepsilon$, the map $T + \varepsilon K$ is injective; in particular, an isomorphism.

**Proof.** If $D = H_0 \oplus D_1$ is the domain of $T$, then because $T_{11} : D_1 \to H_1$ is an isomorphism, we have a bound $\|T_{11}x_1\| \geq C_T\|x_1\|$ for some $C_T > 0$ (where the norms are defined on the appropriate domains). Similarly we have $\|K_{00}x_0\| \geq C_K\|x_0\|$ for some $C_K > 0$. Now suppose $(T + \varepsilon K)(x_0 + x_1) = 0$ for some $x = (x_0, x_1) \in H$. This means precisely that

$$K_{00}x_0 + K_{10}x_1 = 0,$$

$$T_{11}x_1 + \varepsilon(K_{01}x_0 + K_{11}x_1) = 0.$$  

These two equations give us the bounds

$$C_K\|x_0\| \leq \|K_{00}x_0\| = \|K_{10}x_1\| \leq \|K\|\|x_1\|,$$

$$C_T\|x_1\| \leq \|T_{11}x_1\| = \varepsilon\|K_{01}x_0 + K_{11}x_1\| \leq \varepsilon\|K\|(\|x_0\| + \|x_1\|).$$

Combining these two inequalities we then have

$$C_T\|x_1\| \leq \varepsilon\|K\|(1 + \|K\|/C_K)\|x_1\|.$$  

As soon as

$$\varepsilon < \frac{C_T}{\|K\|(1 + \|K\|/C_K)},$$

this implies that $x_1 = 0$, and hence that $x_0 = 0$. \hfill \blacksquare

**Proof of Theorem 3.6.** By the discussion in the proof of Proposition 3.4, there is an open subset $U$ of $\mathcal{P}$ so that all reducibles are cut out transversely in their own locus. For $\pi \in U$, there are thus a finite number of reducible $\pi$-critical orbits. Recall that $cs$ and $SU(2)$-holonomy perturbations are invariant under the action of $SO(3)$, but not of $H^1(Y; \mathbb{Z}/2)$.

The normal Hessian to an $SO(2)$-reducible or $O(2)$-reducible fixed point might have nontrivial kernel, a finite-dimensional subspace of the tangent space.

Enumerate an element of each $SO(2)$-reducible critical orbit that is not fully reducible as $[A_i]$, and let $D_i$ denote a small disc inside the kernel of $\text{Hess}_{A_i}^\nu = Z_i$. The manifold $D_i$ is an $SO(2)$-manifold; we may identify the orbit through $D_i$ as a vector bundle over $S^2$ with generic fiber $D_{i1}$; call this neighborhood $V_i$. An $SO(3)$-invariant function on $V_i$ is the same thing as an $SO(2)$-invariant function on $D_i$. At an $SO(2)$-reducible corresponding to a reduction $E \cong \mathbb{R} \oplus \zeta$, the kernel of this operator is $H^1(Y; \zeta)$. The $SO(2)$-action is by scalar multiplication on this complex vector space. The restriction of some $SO(2)$-invariant nondegenerate quadratic function on $Z_i$ gives a smooth function on $D_i$ with Hessian at the origin equal to the identity. Note that this implies that there are an even number of positive and negative eigenvalues.

We will use holonomy perturbations approximating this smooth function on $D_i$ to correct the fact that the Hessian has kernel. More precisely, if $\gamma_j$ is a sequence of
curves for which \( \text{Hol}_A(\gamma) \) gives an equivariant embedding of the neighborhoods \( V_i \) into \( SU(2)^N \). we choose a function \( h \) on \( SU(2)^N \) which agrees in a neighborhood of each \( A_i \) with \( v \to q(v) \) (identifying a neighborhood of each \( A_i \) with the tangent space at that point). Now pick a sequence of holonomy perturbations \( \pi_n \) arising from approximations of \( h \) and \( \gamma_j \). For sufficiently large \( n \), the restriction of \( D_A \nabla_{\pi_n} \) to \( \ker(\text{Hess}_{A_i}) \) is very close to \( q \).

If we decompose \( K_{A_i,k} \) as the direct sum of \( Z_{i} \) and its orthogonal complement, then one may write \( \text{Hess}_{A_i,\pi} \) in block-matrix form with respect to this splitting; the component \( Z_i \to Z_i \) is the Hessian restricted to the submanifold \( D_i \) at \( A_i \). Therefore, we can apply Lemma 3.7 to \( T = \text{Hess}_{A_i,\pi} \) and \( K = D_A \nabla_{\pi_n} \) for large \( n \) to see that \( \pi + \varepsilon \pi_n \) is a regular perturbation at reducible connections for sufficiently large \( n \) and sufficiently small \( \varepsilon \). A similar argument works for the fully reducible connections, now using a nondegenerate \( SO(3) \)-invariant quadratic form.

Similarly to Proposition 3.4, nondegeneracy for reducible critical points is an open condition\(^6\) in perturbations \( \mathcal{P} \), and we’ve just seen that it is also a dense condition.

Now we have reduced ourselves to furthermore showing that perturbations for which the irreducible critical points are nondegenerate are dense in \( \mathcal{P}_{\varepsilon,U} \). This is an application of the Sard-Smale theorem, using that \( \mathcal{P} \) separates points and tangent vectors. (For the reducible case, see [Flo88] and [Don02, Section 5.5.1].) The openness is slightly more delicate, and uses that because nondegenerate critical points are isolated and therefore reducible critical points are isolated for \( \pi \in U \). This implies that the projection \( p^{-1}(U)^* \to U \) is proper, where the asterisk denotes that we restrict to irreducible connections. This is enough to conclude, as in Proposition 3.4.

**Remark 3.1.** There is not much additional difficulty in proving this theorem for \( SO(3) \)-holonomy perturbations, which are invariant under the \( H^1(Y;\mathbb{Z}/2) \) action. However, one needs to do more bookkeeping at the other kinds of reducibles; those with stabilizer \( \mathbb{Z}/2 \) are the most difficult, being labelled by their Euler classes in twisted cohomology.

**Remark 3.2.** In the above proof, we only used the assumption that \( E \) is weakly admissible — that is, either \( b_1(Y) = 0 \) or \( E \) is admissible — to ensure that the reducible flat connections are cut out transversely in the reducible locus. This is not a serious issue: one may easily first choose a perturbation for which the reducible \( \pi \)-flat connections are cut out transversely in the reducible locus. However, we will later that transversality for the moduli spaces of flowlines is much more restrictive, and forces the weak admissibility assumption on us. That in mind, we see no harm in making the assumption a little early.

Note that the critical set of \( cs + f_\pi \) is canonically identified with that in any Hilbert manifold completion of lower regularity by an elliptic regularity argument; we are not being remiss in leaving the regularity \( k \) out from our notation \( \mathcal{C}_\pi \) for the critical sets.

### 3.5. Signature data of reducible critical points.

Suppose \( Y \) is a rational homology sphere. Fix a lift \( c \) of \( w_2(E) \) to an integral cohomology class; then the

\(^6\)Note that 3.4 shows that nondegeneracy in the reducible locus is an open condition, and here we want nondegeneracy in \( \mathcal{B}_E^\pi \); however, the proof requires only trivial modifications.
Lemma 3.8. Let \( \rho \) be a rational homology sphere and fix an abelian representation \( \rho : \pi_1 Y \to U(1) \). If \( Y \) is the universal abelian cover of \( Y \) — the closed 3-manifold corresponding to \( [\pi_1 Y, \pi_1 Y] \) — then we may identify \( H^1(Y; \rho) \) with the subspace of \( H^1(Y; \mathbb{C}) \) on which \( gx = \rho(g)x \) for all \( g \in H_1(Y) \) (the ‘\( \rho \)-eigenspace’). In particular, if \( H^1(Y; \mathbb{C}) = 0 \), then \( H^1(Y; \rho) = 0 \) for all local coefficient systems \( \rho \).

**Proof.** Let \( C_*(Y; \mathbb{C}) \) be the cellular chain complex of some finite CW-decomposition of \( Y \), and \( C_*(Y; \mathbb{C}) \) the cellular chain complex with \( H_1(Y) \)-action. The cohomology groups \( H^*(Y; \rho) \) are (essentially by definition) isomorphic to the homology groups of the cochain complex

\[
\text{Hom}_{H_1(Y)} \left( C_*(Y; \mathbb{C}), \mathbb{C} \right),
\]

where \( H_1(Y) \) acts on \( \mathbb{C} \) via \( \rho \). By definition, this cochain complex is isomorphic to the \( \rho \)-eigenspace of \( C^*(Y; \mathbb{C}) \). Because the chain complex \( C^*(Y; \mathbb{C}) \) splits as a direct sum of its eigenspaces, labelled by homomorphisms \( H_1(Y) \to U(1) \), we see that the cohomology \( H^*(Y; \mathbb{C}) \) splits as a direct sum of the direct sum of its eigenspaces \( H^*_\rho(Y; \mathbb{C}) \), each of which is isomorphic to the homology of the eigenspace \( C^*_\rho(Y; \mathbb{C}) \), which is then isomorphic to \( H^*(Y; \rho) \).

Proposition 3.9. Let \( Y \) be a rational homology sphere equipped with \( SO(3) \)-bundle \( E \). For \( \varepsilon > 0 \) sufficiently small, the set of perturbations \( \pi \) with nondegenerate reducible critical points and \( |\pi| < \varepsilon \) is an open set, which decomposes as the disjoint union of nonempty open subspaces, each labelled by a function

\[
N_\pi : \text{Red}_{SO(2)}(Y, E) \to \mathbb{Z}_{\geq 0}
\]

with \( N_\pi(\{z_1, z_2\}) \leq \dim \mathbb{C} H^1(Y; \eta_1 \otimes \eta_2^{-1}) \).
Proof. That the set of such perturbations \( \pi \) is open is a mild extension of Proposition 3.4.

Given any \( \varepsilon \)-small perturbation \( \pi \), let \( \beta : [0, 1] \to [0, 1] \) be a function equal to 1 near 0 and equal to 0 near 1, and consider the path \( \pi(t) = \beta(t)\pi \) from \( \pi \) to 0; we have \( \|\pi(t)\| < \varepsilon \) for all \( t \). If

\[
\mathcal{r} = \{z_1, z_2\} \in \text{Red}_{SO(2)}(Y, E)
\]

denotes a connected component of the reducible subset of the configuration space (here \( z_1 + z_2 \) is a fixed integral lift of \( \omega_2 E \) and \( z_1 \neq z_2 \)), by Proposition 3.4, there is a unique \( \pi(t) \)-critical orbit in the component labeled by \( r \), and the proof shows it varies smoothly in \( t \). Choose a particular path \( A(t) \) of framed connections so that \( [A(t)] \) lies in the \( \pi(t) \)-critical orbit in the reducible component labelled by \( r \). We obtain an associated continuous path of self-adjoint Fredholm operators

\[
\text{Hess}_{A(t),\pi(t)} : \Omega^0_k \oplus \Omega^1_k \to \Omega^0_{k-1} \oplus \Omega^1_{k-1}.
\]

There is an associated spectral flow, \( \text{sf}(\text{Hess}_{A(t),\pi(t)}) \in \mathbb{Z} \), defined as the intersection number of the eigenvalues with the line \( \lambda = -c \) for sufficiently small positive \( c \); it is essentially the number of eigenvalues that go from negative to nonnegative, counted with sign. (This definition allows for the operators at \( t = 0, 1 \) to have kernel, as opposed to intersecting with the line \( \lambda = 0 \).) In fact, \( \text{Hess}_{A(t),\pi(t)} \) splits as a direct sum corresponding to the reductions \( \mathbb{R} \oplus (\eta_{1} \otimes \eta_{2}^{-1}) \) (here \( c_1(\eta_{1}) = z_1 \)); the operator in the first component is constant, and in the second component is complex linear, so this spectral flow is actually an even integer. The spectral flow of this path depends only on the ending perturbation \( \pi \), so we define

\[
N_{\pi}(r) = \frac{1}{2} \text{sf}(\text{Hess}_{A(t),\pi(t)}).
\]

The kernel of \( \text{Hess}_{A(t),\pi(t)} \) splits as the direct sum of harmonic 0-forms and a space of 1-forms. Note that \( A(t) \) is \( SO(2) \)-reducible for all \( t \) (and never fully reducible), which implies that the dimension of the space of harmonic 0-forms is constant in \( t \), and so does not contribute to the spectral flow.

The spectral flow depends on \( \pi \) continuously (that is, sufficiently close perturbations give the same integer) when \( \text{Hess}_{A(0),\pi} \) has no kernel other than the harmonic 0-forms. To say that \( \text{Hess}_{A(0),\pi} \) has no kernel other than harmonic 0-forms is precisely to say that \( \pi \) is regular at the critical point \( A(0) \). Therefore, the spectral flow to 0 is a continuous function on the open set of interest, the space of regular perturbations.

As long as \( \varepsilon \) is chosen small enough, the only eigenvalues that are close enough to 0 to cross above or below it are the zero eigenvalues in \( \ker \text{Hess}_{A(1)} \), the unperturbed flat connection in this reducible component. The eigenvalues corresponding to harmonic 0-forms stay constant, so we only need to think about the other zero eigenvalues.

As we are counting the number of eigenvalues that go from negative to nonnegative as we pass from \( \text{Hess}_{A(0),\pi} \) to the unperturbed extended Hessian, the spectral flow must be nonnegative (said another way, the only relevant eigenvalues are the zero eigenvalues at \( t = 1 \), and those can only contribute to spectral flow by going down; reversing the direction, the only way to contribute to the spectral flow is to
have an eigenvalue go upwards to 0 as we approach \( t \to 1 \). Because there are
\[
\dim \mathbb{R} H^1(Y; \eta_1 \otimes \eta_2^{-1})
\]
of these eigenvalues, this spectral flow is bounded above by that number.

Thus we have a continuous map from the space of reducible-regular perturbations to the discrete space
\[
\text{Map}(\text{Red}_{SO(2)}(Y,E), \mathbb{Z}_{\geq 0})
\]
satisfying the stated bound; this decomposes the open set of reducible-regular perturbations as the disjoint union of open subsets labelled by these functions
\[
N_\pi : \text{Red}(Y,E) \to \mathbb{Z}_{\geq 0}
\]
with upper bounds as in the statement of the proposition.

Given any function \( N \) satisfying these bounds, we may construct a small perturbation \( \pi \) so that \( N \) arises as the spectral-flow function \( N_\pi \). Choose an \( S^1 \)-invariant quadratic function with \( 2N_\pi(r) \) positive eigenvalues on the space \( \ker \text{Hess}_A \), where \( A \) is the unique critical point in the component labelled by \( r \); note that on any given complex line, this function takes the form \( cr^2 \) where \( r \) is the radius, for some constant \( c \).

Pick a holonomy perturbation that well approximates that quadratic function, thinking of \( \ker \text{Hess}_A \) as an \( S^1 \)-submanifold of \( \tilde{B}_{E,k} \) by exponentiating. Because holonomy perturbations are invariant under the full gauge group, the same quadratic function is introduced at the reducible labelled \( x \cdot r \). Doing this as we vary over \( \text{Red}_{SO(2)}(Y,E) \) gives us a perturbation which restricts to a nondegenerate quadratic function, with the appropriate number of eigenvalues at each reducible critical point. Then if we choose \( c \) sufficiently small, the perturbation \( c \cdot \pi \) will be nondegenerate at each reducible and have \( c \cdot \| \pi \| < \epsilon \).

Note that there is no analogous discussion in the components \( \text{Red}_a(Y,E) \) containing a fully reducible point, because this point is the unique critical point in that component and is nondegenerate for all small \( \pi \): the spectral flow is trivial. ■

Remark 3.3. It is likely the case that the open set labelled by \( N_\pi : \text{Red}_{SO(2)}(Y,E) \to \mathbb{Z}_{\geq 0} \) is connected, but we will not find this necessary, so do not prove it here. In a sense similar to Remark 5.3, the ‘space of regular perturbations with fixed signature data’ is contractible. This will not be true in the 4-dimensional setting.

Remark 3.4. If one works with \( SO(3) \)-holonomy perturbations, which are invariant under \( H^1(Y;\mathbb{Z}/2) \), one needs to do this bookkeeping for additional types of reducibles, each with different demands on the signature data.

The essential count turns out not to be the number of negative eigenvalues, but rather the signature of the corresponding real vector space. As long as we remember the dimension of the underlying vector space, this is equivalent information.

**Definition 3.3.** Let \( (Y,E) \) be an \( SO(3) \)-bundle over a rational homology sphere, and fix a complex line bundle \( \lambda \) with \( w_2(E) = c_1(\lambda) \) (mod 2), which induces a bijection of \( \text{Red}_{SO(2)}(Y,E) \) with pairs \( \{z_1,z_2\} \subset H^2(Y;\mathbb{Z}) \) with \( z_1 + z_2 = c_1(\lambda) \); the choice of \( \lambda \) is essentially irrelevant. We write \( \eta_1 \) and \( \eta_2 \) for the corresponding complex line bundles.

We say that a signature datum on \( (Y,E) \) is a choice of function
\[
\sigma : \text{Red}_{SO(2)}(Y,E) \to 2\mathbb{Z}
\]
with
\[-\dim \mathbb{R}H^1(Y; \eta_1 \otimes \eta_2^{-1}) \leq \sigma_\pi({\{z_1, z_2}\}) \leq \dim \mathbb{R}H^1(Y; \eta_1 \otimes \eta_2^{-1})\]
and
\[\sigma_\pi({\{z_1, z_2}\}) \equiv \dim \mathbb{R}H^1(Y; \eta_1 \otimes \eta_2^{-1}) \pmod{4}.

Given a small regular perturbation \(\pi\), the associated signature datum \(\sigma_\pi\) is
\[\sigma_\pi({\{z_1, z_2\}}) = \dim \mathbb{R}H^1(Y; \eta_1 \otimes \eta_2^{-1}) - 4N_\pi({\{z_1, z_2\}}).

The set of signature data is denoted \(\sigma(Y, E)\).

The name ‘signature datum’ refers to the fact that \(\sigma\) essentially chooses the symmetric bilinear form on each complex vector space \(\ker \text{Hess}_{\pi}^*\) that we perturb in the direction of (which are determined by their signature). The constraints are precisely those that the signature of an \(S^1\)-invariant bilinear form on a complex vector space must satisfy.

Remark 3.5. If the universal abelian cover \(\tilde{Y}\) of \(Y\) (corresponding to the subgroup \([\pi_1 Y, \pi_1 Y] \subset \pi_1 Y\)) is a rational homology sphere, then \(|\sigma(Y, E)| = 1\) for all \(E\). If the cover \(\tilde{Y}'\) corresponding to the subgroup \([\pi_1 Y, \pi_1 Y]^2\) is a rational homology sphere, then \(|\sigma(Y, \text{triv})| = 1\).
4. Moduli spaces of instantons

4.1. Moduli spaces for cylinders and cobordisms. In Definition 2.1, we defined the configuration space of framed connections on a bundle \( \pi^*E \) over a cylinder with specified limits as a quotient of \( \hat{A}_{k,\delta}(\alpha_- , \alpha_+) \). The trivial vector bundle over this configuration space with fiber \( \Omega_{k-1,\delta}^+(\text{End}(\pi^*E)) \) carries a compatible linear action of \( G_{k+1,\delta}^h(A_-, A_+) \); it therefore descends to a vector bundle \( S_{k-1,\delta} \) over the quotient (\( S \) for ‘self-dual’) with fiber isomorphic to \( \Omega_{k-1,\delta}^+ \). The smoothly varying section \( \hat{A} \mapsto F_A^+ \), which is invariant under both the gauge group action on the left and \( SO(3) \) on the right, descends to an \( SO(3) \)-invariant smooth section of \( S_{k-1,\delta} \).

**Definition 4.1.** The (unperturbed) (even) moduli space of framed instantons on the bundle \( \pi^*E \) with limits \( \alpha_- , \alpha_+ \) is the \( SO(3) \)-invariant subspace

\[
\hat{\mathcal{M}}_E(\alpha_- , \alpha_+) \subset \hat{B}_{k,\delta}^e(\alpha_- , \alpha_+),
\]

consisting of gauge equivalence classes of framed connections \( (A, p) \in \hat{A}_{k,\delta}(A_-, A_+) \) with \( F_A^+ = 0 \), where \( A_\pm \) are connections whose gauge equivalence class is \( \alpha_\pm \). Equivalently, it is the zero set of the section \( F_A^+ \) of \( S_{k-1,\delta} \). There are equivariant end-point maps \( \text{ev}_\pm : \hat{\mathcal{M}}_E(\alpha_- , \alpha_+) \to \alpha_\pm \). We denote the quotient \( \hat{\mathcal{M}}_E/SO(3) \) by \( \mathcal{M}_E(\alpha_- , \alpha_+) \).

In our notation for the moduli space, we drop the symbols \( k, \delta, \) and \( e \). The first two are unnecessary, because as long as the perturbation \( \pi \) is of regularity \( L^2 \), the moduli spaces for all \( k \leq j \) large enough that point-evaluation is continuous are canonically identified by an elliptic regularity result; similarly \( \delta \) is irrelevant so long as it is taken sufficiently small that it does not agree with any of the eigenvalues of the limiting operator \( \hat{\text{Hess}}_{\alpha,\pi} \). The even moduli space is not the same as its quotient by the full gauge group (which makes sense when \( \pi \) is invariant under the full gauge group), but we will never work with non-even moduli spaces, so it is an unnecessary label.

Note that if the \( A_k \) are not flat connections, there are no unperturbed instantons with those limits (and \( L^2 \) curvature). If \( \pi \) is a regular perturbation, so that all critical points of \( cs + f_\pi \) are nondegenerate, we instead work with perturbed moduli spaces of instantons; we will need to do this to guarantee these moduli spaces are smooth manifolds.

Write \( \mathcal{P} \) for the Banach space of perturbations of Section 3.1. Given \( \pi \in \mathcal{P} \) corresponding to \( f_\pi : \hat{B}_{k}^e \to \mathbb{R} \), the natural perturbation on the connections \( A \) over \( \mathbb{R} \times Y \) is

\[
(\hat{\nabla}_\pi)(A) = (dt \wedge (\nabla f_\pi)(i_t^* A))^+, \nabla_\pi \text{ is pullback to the slice } \{t\} \times Y.
\]

The operator \( \hat{\nabla}_\pi \) gives a well-defined, smoothly varying section of the trivial \( \Omega_{k-1,\delta}^{2,+}(\text{End}(\pi^*E)) \)-bundle over \( \hat{A}_{k,\delta}(A_-, A_+) \), invariant under the action of \( G_{k+1,\delta}^h(A_-, A_+) \), and the rest of Theorem 3.1 carries over with the obvious modifications (including a version with dependence on \( \pi \)).

In particular, because the above perturbations are invariant under the action of the full 4-dimensional gauge group \( G_{k+1,\delta}^h(A_-, A_+) \), as well as the \( SO(3) \)-action, \( F_A^+ + \hat{\nabla}_\pi(A) \) defines a smooth section

\[
(2) \quad \mathcal{P} \times \hat{B}_{k,\delta}^e(\alpha_- , \alpha_+) \to \mathcal{P} \times S_{k-1,\delta};
\]
that $F_A^+ + \widehat{\nabla}_\pi(A) \in \Omega^{2+}_X(\text{End}(\pi^*E))$ follows from the assumption that $A_t$ decays exponentially on the ends to connections with $*F_{A_t} = -\widehat{\nabla}_\pi(A_t)$.

As in Definition 4.1, we say that the perturbed moduli space of framed instantons for the perturbation $\pi$, written $\mathcal{M}_{E,\pi}(\alpha_- \ldots \alpha_+)$, is the set of gauge equivalence classes of framed connections with $F_A^+ = -\nabla_\pi(A)$. As in the unperturbed case, this is only possible if $*F_{A_t} = -\nabla_\pi(A_t)$, and so the limiting connections are critical points for $cs + f_\pi$.

Using the gradient equation, we can define a useful notion of topological energy for an instanton on the cylinder.

**Proposition 4.1.** For a $\pi$-perturbed instanton $A$, the expression

$$\int \|F_A + *_3\nabla_\pi(A(t))\|^2$$

depends only on the endpoints $\alpha_- \ldots \alpha_+$ of $A$, and the homotopy class $z \in \pi_1(\mathcal{B}_E, \alpha_- \ldots \alpha_+)$ that $A$ traces out; it is equal to $2(\text{cs}_\pi(\alpha_-) - 2\text{cs}_\pi(\alpha_+))$, and we use this to define

$$\mathcal{E}_z^\pi(\alpha_- \ldots \alpha_+) = 2(\text{cs}_\pi(\alpha_-) - \text{cs}_\pi(\alpha_+)),$$

the topological energy of $(\alpha_- \ldots \alpha_+, z)$.

Whenever there is an instanton going from $\alpha_-$ to $\alpha_+$ in the homotopy class $z$, we have $\mathcal{E}_z^\pi(\alpha_- \ldots \alpha_+) \geq 0$ with equality if and only if $A$ is constant. If $z_i$ is a homotopy class from $\alpha_i$ to $\alpha_{i+1}$, then

$$\mathcal{E}_{z_1 \ast z_2}^\pi(\alpha_1, \alpha_3) = \mathcal{E}_{z_1}^\pi(\alpha_1, \alpha_2) + \mathcal{E}_{z_2}^\pi(\alpha_2, \alpha_3).$$

If $1 \in \pi_1(\mathcal{B}_E)$ is the positive generator, then $\mathcal{E}_{z_1 \ast 1}^\pi(\alpha_- \ldots \alpha_+) = 64\pi^2 + \mathcal{E}_z^\pi(\alpha_- \ldots \alpha_+)$.  

**Proof.** The given expression is gauge invariant, so we may assume $A$ is in temporal gauge and write $A = A(t)$. The curvature $F_A = F_{A(t)} + dt \wedge A'(t)$, and so we may write this integral as

$$\int \|A'(t)\|^2 + \|F_{A(t)} + \nabla_\pi(A(t))\|^2 dt.$$

Because $F_A + dt \wedge \nabla_\pi(A(t)) = F_A + *(3\nabla_\pi(A(t)))$ is anti-self-dual, and $*(dt \wedge \omega) = *_3\omega$ for a 1-form $\omega$ pulled back from $Y$, we see that $A'(t) = -*_3 F_{A(t)} - \nabla_\pi(A(t)) = -\nabla(\text{cs} + f_\pi)(A(t))$. Using that $A(t)$ is a gradient flowline, we may rewrite the integral as

$$\int \|A'(t) + \nabla(\text{cs} + f_\pi)(A(t))\|^2 - 2\langle A'(t), \nabla(\text{cs} + f_\pi)(A(t)) \rangle dt$$

$$= -2 \int \langle A'(t), \nabla(\text{cs} + f_\pi)(A(t)) \rangle dt$$

$$= -2 \int d(\text{cs} + f_\pi)(A'(t))$$

$$= -2 \left((\text{cs} + f_\pi)(A(\alpha)) - (\text{cs} + f_\pi)(A(-\alpha))\right) \in \mathbb{R},$$

the desired result. Now it is clear that this only depends on the connected component of the connection $A$. (Here we are using the path to pin down the real lift of this difference of Chern-Simons values.)

That $\mathcal{E}_0^\pi(\alpha, \alpha) = 0$ is clear from the existence of the constant trajectory. Additivity is clear by picking a connections $A$, for each that are constant sufficiently
far down the cylindrical ends, and gluing those together; the difference in \( cs + f_\pi \) is additive.

Lastly, we need to determine \( E_1^c(\alpha, \alpha) \); by definition, this is the same as determining \( 2(cs + f_\pi)(g(A)) - 2(cs + f_\pi)(A) \) for a gauge transformation \( g \) generating \( \pi_0 G_E^c \). Because \( f_\pi \) is invariant under the full gauge group, we are only asking to determine the difference in the Chern-Simons functional. Pick a trajectory \( A \) going from \( A \) to \( g(A) \), constant near the ends. By definition, this is given by computing \( 2 \int_{S^1 \times Y} \text{Tr}(F_A^2) \). Because \( A \) differs from \( g(A) \) by a gauge transformation, we may glue these together to get a connection on an \( SO(3) \)-bundle \( E \) over \( S^1 \times Y \), the bundle \( E \) being the mapping torus of the automorphism \( g \); noting that \( F_A^2 \) is zero near the ends, this implies we may compute the curvature integral just as well over this closed 4-manifold, where it is equal to \( -2 \cdot 8\pi^2 p_1(E) \). We choose the positive generator of \( \pi_1(\tilde{\mathcal{B}}_E) \) to be the one that makes this integral positive; it remains to compute it. This is computed more generally in [KM11b, Equation (24)], and in this case the minimal \( p_1(E) \) is 4, as desired. Because of the slight difference that we use specifically even gauge transformations, we provide a short proof, given in the lemma that follows.

\[ \text{Lemma 4.2.} \] Let \( E \) be an \( SO(3) \)-bundle over a \( 3 \)-manifold \( Y \). Given an even gauge transformation \( g \in \Gamma(\text{Aut}(E)) \), the minimal first Pontryagin class of the mapping torus of \( g \) on \( S^1 \times Y \) is 4.

\[ \text{Proof.} \] The bundle \( E \) has \( w_2 E \in H^2(S^1 \times Y; \mathbb{Z}/2) \cong H^2(Y; \mathbb{Z}/2) \oplus H^1(Y; \mathbb{Z}/2) \). The first factor of \( w_2 E \) is precisely the second Stiefel-Whitney class of its restriction to \( Y \), which is \( w_2 E \). The second term of \( w_2 E \) is precisely the obstruction class \( o(g) \in H^1(Y; \mathbb{Z}/2) \) for lifting a section of \( \text{Aut}(E) \) to a section of \( \text{Aut}(E) \); to see this, pick a loop \( \gamma \) in \( Y \) and consider the corresponding torus in \( S^1 \times Y \). The map \( \gamma \mapsto w_2(\gamma)|_{S^1 \times \gamma} \) defines a homomorphism \( H_1(Y; \mathbb{Z}) \to \mathbb{Z}/2 \). If \( o(g) = 0 \), then the restriction of \( g \) to \( \gamma \) lifts to \( \tilde{\text{Aut}}(E) \), and so is homotopic to the identity (because \( \pi_1(\tilde{\text{Aut}}(E)) = 0 \)), so the corresponding bundle on the torus \( S^1 \times \gamma \) is trivial.

Conversely, every bundle over \( S^1 \times Y \) may be constructed as the mapping torus of a bundle over \( Y \), and both \( w_2(E) \) and \( o(g) \) of the bundle and mapping are determined by \( w_2 E \). Given fixed \( w_2(E) = w_2(E) \oplus o(g) \), the Dold-Whitney theorem identifies the possible values of \( p_1(E) \) as those elements of \( H^4(S^1 \times Y; \mathbb{Z}) \) with \( p_1(E) \equiv (w_2 E)^2 \) (mod 4), where the mod 4 indicates we use the Pontryagin square on even-dimensional cohomology \( H^2(X; \mathbb{Z}/2) \to H^4(X; \mathbb{Z}/4) \). Expanding this, we obtain \( p_1(E) = [S^1] \cdot 2o(g) \cdot w_2(E) \in H^4(S^1 \times Y; \mathbb{Z}/4) \). In particular, as long as \( o = 0 \), the only condition is \( p_1(E) \equiv 0 \) (mod 4).

\[ \text{Corollary 4.3.} \] Given a \( 3 \)-manifold \( Y \) equipped with \( SO(3) \)-bundle \( E \) and a regular perturbation \( \pi \), if \( \alpha \) is a \( \pi \)-critical orbit, the constant connection is the unique \( \pi \)-perturbed instanton \( A \) in the component labelled by the trivial homotopy class \( 0 \in \pi_1(\tilde{\mathcal{B}}_E, \alpha) \).

The constant connection is always a solution to the perturbed ASD equations. Because \( E_1^c(\alpha, \alpha) = 0 \), and the only connections with energy equal to zero are constant, so is any solution of the perturbed ASD equations.

For a Riemannian manifold \( W \), with two cylindrical ends, orientably isometric to \( (-\infty, 0] \times Y_1 \) and \( [0, \infty) \times Y_2 \), written as \( W : Y_1 \to Y_2 \), we can no longer consider
constant perturbations (pulled back from $Y$). Further, we need to be able to interpolate between perturbations, and have enough available to prove transversality results. As before, the precise formulation of the perturbations here is similar to that in [KM11b, Section 3.2].

Fix once and for all a smooth function $\beta_0 : [0, \infty) \to [0, 1]$ with $\beta_0(x) = 0$ for $x$ near 0 and $\beta_0(x) = 1$ for $x \geq 1$.

Let $\beta$ be a smooth bump function $[0, \infty) \to [0, 1]$ with support contained in $(1, 2)$. Given a perturbation $\pi \in P_{E_2}$, we may define the associated 4-dimensional perturbation $\hat{\nabla}_{\pi, \beta}$ on $[0, \infty) \times Y_2$ as

$$(\hat{\nabla}_{\pi, \beta})(A) = (dt \wedge \beta(t)\nabla_\pi(A(t)))^+,$$

and identically zero elsewhere, and similarly for a perturbation in $P_{E_1}$. Choose a countable family of bump functions $\beta_i$ so that for each rational $q$ and length $\ell > 0$, there is some $\beta_i$ with $\beta_i(q) = 1$ and whose support is contained in an interval of length $\ell$. Taking weighted $L^1$ sums of these (with weights growing sufficiently quickly) along with the same countable set of holonomy perturbations $\pi$, as we did below Theorem 3.1, we obtain 4-dimensional spaces of perturbations $P_{E_i}^{(4)}$; we demand that the $K_i$ grow so quickly that corresponding map $\pi(t) : [1, 2] \to \hat{P}_{E_i}$ is smooth.

We will write $\nabla_\pi : \hat{A}_{E_i} \to \Omega^{1}_{k-1}(Y_i; \mathfrak{g}_{E_i})$ for the map given by the appropriate sum of $\beta_i(t)\nabla_\pi$, corresponding to $\pi \in P_{E_i}^{(4)}$, and still denote the section $\hat{A}_{E_i}^{(4)} \to \Omega^{2, +}_{k-1, \delta}$ as $\nabla_\pi$.

An end perturbation on $W$ is labelled by is

$$(\pi_{-}, \pi_{1}, \pi_{2}, \pi_{+}) \in \mathcal{P}_{E_1} \times \mathcal{P}_{\mathcal{Y}_1}^{(4)} \times \mathcal{P}_{\mathcal{Y}_2}^{(4)} \times \mathcal{P}_{E_2} =: \mathcal{P}^{(4)}.$$

The terms $\pi_{\pm}$ denote the constant perturbations on the ends, and the $\pi_{i}$ are perturbing terms on a compact piece of the end, as above. (While $\mathcal{P}^{(4)}$ depends on $W$, this dependence is implicit and should be clear from context.)

Precisely, if $\pi = (\pi_{-}, \pi_{1}, \pi_{2}, \pi_{+})$, write the corresponding element $\hat{\nabla}_\pi(A)$ on the end $(-\infty, 0] \times Y_1$ as

$$(\hat{\nabla}_{\pi_{-}})(A) = (dt \wedge \beta_0(-t)\nabla_{\pi_{-}}(A(t)))^+ + \hat{\nabla}_{\pi_1}(A(t)),$$

on the end $[0, \infty) \times Y_2$ as

$$(\hat{\nabla}_{\pi_{+}})(A) = (dt \wedge \beta_0(t)\nabla_{\pi_{+}}(A(t)))^+ + \hat{\nabla}_{\pi_2}(A(t)),$$

and identically zero elsewhere.

We will frequently have fixed regular perturbations $\pi_i$ on the 3-manifold $Y_i$, and be interested in varying only the compactly supported end perturbations. We denote this space of compactly supported perturbations as $\mathcal{P}_{c}^{(4)}$. One may either think of this as an affine space that depends on the implicit, fixed $\pi_i$, or the literal vector space that they are affine over. In any case, we do not use the $\pi_i$ in our notation, and it will be clear from context what is meant.)

These end perturbations are essentially the same as the perturbations used in [KM11b], though we allow the bump function weighting the compactly supported perturbations to vary (in order to make some technical arguments easier).

We will need one further type of perturbation, which is mainly useful to achieve transversality at flat connections. This is similar to the notion of holonomy perturbations along thickened loops in [D'87] and [Fre04].
Definition 4.2. Let $W$ be a compact Riemannian manifold with boundary and $W^\circ$ its interior. A collection of thickened loops is a finite set of embeddings $\gamma_i : S^1 \to W^\circ$, equipped with small positive real numbers $\epsilon_i$ so that the $\epsilon_i$-neighborhoods of $\gamma_i(S^1)$ are disc bundles entirely contained inside $W^\circ$. We further choose a lift of $E|_{\gamma_i}$ to an $SU(2)$-bundle.

We write the corresponding $\epsilon$-sphere bundle as $S_\epsilon(i)$. For some small $\delta_i$, we have that the exponential map gives a diffeomorphism from a neighborhood of $S_\epsilon(i)(\epsilon_i/2)$ to $(-\delta_i, \delta_i) \times S_\epsilon(i)(\epsilon_i/2)$, along which we may write the metric as $dt^2 + g_t$, where $g_t$ is a continuously varying family of metrics. The given lift of $E$ along $\gamma_i$ naturally induces one on $D_\epsilon(i)$ and hence $S_\epsilon(i)(\epsilon_i/2)$.

Once and for all fix a smooth bump function $\beta_2 : (-1,1) \to [0,1]$ which is 1 near 0 and 0 near $-1,1$. We then have corresponding bump functions $\beta_\epsilon : (-\epsilon, \epsilon) \to [0,1]$ given by $\beta_\epsilon(x) = \beta(x/\epsilon)$. By abuse of notation we continue to denote these as $\beta_2$.

Given an $SU(2)$-holonomy perturbation on $S_\epsilon(i)(\epsilon_i/2)$, we may construct a gauge-invariant map $A(W) \to \Omega^{2,+}(W)$ by sending $A \mapsto \beta_2(t) \left( dt \wedge \nabla_{g_{\beta_2(t)}}^A \right)^+$ on the neighborhood diffeomorphic to $(-\delta_i, \delta_i) \times S_\epsilon(i)(\epsilon_i/2)$, and zero elsewhere. This is called an interior holonomy perturbation, and we say that this is adapted to the thickened loop $(\gamma_i, \epsilon_i)$.

Given a collection of thickened loops, and finitely many adapted perturbations to each, we say that the resulting finite set of interior holonomy perturbations on $A(W)$ is adapted to the collection of thickened loops.

Unlike the usual setup for perturbation spaces, we do not have a single infinite-dimensional space in which all of our perturbations lie. This is by design. If we took sums of holonomy perturbations along intersecting loops, the value at the intersection points of the corresponding $(-\delta, \delta)$ depends on the value on both of the two factors. However, we will later need to argue that solutions to the perturbed equations (and their linearizations, and their duals) have a unique continuation property, and here it is crucial that on these specified neighborhoods the perturbed ASD equations take the form of an ODE; but if two of our neighborhoods overlapped, this is simply not true.

A fixed finite collection of interior holonomy perturbation adapted to a collection of thickened loops will be denoted $L$; the corresponding free vector space is denoted $V_L$, and carries a linear map to the space of smooth, $G$-invariant sections $A(W) \to \Omega^{2,+}$. Once we have fixed $L$, we write $P_L^{(4)} = P^{(4)} \oplus V_L$ for the space of all holonomy perturbations on $W$ so that those supported in the complement of the ends are contained linear combinations of those in $L$, and similarly $P_L^{(4)} \cdot c = P^{(4)}_c \oplus V_L$ for the space of compactly supported perturbations on $W$, including the perturbations $V_L$ on the interior.

These perturbations satisfy the following properties, analogous to those of [KM11b, Proposition 3.7] or [Lin18, Definition 3.2.7].

Proposition 4.4. Suppose $\pi_\pm$ are 3-manifold perturbations on $(Y_\pm)$, with fixed critical orbits $\alpha_\pm$. 
The 4-dimensional perturbation on the Riemannian manifold $W$ with cylindrical ends defined above are maps

$$\widehat{\nabla}_\pi : A_{E,k,\delta}(\alpha_-, \alpha_+) \to \Omega_{E,k,\delta}^2(W; g_E)$$

such that:

1. $\widehat{\nabla}_\pi$ is smooth in $A$, and $D_A \widehat{\nabla}_\pi$ defines a smooth section of $\text{Hom}(T_j, T_j)$ for any $j \leq m$.

2. $\|D_A \widehat{\nabla}_\pi\|_{L^1} \leq K$, independent of $A$.

3. $\|D_A \widehat{\nabla}_\pi\| \leq K'$ independent of $A$, where here we are taking the operator norm with respect to the $L^2_{k,\delta}$ norm.

4. If $A$ and $A'$ agree on the collars of $W$ and on each $(-\delta, \delta) \times S(\epsilon_i/2)$, then $\widehat{\nabla}_\pi(A) = \widehat{\nabla}_\pi(A')$.

5. Suppose $\sigma$ is a gauge transformation of $E$, defined away from some finite set of points; and suppose $A_i$ are globally defined connections with $\sigma^* A_1 = A_2$ where defined. Then

$$\widehat{\nabla}_\pi(A_1) = \widehat{\nabla}_\pi(A_2).$$

The proofs are straightforward modifications of those for the 3-dimensional case; the only new statement here is the last one, and this follows because on the ends, $(\nabla_\pi)(A(t))$ defines a continuous (in fact smooth) map $f(A(t)) : [1, 2] \to \Omega^1(Y_i)$ so that $f(A(t))$ only depends on the gauge equivalence class of $A(t)$. Because there are only finitely many slices for which $\sigma$ is not globally defined, we see that $f(A(t)) = f(A'(t))$ for all but finitely many $t$; because these are continuous functions, they agree everywhere. An essentially identical argument applies for interior holonomy perturbations.

**Definition 4.3.** In this section, and in what follows, we used the following notation for Banach spaces of perturbations.

- On a 3-manifold $Y$ equipped with $SO(3)$-bundle $E$, the space $P_E$ given by weighted $L^1$ sums of cylinder functions.

- On a compact interval $[1, 2] \times Y$ inside the cylinder $\mathbb{R} \times Y$, the space $P_E^{(4)}$ of weighted $L^1$ sums of

$$\beta \cdot (dt \wedge \nabla_\pi(A(t)))^+,$$

for $\beta$ one of a fixed countable set of bump functions supported in $(1, 2)$ so that for any given point in the interval, some bump function has arbitrarily small support and has $\beta = 1$ there,

- On a 4-manifold $(W, E)$, with cylindrical ends $(Y_1, E_1)$ and $(Y_2, E_2)$, the space

$$P^{(4)} = P_{E_1} \oplus P_{E_2}^{(4)} \oplus P_{E_1}^{(4)} \oplus P_{E_2},$$

where the pieces $P_{E_i}^{(4)}$ are supported in $[1, 2] \times Y_i$ in the appropriate end, and the spaces $P_{E_i}$ contribute a perturbation of the form

$$\beta_0(t) (dt \wedge \nabla_\pi(A(t)))^+$$
on the end, where \( \beta_0 \) is a cutoff function supported in \((0, \infty)\) and 1 near \( \infty \) (with \( W \) left implicit in the notation \( P^{(4)} \)).

- The affine subspace \( P_c^{(4)} \) of perturbations with \((\pi_-, \pi_+) \in P_{E_1} \oplus P_{E_2} \) fixed (but left implicit in notation),

- The finite-dimensional space \( V_L \) free on a finite set of holonomy perturbations adapted to a collection of thickened loops,

- The spaces \( P^{(4)}_L = P^{(4)} \times V_L \) and \( P^{(4)}_{L,c} = P^{(4)}_c \times V_L \).

Fix \( \epsilon > 0 \) so that for any reducible flat connection \( \alpha \), the only eigenvalues of \( \text{Hess}_{\alpha,0}^{\nu} \) with absolute value at most \( \epsilon \) are the elements of the kernel; one may fix a neighborhood \( 0 \in U_\epsilon \subset \mathcal{P}_Y \) so that this remains true for all \( \pi \in U_\epsilon \) and any reducible \( \pi \)-flat connection \( \alpha \).

so that \( \dim \text{Eig}_{\pi,\epsilon}(\text{Hess}_{\alpha,\pi}^{\nu}) = 0 \) for all \( \pi \in U_\epsilon \) and \( \alpha \) a \( \pi \)-flat connection. In particular, this means that

\[
\text{dim} \text{Eig}_{\pi,\epsilon,\lambda}(\text{Hess}_{\alpha,\pi}^{\nu}) = \dim \ker \text{Hess}_{\alpha,0}^{\nu}
\]

for all such \((\pi, \alpha)\).

If we let \( \delta \) be a constant \( 0 < \delta \ll \epsilon \), then we denote by \( P_{E,\delta} \subset U_\epsilon \) the subset of those \( \pi \in U_\epsilon \) so that \( \text{Hess}_{\alpha,\pi}^{\nu} \) has no eigenvalues of absolute value at most \( \delta \), for any \( \pi \)-flat connection \( \alpha \) (not necessarily reducible). This subset does not include 0, but by Theorem 4.34, \( \bigcup_{\delta \to 0} P_{E,\delta} \) is dense in \( U_\epsilon \). There are corresponding open subspaces \( P^{(4)}_\delta \) and \( P_{L,\delta}^{(4)} \) for which the perturbations on the 3-manifold at \( \infty \) lie in \( P_{E,\delta} \). These spaces will be important for the later weighted Sobolev theory.

So long as \( \delta \) is sufficiently small, there is a union of components of \( P_{E,\delta} \) so that the enumeration of reducible flat connections of Proposition 3.4 holds; this follows because modifying the count of reducible flat connections would require one of them to become critical, and for \( \delta \) sufficiently small, there is some \( \pi \in P_{E,\delta} \) arbitrarily close to zero. Additionally, we may decompose \( P_{E,\delta} \) into connected open sets depending on the spectral flow of \( \text{Hess}_{\alpha,\pi}^{\nu} \) for a generic path \( 0 \to \pi \in P_{E,\delta} \) at the reducible \( \pi \)-flat connections \( \alpha \), precisely as in Proposition 3.9.

We still have a notion of analytic energy of a configuration on \( W \), but it is no longer a topological invariant, and will mostly be useful in keeping track of what happens during compactification.

**Definition 4.4.** Let \( W \) be a Riemannian manifold with cylindrical ends, equipped with an \( \text{SO}(3) \)-bundle and a perturbation \( \pi = (\pi_-, \pi_1, \pi_2, \pi_+) \) as above. Let \( \alpha_\pm \) be critical orbits with respect to \( \pi_\pm \). The analytic energy of a connection \( A \in \overline{\mathcal{B}}_{E,k,\delta}(\alpha_-, \alpha_+) \) is defined to be

\[
\mathcal{E}_{\text{an}}^{\pi}(A) = \mathcal{E}_{\text{top}}^{\pi}(A) + \int \| F_A^+ + \nabla_\pi(A) \|^2.
\]

This agrees with \( \mathcal{E}_{\text{top}}^{\pi}(A) \) for \( \pi \)-ASD connections, and agrees with \( \int \| F_A + s_\lambda \nabla_\pi(A(t)) \|^2 \) for connections on the cylinder with constant perturbation.

**4.2. Linear analysis and index theory.** Throughout this section, a perturbation \( \pi \) on \( W \) is fixed so that the limiting perturbations \( \pi_i \) on \((Y_i, E_i)\) have finitely
many critical orbits in $\mathcal{C}_\pi$, all nondegenerate and such that each $\text{Hess}^\nu_{\alpha,\pi}$ has all eigenvalues of absolute value larger than a fixed constant $\delta$ (independent of $\alpha$).

We fix two such critical orbits, $\alpha_{\pm} = [A_{\pm}, E_b]$. Here recall that the elements of $\alpha_{\pm}$ vary over gauge equivalence classes of pairs $(A_{\pm}, p)$, where $p \in E_b$ is a point in the fiber of $E$ above the basepoint $b$, thought of as a framing of $E$ above that point.

To analyze the local structure of the moduli spaces around a framed instanton $(A, p)$ on $\mathbb{R} \times Y$ framed above $(0, b)$, we should restrict to the framed Coulomb slice $(A + \ker(d^*_A)) \times (\pi^*E)_b \subset \tilde{A}_{k,\delta} (A_-, A_+)$: this is just the usual Coulomb slice with an additional $SO(3)$ coordinate for the framing. Every framed connection sufficiently close to $(A, p)$ is gauge equivalent to one in the framed Coulomb slice, and the representation is unique up to the action of $\Gamma_A \rightarrow \text{Aut}(E_b) \cong SO(3)$ (the last isomorphism depending on the choice of framing at the basepoint). This acts trivially on the connection coordinate, and by translation in the framing coordinate.

We also find it convenient to pass to a smaller $\Gamma_A$-invariant open set in the framing coordinate, so that our subset is identified with

$$
(\pi, A + a, \omega \xi_p) \mapsto u (d_A a + a \wedge a)^+ + u \left( \hat{\nabla}_\pi (A + a) - \hat{\nabla}_\pi (A) \right).
$$

We can thus identify the derivative of the perturbed map (for fixed $\pi$ and at $(A, p)$ a $\pi$-perturbed framed instanton) $\tilde{K}^{(4)}_{k,\delta} \rightarrow \Omega^{2,\delta}_k$ as the perturbed ASD operator,

$$
d_A^\nu a + (D_A \nabla_\pi)(a \otimes 0 : k^{(4)}_{k,\delta} \oplus \mathfrak{g}_A^1 \rightarrow \Omega^{2,\delta}_k;
$$

we write this as $D_{A,\pi}$, and the corresponding perturbed normal ASD operator as

$$
D_{A,\pi}^\nu : K^{(4)}_{k,\delta} \rightarrow \Omega^{2,\delta}_k.
$$

Just as in §3.2, we gain control over the normal ASD operator with domain $K^{(4)}_{k,\delta}$ by expressing it as a summand of a larger elliptic operator and including a gauge fixing condition. Define $Q_{A,\pi}$, the perturbed extended ASD operator, by

$$
Q_{A,\pi} : \Omega^1_{k,\delta} (\pi^* \mathfrak{g}_E) \rightarrow \Omega^0_{k-1,\delta} (\pi^* \mathfrak{g}_E) \oplus \Omega^{2,\delta}_k (\pi^* \mathfrak{g}_E),
$$

given by

$$
Q_{A,\pi} (a, \xi) = (d_A^\nu a, d_A^\nu a + D_A \hat{\nabla}_\pi (a)).
$$

This is the operator usually used in the instanton theory for integer homology spheres, as applied to weighted Sobolev spaces.
Now recall that we have an $L^2$-orthogonal splitting $\Omega^1_{k, \delta}(\mathcal{g}_E) = \text{Im}(d_A) \oplus \ker(d_A^*)$; that every element may be written uniquely as a sum in this way is essentially the statement that the map $\Delta_A : \Omega^0_{k+1, \delta}(\mathcal{g}_E)/\mathcal{g}_A \to \Omega^0_{k-1, \delta}(\mathcal{g}_E)/\mathcal{g}_A$ is an isomorphism, assuming $\delta$ is not an eigenvalue of the Laplacian of the limiting connections, which is true as long as $\delta$ is chosen sufficiently small.

Now in this splitting $\Omega^1 = \text{Im}(d_A) \oplus \ker(d_A^*)$, and rewriting the first term as $\Omega^0_{k+1, \delta}/\mathcal{g}_A$ under the isomorphism $d_A^*$, we may write

$$Q_{A, \pi} = \begin{pmatrix} \Delta_A & 0 \\ 0 & D_{A, \pi}^{\nu} \end{pmatrix}. $$

The fact that $D_{A, \pi}^{\nu}$ takes values in $\ker(d_A^*)$ is the linearization of the statement that $F_A^+ + \tilde{\nabla}_\pi(A)$ is a gauge invariant quantity. In particular, the normal ASD operator is Fredholm, and because the index of the top left operator is $-\dim \mathcal{g}_A$, the index of the normal ASD operator $D_{A, \pi}^{\nu}$ is the index of $Q_{A, \pi}$ plus the dimension of $\mathcal{g}_A$.

We recall the basic Fredholm property from [Don02, Sections 3.2 and 3.3.1].

**Proposition 4.5.** Suppose $W$ is a Riemannian 4-manifold with cylindrical ends, isometric to $(-\infty, 0] \times Y_1$ and $[0, \infty) \times Y_2$, equipped with an SO(3)-bundle $E$ restricting to the pullbacks of fixed SO(3)-bundles $E_i$ on the ends. In this situation, we say that $(W, E)$ is a cobordism from $(Y_1, E_1)$ to $(Y_2, E_2)$. Suppose $\pi$ is a fixed perturbation on $E$, restricting to regular perturbations $\pi_i$ on the ends, and $A_i$ are fixed nondegenerate critical points of $c\text{S}_{Y_i} + f_{\pi_i}$.

Let $A$ be a choice of connection in $\mathcal{A}_{E, k, \delta}(A_1, A_2)$.

If $\delta > 0$ is less than the absolute value of any eigenvalue of $\tilde{\text{Hess}}_{A_i, \pi}^{\nu}$, then $Q_{A, \pi}$ is Fredholm, and has index independent of such $\delta$. If the $A_i$ are irreducible, we may even take $\delta = 0$.

There are two essential points. The first is that when a connection $A = A(t)$ on the cylinder is in temporal gauge, the operator $Q_{A, \pi}$ can be written in the form

$$Q_{A, \pi} = \frac{d}{dt} + \tilde{\text{Hess}}_{A(t), \pi}. $$

(See [Don02, Section 2.5].) Secondly, if $e^{\sigma(t)} = f_\delta(t)$ is the function used to define the weighted Sobolev spaces, multiplication by $f_\delta$ is an isometry $L^2_{k, \delta} \to L^2_{k-1, \delta}$, and a first order linear differential operator $D$ of the form $\frac{d}{dt} + L_\delta$ is taken under this isometry to $\frac{d}{dt} + L_t - \sigma'(t)$. So to study our operator $Q_{A, \pi}$ on weighted Sobolev spaces, we should equivalently study $\frac{d}{dt} + \tilde{\text{Hess}}_{A(t), \pi} - \sigma'(t)$ where $\sigma(t) = -\delta t$ for $t < 0$ and $\sigma(t) = \delta t$ for $t > 0$. (This is well-explained in [Don02, Section 3.3.1] and the beginning of [Lin18, Section 3.3].) Once we’ve done this, the above result for the ASD operator is a consequence of general theory; it is proven for $L_t$ a family of almost self-adjoint first order differential operators as [KM07, Proposition 14.2.1].

An important consequence of this description is that for $A = A(t)$ in temporal gauge, we can describe the index of $Q_{A, \pi}$ as the spectral flow of the family of operators $\tilde{\text{Hess}}_{A(t), \pi} - \sigma'(t)$ between $\tilde{\text{Hess}}_{A_1, \pi} + \delta I$ and $\tilde{\text{Hess}}_{A_2, \pi} - \delta I$, the (algebraic) intersection number of the paths the eigenvalues take with $0 \in \mathbb{R}$.

Because each $\mathcal{A}_{E, k, \delta, z}(A_1, A_2)$ is connected (even contractible) and index is a homotopy invariant of self-adjoint Fredholm operators, the index of $Q_{A, \pi}$ only depends on the homotopy class $z$, not the actual choice of connection $A$. For the
same reason, this also agrees with the index of $Q_{A,\nu}^A$, for any other perturbation which restricts to the same perturbation sufficiently far on the ends.

It is perhaps worth observing that the index of $D_{A,\nu}^\alpha$, defined to be the derivative of the section operator $\hat{B}_{E,k,\delta}^A \to S_{k-1,\delta}$ normal to an orbit, does have its index jump as we pass from irreducibles to reducibles. This makes sense, thinking of $D^\nu$ as measuring the expected codimension of the orbit through $A$ in the entire moduli space: the codimension is larger at smaller-dimensional orbits.

Following the definition after [KM11b, Lemma 3.13], we use this to give the following definition.

**Definition 4.5.** In the situation of Proposition 4.5, let $z$ denote a connected component of $A_{E,k,\delta}(\alpha, \beta)$. We write the unframed grading $\text{gr}^W_z(\alpha, \beta) = \text{ind}(Q_{A,\nu}^z)$ for any choice of $A$ in the component $z$, and the relative grading between the orbits $\alpha, \beta$, with respect to the path $z$, is $\text{gr}_z(\alpha, \beta) = \overline{\text{gr}}_z(\alpha, \beta) + 3 - \dim \alpha$.

When $W$ is the cylinder with the constant perturbation, we drop the superscript $W$.

The relative grading here will be quite natural in the definition of the framed instanton Floer complex, whose differential is defined in terms of fiber products with moduli spaces; $\text{gr}^e_{z}(\alpha, \beta)$ is the expected dimension of a fiber of the map

$$\text{ev} : \hat{M}_{E, \pi, z}(\alpha, \beta) \to \alpha.$$ 

It is worth noting here that $3 - \dim \alpha = \dim \mathfrak{g}_\alpha = \dim \Gamma_\alpha$.

The more immediate point of this definition is that the relative grading is additive. In the particular case that $W$ is isometric to the cylinder $\mathbb{R} \times Y$, $z$ corresponds to a relative homotopy class between $[\alpha]$ and $[\beta]$ in $B_{E,k}^\nu$. If $w$ is a path from $[\beta]$ to $[\gamma]$, then

$$\text{gr}_z(\alpha, \beta) + \text{gr}_w(\beta, \gamma) = \text{gr}_{z \cdot w}(\alpha, \gamma)$$

($z \cdot w$ the concatenated path). This follows by computing these as a spectral flow: if $A_z(t) = A_z \in A_{E,\delta,\pi}(\alpha, \beta)$ is in temporal gauge and $A_z(t) = \alpha$ (resp. $\beta$) for $t \to 0$ (resp $t \to 0$), the index of $Q_{A_z,\pi}$ is the spectral flow of the path $\text{Hess}_{A_z(t),\pi} - \sigma(t)$, where $\sigma'(t) = -\delta$ for $t \to 0$ and $\sigma'(t) = \delta$ for $t \to 0$. Making a similar choice of $A_w$, if we glue together $A_z$ and $A_w$ sufficiently far out on the ends, we can find a connection $A_{z \cdot w}$ in the component corresponding to $z \cdot w$ as the concatenation of the paths $A_z(t)$ and $A_w(t)$. However, we cannot concatenate the corresponding paths of self-adjoint operators yet; for large $t$, the first ends at $\text{Hess}_{\beta,\pi} - \delta$ and the second begins at $\text{Hess}_{\beta,\pi} + \delta$. To actually concatenate them, we must traverse the path

$$\text{Hess}_{\beta,\pi} - (1 - 2t)\delta;$$

doing so changes $(3 - \dim \beta) = \dim \ker(\text{Hess}_{\beta,\pi})$ negative eigenvalues to positive, and so

$$\text{gr}_z(\alpha, \beta) + (3 - \dim \beta) + \text{gr}_w(\beta, \gamma) = \text{gr}_{z \cdot w}(\alpha, \gamma).$$

Additivity of this grading for general cobordisms is also true. We record this as a proposition.
Proposition 4.6. Suppose we have cobordisms \((W_1, E_1)\) from \((Y_1, E_1)\) to \((Y_2, E_2)\) and \((W_2, E_2)\) from \((Y_2, E_2)\) to \((Y_3, E_3)\), equipped with paths \(\gamma_1 : \mathbb{R} \to W_1\) and \(\gamma_2 : \mathbb{R} \to W_2\) between the basepoints of the \(Y_i\). Suppose the \(W_i\) are equipped with perturbations \(\hat{\pi}_i\), which restrict to regular perturbations \(\pi_j\) on each \(Y_j\), and furthermore suppose each \((Y_j, E_j, \pi_j)\) is equipped with some \(\pi_j\)-flat connection \(A_j\).

We can define the composed cobordism \((W_{12}, E_{12})\) from \((Y_1, E_1)\) to \((Y_3, E_3)\), identifying \((T, x) \times Y_2\) with \((-\infty, -T) \times Y_2\) on the cylindrical ends; there is a corresponding perturbation \(\hat{\pi}\) interpolating between \(\pi_1\) and \(\pi_3\). We denote by \(z_i\) a component of \(\hat{A}_{E_{12}}(A_1, A_{i+1})\); there is a component \(z_1 * z_2\) corresponding to gluing representative connections of these components along the ends. In this situation, we have

\[
g^{W_1}(\alpha_1, \alpha_2) + g^{W_2}(\alpha_2, \alpha_3) = g^{W_{12}}(\alpha_1, \alpha_3).
\]

This follows from the additivity theorem of the index when the limiting operators over the ends have no kernel ([Don02, Proposition 3.9]) and the relation of operators on weighted Sobolev spaces to unweighted spaces, given by conjugating by the weighting function: the operator \(Q_{A_1, \pi_1}\) on its positive end, after conjugating by the weighting function \(e^{\sigma}\), takes the form \(\frac{1}{m} + \Hess_{A_2, \pi_2} - \delta\) and the form \(\frac{1}{2} + \Hess_{A_2, \pi_2} + \delta\) on the negative end of \(W_2\); to glue these we first need to interpolate between \(\delta\) and \(\delta\), moving the \((3 - \dim \alpha_2)\) as \(\dim \ker(\Hess_{A_2, \pi_2})\) negative eigenvalues across 0; this is observed as [Don02, Proposition 3.10], identifying the index on weighted spaces with what he denotes \(\text{ind}^+(P)\).

4.3. Uhlenbeck compactness for framed instantons. The following definition is precisely [KM07, Definition 16.1.1]. Note that there are no framings involved yet.

Definition 4.6. Let \(Y\) be a Riemannian 3-manifold, equipped with \(SO(3)\)-bundle \(E\) and regular perturbation \(\pi\); then \(\pi\) has finitely many critical points in \(B_{E, k}\), which we write a generic point of as \(\alpha\). We say that a trajectory from \(\alpha_-\) to \(\alpha_+\) is an equivalence class of nonconstant \(\pi\)-perturbed instanton \(A \in \mathcal{M}_E(\alpha_-; \alpha_+) \subset \mathcal{B}^\pi_{\star \pi, E, \delta}(\alpha_-; \alpha_+)\) on \(\mathbb{R} \times Y\) under the translation action.

The homotopy class of a trajectory is the element of \(\pi_1(\mathcal{B}^\pi_{\star \pi, E, \delta}(\alpha_-; \alpha_+))\) it traces out; these are in noncanonical bijection with \(\mathbb{Z}\). The topological energy of a trajectory was defined above as \(2((cs + f_x)(ev_\pi A) - (cs + f_x)(ev_\pi A))\); even though the Chern-Simons functional of an individual connection is only defined in \(\mathbb{R}/8\pi^2\mathbb{Z}\), this difference of boundary components of a connection on a cylinder is defined in \(\mathbb{R}\).

A broken trajectory from \(\alpha_-\) to \(\alpha_+\) consists of a finite sequence of \(\pi\)-perturbed instantons \(A_i\) on \(\pi^*E\) over \(\mathbb{R} \times Y\) (say \(1 \leq i \leq n\), with

\[
ev\alpha A_1 = \alpha_-, \quad ev\alpha A_n = \alpha_+, \quad \text{and} \quad ev\alpha A_i = ev\alpha A_{i+1} \text{ for } 0 < i < n.
\]

The homotopy class of a broken trajectory is the composite of the homotopy classes given by the individual trajectories; the energy of a broken trajectory is defined as

\[
E^\pi(A_1, \cdots, A_n) = \sum_{i=1}^n E^\pi(A_i).
\]

We topologize this exactly as in [KM07, Page 276]. As per the author’s taste, we present this in terms of sequences, following [Don02, Page 116]: give the space of broken trajectories the final topology so that the following sequences of unbroken
instantons, and their natural generalizations to sequences of broken instantons, converge to their stated limits: $A_n$, where $i \in \mathbb{N}$, converges to the broken trajectory $(B_1, \cdots, B_n)$, if there is a sequence $(T^1_i, \cdots, T^n_i)$ of real numbers with $T^1_i \leq T^{j+1}_i$ for $0 < j < n$, and the successive differences $T^{j+1}_i - T^j_i \to \infty$ as $i \to \infty$, so that the pullbacks $\tau^i_j A$ converge as $i \to \infty$ to $B_j$ in the $L^2_{k,\delta}$ topology. Note that the energy of a trajectory is continuous with respect to this ‘chain convergence’.

This form of noncompactness, trajectories breaking into a composite of lower-index trajectories, is familiar in Morse theory. There is another kind of noncompactness familiar in the instanton theory: Uhlenbeck bubbling.

**Definition 4.7.** Let $(Y, E, \pi)$ be as above. An ideal instanton is a solution $A$ to the $\pi$-perturbed ASD equations, along with a finite (possibly empty) collection of points $x_i \in \mathbb{R} \times Y$ and integer weights $k_i \geq 1$ at each point. An ideal trajectory is an $\mathbb{R}$-equivalence class of nontrivial ideal instantons, where nontrivial means either that $A$ is nonconstant or that \{x_i\} is nonempty; equivalently, a nontrivial ideal instanton is one so that $\tau^i_j A$ is not gauge equivalent to $(A, x, k)$ for any $t$.

Similarly, a broken ideal trajectory is a sequence of ideal trajectories $(A_i^j)_{i=1}^n$ with $ev_+ A_i = ev_-. A_{i+1}$.

The energy of an ideal instanton $(A, x, k)$, where $1 \leq i \leq n$, is $E^\pi(A) + 16\pi^2 \sum k_i$. If the homotopy class of the instanton $A$ is $z$, the homotopy class of the ideal instanton $(A, x, k)$ is $z + \sum k_i$. The energy and homotopy class of a broken ideal instanton are defined to be additive under concatenation.

We say that a sequence of $\pi$-perturbed instantons $A_n$ converges to an ideal instanton $(A, x, k)$ if there is a sequence of gauge transformations $\sigma_n$ defined on $(\mathbb{R} \times Y) \setminus \{x_i\}$ such that $\sigma_n A_n \to A$ in the $L^2_k$ topology on compact subsets of $(\mathbb{R} \times Y) \setminus \{x_i\}$, and such that the density measures

$$2|F_{A_n}|^2 \to 2|F_A|^2 + \sum_{i=1}^n 64\pi^2 k_i \delta_{x_i}.$$ 

There is then a natural extension of this to a definition of convergence to (and of) broken ideal trajectories. The space of broken ideal trajectories from $\alpha$ to $\beta$ in the homotopy class $z$ is written $\mathcal{M}_{E, z, \pi}(\alpha, \beta)$. Note that this does not include constant trajectories, or broken ‘trajectories’ for which one of the components is constant.

The Uhlenbeck compactness theorem for the ASD equations on the cylinder is the following.

**Proposition 4.7.** The subspace of broken ideal trajectories in $\mathcal{M}_{E, z, \pi}(\alpha, \beta)$ with a fixed energy bound $E^\pi(A) \leq C$ is compact.

We do not repeat the proof, which can be seen in [Don02, Section 5.1]; the proof of the corresponding statement for compact cylinders is [KM11b, Proposition 3.20].

A somewhat stronger statement, Proposition 4.8 below, is true; we will only use it briefly, but find it to be somewhat interesting.

As in Definition 4.3, for any real numbers $\epsilon \gg \delta > 0$, we let $\mathcal{P}_{E, \delta} \subset \mathcal{P}_E$ be the open subset of perturbations on $(Y, E)$ so that, for each $\pi$-critical point $\alpha$, all eigenvalues of $Hess_{\alpha, \pi}$ have absolute value larger than $\delta$, and so that for every pair $(\pi, \alpha)$ of perturbation and $\pi$-flat reducible connection, the operator $Hess_{\alpha, \pi}$ has no eigenvalues of absolute value $\epsilon$. (In particular, we assume $\pi$ is a regular
perturbation.) Because the projection $\mathcal{C}_\pi \to \mathcal{P}_{E,\delta}$ of the parameterized critical set (in $\mathcal{B}_{\pi}^e$) to the space of perturbations is a proper submersion, it is in particular a locally trivial fiber bundle. Therefore, for some small open set $U$ around any $\pi_0 \in \mathcal{P}_{E,\delta}$, we have a canonical bijection $\mathcal{C}_\pi_{\pi_0} \cong \mathcal{C}_\pi$ for any $\pi \in U$; in fact, if we fix $\alpha \in \mathcal{C}_\pi_{\pi_0}$, we may choose a map $s_\alpha : U \to \mathcal{A}_{E,k}$, so that $s(\pi)$ is a $\pi$-flat connection which is identified under the above bijection with $\alpha$. Choose once and for all, for each homotopy class $z$, a map $r_z : \mathcal{A}_{E,k} \times \mathcal{A}_{E,k} \to \mathcal{A}^{(4)}_{\pi_0} \mathcal{E}_{E,z,k,\delta}$, sending $(A_-, A_+)$ to a connection which is constant at $A_-$ for $t \leq -1$ and constant at $A_+$ for $t \geq 1$ and in the homotopy class $z$.

We may use the $s_\alpha$ to define the parameterized configuration space $\mathcal{P}_\delta \mathcal{A}^{(4)}_{\pi,\mathcal{E},z,k,\delta}$, whose elements $(\pi, \alpha_\pm, A)$ consist of a perturbation $\pi \in \mathcal{P}_{E,\delta}$, a choice of two $\pi_\pm$-flat connections $\alpha_\pm$, and a connection $A \in \mathcal{A}^{(4)}_{\pi,\mathcal{E},z,k,\delta}(\alpha_-, \alpha_+)$. This set is given the structure of a smooth Banach manifold by patching together charts of the form 

$$U \times \mathcal{A}_{\pi,\mathcal{E},z,k,\delta}(\alpha_-, \alpha_+) \cong r(z, s_{\alpha_-, \pi}, s_{\alpha_+, \pi}) + \Omega^1_{E,k,\delta}(g_E).$$

Write 

$$\mathcal{P}_\delta \mathcal{A}^{(4)}_{\pi,\mathcal{E},z,k,\delta} = \mathcal{P}_\delta \mathcal{A}^{(4)}_{\pi,\mathcal{E},k,\delta} \times \pi^* \mathcal{E}_{(0,b)}$$

for the parameterized space of framed connections, which inherits a smooth structure and a smooth right action of $SO(3)$. It carries a smooth projection map $\mathcal{P}_\delta \mathcal{A}^{(4)}_{\pi,\mathcal{E},z,k,\delta} \to \mathcal{P}_{E,\delta}$. It carries the action of a bundle of Banach Lie groups over $\mathcal{P}_{E,\delta}$; the fiberwise quotient gives a topological space $\mathcal{P}_\delta \mathcal{B}_{\pi,\mathcal{E},z,k,\delta}$. We may define the parameterized moduli space $\mathcal{P}_\delta \mathcal{M} \subset \mathcal{P}_\delta \mathcal{B}_{\pi,\mathcal{E},z,k,\delta}$ as the equivalence classes of triples $(\pi, A, p)$, where $\pi \in \mathcal{P}_{E,\delta}$ is a perturbation, $A$ is a $\pi$-perturbed instanton, and $p$ is a framing.

Because the perturbed ASD equations do not depend on the framing, this set inherits the right $SO(3)$-action. The quotient of $\mathcal{P}_\delta \mathcal{M}$ by this $SO(3)$ action is the parameterized moduli space of instantons, which we denote $\mathcal{P}_\delta \mathcal{M}$. If we take the quotient by the $\mathbb{R}$ action and throw out the constant trajectories, we may extend this to a space $\mathcal{P}_\delta \mathcal{M}$, the parameterized moduli space of broken trajectories. (We will soon discuss the version of this appropriate to the framed setting, which is slightly more subtle.)

**Proposition 4.8.** Let $\mathcal{P}_\delta \mathcal{M}^{\leq C}$ be the subspace of $\mathcal{P}_\delta \mathcal{M}$ consisting of those pairs $(\pi, A)$ so that $\mathcal{E}(A) \leq C$. Then the projection map $\mathcal{P}_\delta \mathcal{M}^{\leq C} \to \mathcal{P}_\delta$ is proper.

**Proof.** Suppose we have a sequence $(\pi_n, A_n)$ of perturbations and unbroken instantons so that $\pi_n \to \pi$. We want to show that there is a subsequence of $A_n$ which converges to a broken ideal $\pi$-trajectory. (The general case where $A_n$ is itself a broken ideal trajectory provides no further difficulty.) First, because the possible ev$_\pm$ take values in a finite set, choose a subset of $A_n$ so that, for $n$ large, ev$_\pm A_n$ correspond to $\alpha_\pm \in C_{\pi} \cong C_{\pi_0}$. We may now apply [Don02, Lemma 4.3], which establishes that any $\pi$-instanton with sufficiently small energy and ev$_\pm A_n$ is gauge equivalent on $(-\infty, 0) \times Y$ to $\alpha + a$ for some $a$ with a uniform bound on $|a(t)|e^{-\delta t}$, as well as the derivatives $|\nabla^{(l)}(a)(t)|e^{\delta t}$ for $l \leq k$. The constants in these uniform bounds are bounded for the convergent sequence $\pi_n \to \pi$, and $\delta$ is fixed. In particular, for our solutions $A_n$, gauge equivalent to $\alpha_n + a_n$, this is enough for the Arzela-Ascoli and dominated convergence argument in Lemma 5.1.
of the same to imply that \( a_n \to a \) for some function \( a \) satisfying the same bounds; in fact we must have \( F_{A^\pi_n}^+ + \nabla_\pi (\alpha + a) = 0 \) as this is the pointwise limit of the corresponding equations \( F_{A_n}^+ + \hat{\nabla}_\pi (A_n) = 0 \).

Now that we have control over the ends, everything else is standard: a uniform bound on \( \tilde{E}^{\pi_n} (C) \) implies a uniform bound on the \( L^2 \) norm of \( F_A \) on compact sets, so one has a limit on compact sets after accounting for bubbling, and then a limit on the whole line to a broken ideal trajectory; this uses that there is a uniform positive lower bound on the minimal energy of a nontrivial \( \pi_n \)-instanton. To see this, recall that the energy of an instanton may be written as \( 2 \int |A'(t)|^2 \) as in the proof of Proposition 4.1. If \( A_n \) is a sequence of nontrivial \( \pi_n \)-instantons with \( \tilde{E}^{\pi_n} (A_n) \to 0 \), this implies that the distance between the endpoints \( \alpha_\pm \) goes to zero. But \( \alpha_\pm \to \alpha \), so \( \alpha = \alpha_+ \); but then because \( \alpha \) is sent to \( \alpha_\pm \) under the bijection \( \mathbb{C}_{\pi_n} \cong \mathbb{R} \), we see that \( \alpha = \alpha_\pm \) for large \( n \). However, this would imply that \( \tilde{E}^{\pi_n} (A_n) \), which is a multiple of \( 64 \pi^2 \), is actually zero; but we assumed \( A_n \) was nontrivial.

The reason we restrict to perturbations in \( \mathcal{P}_{E,\delta} \) is for the definition of the parameterized moduli space as \( L^2_{E,\delta} \) connections; we always want to take \( \delta \) less than the eigenvalues of the extended Hessian, and otherwise would need to choose \( \delta \) depending on the perturbation.

We conclude the discussion of compactness for cylinders by defining the object of interest to us: the compactification of the framed moduli space of trajectories.

**Definition 4.8.** Let \( Y \) be a Riemannian 3-manifold, equipped with \( \text{SO}(3) \)-bundle \( E \) and regular perturbation \( \pi \). We say that a framed ideal trajectory from \( \alpha_- \) to \( \alpha_+ \) is an equivalence class of \( \pi \)-perturbed framed instanton \( (A, p) \in \hat{\mathcal{B}}}^{\pi_E, k, \delta} (\alpha_-, \alpha_+) \) equipped with a (possibly empty) points of points \( x \in \mathbb{R} \times Y \) and integer weights \( k_x > 0 \), where none of the \( x \) are the basepoint \((0, b)\). We demand the ideal instanton is nontrivial, in the sense that either the set of points \( x_i \) is nonempty or the trajectory \( A \) is nonconstant.

A deframed ideal trajectory from \( \alpha_- \) to \( \alpha_+ \) is an ideal trajectory from \( \alpha_- \) to \( \alpha_+ \) so that \((0, b)\) is a weight-point \( x \) with \( k_{(0, b)} > 0 \).

A framed broken trajectory from \( \alpha_- \) to \( \alpha_+ \) is a finite sequence whose elements are either framed ideal trajectories \( (A, p_i, x_i, k_i) \) or deframed ideal trajectories \( (A, x_i, k_i) \) with

\[
\operatorname{ev}_-(A_1, p_1) \in \alpha_-, \quad \operatorname{ev}_+(A_n, p_n) \in \alpha_+,
\]

\[
\operatorname{ev}_+(A_i, p_i) = \operatorname{ev}_-(A_{i+1}, p_{i+1}) \text{ for } 0 < i < n, \text{ when } A_i, A_{i+1} \text{ are both framed.}
\]

The set of framed broken trajectories in the homotopy class \( z \) from \( \alpha_- \) to \( \alpha_+ \) is written \( \tilde{\mathcal{M}}_{E, \pi} (\alpha_-, \alpha_+) \). We write a generic element as \((A, p)\), even though \( A \) may be broken and not every piece must be framed.

This set is topologized as follows. A sequence \((A_n, p_n)\) of framed instantons converges to a framed ideal instanton \((A, p, x, k)\) if there is a sequence of gauge

\[7\text{Here deframed is meant to indicate that the framing has been removed via the placement of a } \delta \text{-mass at the basepoint; not all the weight-sets are allowed, and in particular a non-ideal instanton without a framing is not a 'deframed ideal trajectory'}.\]
transformations $\sigma_n$, defines on $(\mathbb{R} \times Y) \setminus \{x_i\}$, and in particular defined on the base-point $(0, b)$, so that $\sigma_n^* A \to A$ converge in the sense of ideal instantons above, and $\sigma_n^* p_n \to p$.

However, if the underlying trajectory converges to an ideal trajectory with non-trivial weight at $(0, b)$, so that the $\sigma_n$ are undefined at $(0, b)$, then it doesn’t make sense to compare $\sigma_n^* p_n$ and $p$. In this situation we lose the framing via bubbling at the basepoint. In this case, the sequence $(A_n, p_n)$ of framed instantons converges to a deframed ideal trajectory. Incorporating this into the topology on broken trajectories is straightforward.

**Proposition 4.9.** The space $\overline{\mathcal{M}}_{E, z, \pi}(\alpha_-, \alpha_+)$ is compact.

**Proof.** We begin by recalling that there is a surjective map

$$\overline{\mathcal{M}}_{E, z, \pi}(\alpha_-, \alpha_+) \to \overline{\mathcal{M}}_{E, z, \pi}(\alpha_-, \alpha_+).$$

This map is proper: if a sequence $(A_n, p_n)$ of framed broken trajectories has underlying sequence of broken ideal trajectories converge to $A$, the framings are either incomparable (and that limit component of $A_n$ is an deframed ideal trajectory), or (an appropriate sequence of translations of) $\sigma_n^* p_n$ are all defined, and live in the compact space $E_b \cong \text{SO}(3)$; so some subsequence converges, as desired. Doing this for the finitely many components of the limit $A$ constructs an element of $\overline{\mathcal{M}}$ that a subsequence of $(A_n, p_n)$ converges to. Because we have constructed a proper map to a compact space, the total space $\overline{\mathcal{M}}$ is compact. 

Similarly, the projection from the parameterized moduli space $\mathcal{P}_{E, \delta} \mathcal{M}^{\leq C}$ to $\mathcal{P}_{E, \delta}$ is proper.

There are straightforward extensions of these to moduli spaces on cobordisms, which we now state. In what follows, we write $\mathcal{P}_{E, \delta}^{(4)}$ for the subspace

$$\mathcal{P}_{E, \delta} \oplus \mathcal{P}_{E_1}^{(4)} \oplus \mathcal{P}_{E_2}^{(4)} \oplus \mathcal{P}_{E_2, \delta},$$

and similarly $\mathcal{P}_{L, \delta}^{(4)} = \mathcal{P}_{L}^{(4)} \times V_L$, where $V_L$ is a finite set of interior holonomy perturbations adapted to a collection of thickened loops, as in Section 4.1. For the rest of this section we fix a specific $L$. (As usual, the manifold $W$ is left implicit in the notation.)

**Definition 4.9.** Let $W$ be an oriented Riemannian 4-manifold, with cylindrical ends oriented isometric to $(-\infty, 0] \times Y_1$ and $[0, \infty) \times Y_2$, equipped with an $\text{SO}(3)$-bundle $E$ and specified isomorphisms to the pullback of bundles on $Y_i$ over the ends. Suppose $W$ is equipped with an embedding $\gamma : \mathbb{R} \to W$ which agrees for $|t|$ sufficiently large with $(t, b_1)$ or $(t, b_2)$, depending on the sign of $t$, and write $b = \gamma(0)$ as the basepoint of $W$. Suppose $W$ is equipped with a perturbation $\pi$, restricting to fixed regular perturbations $\pi_i$ on the ends.\(^8\) Connected components of the space of connections on $W$ limiting to $\alpha_\pm$, critical points of $\pi_\pm$, are in bijection with $\mathbb{Z}$, and labeled by $z$.

Then a broken ideal $W$-trajectory from $\alpha_-$ to $\alpha_+$ in the homotopy class $z$ is a triple of a broken ideal $\pi_-$-trajectory on $Y_1$ from $\alpha_-$ to some $\beta$ (possibly constant, if $\alpha_- = \beta$), an ideal $\pi_-$-instanton on $W$ from $\beta$ to some $\gamma$, and a broken ideal $\pi_+$-trajectory on $Y_2$ from $\gamma$ to $\alpha_+$ (possibly constant, if $\gamma = \alpha_+$). The homotopy class is

\(^8\)That is, $\pi$ is an element of $\mathcal{P}_{E}^{(4)} \times V_S$; that is, a sum of a perturbation supported in the collar and finitely many interior holonomy perturbations adapted to a collection of thickened loops.
the composite of the corresponding homotopy classes of non-ideal instantons, then summing the weights. We denote the set of broken ideal $W$-trajectories from $\alpha_-$ to $\alpha_+$ in the homotopy class $z$ as $\mathcal{M}_{E,z}^{W,\pi}(\alpha_-\alpha_+)$.

An framed broken ideal $W$-trajectory is the same, but with in addition a framing on each component where there is no weight at the basepoint of $Y_i$ or $W$, and so that the neighboring evaluations agree:

$$\text{Hol}_{A_{\gamma_i}}^{0,-\infty}(p_i) = \text{Hol}_{A_{\gamma_{i+1},\gamma_{i+1}+1}}^{0,-\infty}(p_{i+1}).$$

We denote the set of framed broken ideal $W$-trajectories from $\alpha_-$ to $\alpha_+$ in the homotopy class $z$ as $\mathcal{M}_{E,z}^{W,\pi}(\alpha_-\alpha_+)$. We similarly denote the parameterized space by

$$\mathcal{P}_{L,\delta}\mathcal{M}_{E,z}.$$ 

Note in particular that we include the ‘deframed trajectories’ here as in the cylindrical case, and at these, we have lost the framing at the basepoint. Here $(W,E)$ is collapsed in the notation to simply the bundle $E$.

A sequence of instantons $A_n$ on $W$ from $\alpha_-$ to $\alpha_+$ converges to a broken trajectory $(\mathcal{A}_-,\mathcal{A}_,\mathcal{A}_+)$ if, for each end, there is a sequence of translations $T_n^\epsilon$ with $T_n^{\epsilon+1} - T_n^\epsilon \to \infty$ on the negative end and $\to -\infty$ on the positive end so that $\tau_{T_n^\epsilon}^* A_n \to A_\infty$ on the negative end, and correspondingly on the positive end. Note that the assumption that $T_n^\epsilon \to \infty$ implies that these translates exhaust the tube. One similarly accounts for the topology of the bubbling phenomenon.

We have the following analogue of Proposition 4.8 for $W$-trajectories, which will be useful to us. The proof is a verbatim combination of what was already said in the case of the ends (that is, in the cylindrical case), as well as a discussion of what happens on compact subsets of $W$.

**Proposition 4.10.** Let $\mathcal{P}_{L,\delta}\mathcal{M}_{E}^{\leq C}$ be the subspace of $\mathcal{P}_{L,\delta}\mathcal{M}_{E}$ consisting of those pairs $(\pi,A)$ so that $\mathcal{E}^\pi(A) \leq C$. Then the projection map $\mathcal{P}_{L,\delta}\mathcal{M}_{E}^{\leq C} \to \mathcal{P}_{L,\delta}^{(0)}$ is proper.

**Proof.** Fix $(A_n,\pi_n)$ with $\pi_n \to \pi$. We need to show that there is some sequence of gauge transformations, defined on the complement of some finite set, so that there is a connection $A$ and a subsequence (still written $A_n$) so that $\sigma_n^* A_n \to A$. We have already dealt with the ends in the discussion of the cylinder. What remains is to show that a uniform energy bound gives a uniform bound in curvature over compact sets (which implies that $A_n$ has a convergent subsequence on the interior modulo gauge) and that if $A_n$ converges to an ideal $\pi$-perturbed instanton, the weight of the $\delta$-mass at the ideal point $x$ is still the same: $64\pi^2$.

First, consider the restriction of $A$ to the complement of the ends, a compact manifold $W'$ with boundary $Y$. The perturbations vanish on the boundary of $W'$ (and on the interior are the previously discussed interior holonomy perturbations). On the ends the perturbations are of the form (eg, for $Y_i$) $\pi_i(t)_n + \beta_0(t)(\pi_-)_n$, where $\beta_0 : [0,\infty) \to [0,1]$ is zero near 0, has $\beta_0(t) = 1$ for $t \geq 1$, and $\pi_i(t)_n$ is a sequence of compactly supported $C^1$ function $(1,2) \to \mathcal{P}_{E_i}$ converging toward $\pi_i(t)$. 
Remember that for a $\pi$-instanton $A$ (these chosen arbitrarily), its analytic energy $E_\pi^{an}(A)$ was defined to be equal to

$$E_\pi^{\text{top}}(A) = 2 \left( (cs + f_{\pi_-})(\alpha_-) - (cs + f_{\pi_+})(\alpha_+) \right).$$

Cutting this off at the boundary of the ends (we say $A$ restricted to $\{0\} \times Y_1$ is $\alpha_1$, and restricted to $\{0\} \times Y_2$ is $\alpha_2$), where the perturbations are all zero, this decomposes into three pieces:

$$2 \left( (cs + f_{\pi_-})(\alpha_-) - (cs)(\alpha_1) \right)$$
$$+ 2 \left( (cs)(\alpha_1) - (cs)(\alpha_2) \right)$$
$$+ 2 \left( (cs)(\alpha_2) - (cs + f_{\pi_+})(\alpha_+) \right)$$

First we study how the value of $cs + f_{\pi(t)}$ changes along the ends: consider the function $(cs + f_{\pi(t)})(\alpha_1^t)$. Here, $\pi(t) = \pi_1(t) + \beta_0(t)\pi_-$. Then

$$\int d \left( (cs + f_{\pi(t)})(\alpha_1^t) \right) = cs(\alpha_1) - (cs + f_{\pi(0)})(\alpha_{\pi -}).$$

We may expand the integral as

$$\int (cs + f_{\pi(t)})'(\alpha_1^t)dt + \int d(cs + f_{\pi(t)}) \left( \frac{d}{dt} \alpha_1^t \right).$$

Note that the second integral is the same as $\int \langle *F_{\alpha_1^t} + \nabla_{\pi(t)}(\alpha_1^t), \frac{d}{dt} \alpha_1^t \rangle$; by assumption that $\alpha_1^t$ satisfies the time-dependent gradient flow equations, this integral is $\int \| \frac{d}{dt} \alpha_1^t \|^2 dt$. Because $f_{\pi(t)} = \beta_0(t)f_{\pi_0} + f_{\pi(t)}$, the first integral is the same as $\int \beta_0'(t)f_{\pi_0}(\alpha_1^t) + f_{\pi_1(t)}(\alpha_1^t)dt$. This is uniformly bounded above and below because the support of $\beta_0'$ and $\pi'$ are both compact, and the functions $f_{\pi_0}$ and $f_{\pi}$ are bounded. Thus up to a uniformly bounded amount, the first part of the Chern-Simons difference above is $2 \int \| \frac{d}{dt} \alpha_1^t \|^2$; a similar discussion implies the same of the the third part of the Chern-Simons difference and $2 \int \| \frac{d}{dt} \alpha_2^t \|^2$.

Because $\pi_n \to \pi$, we obtain a bound on the energy of any $\pi_n$-instanton $A_n$ on the ends, uniform in $n$.

What remains is the middle piece. Write $\omega(A) \in \Omega_{L^\infty}(W'; g_E)$ for the perturbing term. In $W'$, the fact that $A$ is $\pi$-ASD means that $F_A^\pm = -\omega(A)$. Recall that at each point, $\text{Tr}(F_A^2) = |F_A^-|^2 - |F_A^+|^2$, and in particular we may write

$$\text{Tr}(F_A^2) = |F_A^-|^2 - 2|F_A^+|^2 = |F_A|^2 - 2|\omega(A)|^2.$$

Because $|\omega(A)|_{L^\infty}$ is uniformly bounded and $W'$ is compact, by integrating we see that

$$\int_{W'} \text{Tr}(F_A^2) \geq \int |F_A|^2 - C$$

for some constant $C$. The left-hand side is precisely the Chern-Simons difference of $\alpha_2$ and $\alpha_1$. Because $\omega_n \to \omega$, again we see that this bound can be made uniform in $n$.

Altogether we see that an energy bound gives a bound on the $L^2$ norm of the $F_{A_n}$ on $W'$, as well as a bound on the $L^2$ norms of $\alpha_1^t$ and $\alpha_2^t$. Because the $\alpha_1^t$ satisfy the equations $\frac{d}{dt} \alpha_1^t = -*F_{\alpha_1^t} - \nabla_{\pi(t)}(\alpha_1^t)$, and $|\nabla_{\pi(t)}A|_{L^\infty}$ is uniformly bounded, we get a uniform $L^2$ bound on curvature on any compact piece of $W$. Then as in Proposition 4.8, we may conclude the existence of the appropriate gauge
Now suppose \( A_n \) is a sequence of \( \pi \)-perturbed instantons on \( W \) and suppose there is a sequence of gauge transformations \( \sigma_n^* \), defined on the complement of a finite set of points, so that \( \sigma_n^* A_n \to A \) in \( L^2_k \) on compact sets. Because of gauge invariance in the complement of a finite set, \( \omega_n(A_n) \to \omega(A) \) in the \( L^2_k \) topology, so on compact subsets of the complement of the finite set, \( F_{A_n}^* + \omega_n(A_n) \to F_A^* + \omega(A) \) in \( L^2_{k-1} \); in particular, the latter is zero except possibly at a finite set, and hence is zero everywhere by continuity of \( L^2_{k-1} \) functions; so the limit \( A \) constructed is indeed a solution of the perturbed equations. We will show the loss in curvature mass is still, as expected, a multiple of \( 64\pi \) indeed a solution of the perturbed equations. We will show the loss in curvature.

The space \( \mathcal{M}_{E,z,\pi}^{\leq 0}(\alpha_- \alpha_+) \) of broken ideal trajectories on \((W,E)\) with uniform energy bound is compact.

The following theorem summarizes the content of this section.

**Theorem 4.12.** Let \((Y,E)\) be a 3-manifold equipped with a weakly admissible \( SO(3) \)-bundle and regular perturbation \( \pi \). There is a natural compactification of the space of unparameterized trajectories

\[
\hat{\mathcal{M}}_{E,z,\pi}^0(\alpha, \beta) \subset \mathcal{M}_{E,z,\pi}(\alpha, \beta)
\]

as a compact \( SO(3) \)-space equipped with equivariant endpoint maps.\(^9\) The added strata \( \hat{\mathcal{M}} \backslash \mathcal{M} \) are given as a union of fiber products of moduli spaces of lower dimension (in codimension equal to the number of intermediary orbits between \( \alpha \) and \( \beta \) and strata corresponding to Uhlenbeck bubbling.

The same is true for a cobordism \((W,E)\) equipped with a perturbation \( \pi \) which is regular on the bounding manifolds \((Y_i,E_i)\): the space of framed instantons \( \hat{\mathcal{M}}_{E,z,\pi}(\alpha, \beta) \) has a natural compactification to the compact \( SO(3) \)-space \( \overline{\mathcal{M}}_{E,z,\pi}(\alpha, \beta) \).

We conclude with some discussions on energy bounds. On a cylinder \( \mathbb{R} \times Y \), equipped with a regular perturbation \( \pi \), there are finitely many critical orbits \( \alpha \). For a fixed pair \( \alpha, \beta \), the space of connections from \( \alpha \) to \( \beta \) has \( \pi_1 \mathcal{B}_E^p(\alpha, \beta) \cong \mathbb{Z} \) components. We may define two functions of \( z \in \pi_1 \mathcal{B}_E^p(\alpha, \beta) \). The first is

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\(^9\)Here recall that \( \hat{\mathcal{M}}^0 \) is \( \hat{\mathcal{M}} / \mathbb{R} \), less the constant trajectories.
the topological energy $E^\pi(z)$, defined by taking a connection $A$ in the component labelled by $z$, and evaluating

$$2 ((cs + f_\pi)(\beta) - cs + f_\pi)(\alpha)) \in \mathbb{R};$$

we use the path $A$ and the homotopy lifting property of the covering $\mathbb{R} \to \mathbb{R}/8\pi^2$ to pin down a real lift of this difference. The second is $gr_z(\alpha, \beta) \in \mathbb{Z}$. These are both affine functions; writing $z \in \mathbb{Z}$, we have $E^\pi(z) = 64\pi^2 z + c$ and $gr_z(\alpha, \beta) = 8z + c'$. In particular, we see that $E^\pi(z) - 8\pi^2 gr_z(\alpha, \beta)$ is constant. Therefore an energy bound is equivalent to an upper bound on $gr_z(\alpha, \beta)$.

Suppose that the cobordism $(W, E) : (Y_1, E_1) \to (Y_2, E_2)$ is equipped with a perturbation $\pi$ so that its restriction $\pi_k$ to the ends is regular, and all $\pi$-perturbed instanton moduli spaces up to some energy bound $C$ are cut out transversely. The set $\pi_0 \left( \hat{B}_E^\pi(\alpha, \beta) \right)$ of components of the space of trajectories from $\alpha$ to $\beta$ is affine over both $\pi_1(\hat{B}_{E_1}, \alpha)$ and $\pi_1(\hat{B}_{E_2}, \beta)$, and hence we may write the set of components as affine over $\mathbb{Z}$. Then as before $E^\pi_W(z)$ is affine with distortion $64\pi^2$ and $gr_z(\alpha, \beta)$ is affine with distortion 8; so again we find that an energy bound is equivalent to a bound on $gr_z(\alpha, \beta)$. In what follows we will often only care about the case $gr_z(\alpha, \beta) \leq 7$; there is a corresponding energy bound.

Because components of the moduli space with $gr_z(\alpha, \beta) < 0$ are empty when cut out transversely, and bubbling decreases $gr_z(\alpha, \beta)$ by multiples of 8, we have the following.

**Corollary 4.13.** When $gr_z(\alpha, \beta) \leq 7$, the compactification $\overline{M}_{E, z, \pi}(\alpha, \beta)$ has no strata corresponding to Uhlenbeck bubbling.

### 4.4. Reducible instantons on the cylinder and cobordisms

The first main result of this section is the following.

**Proposition 4.14.** Let $E$ be an $SO(3)$-bundle over a rational homology sphere $Y$ equipped with a Riemannian metric. Let $P_E$ denote a Banach space of perturbations on $(Y, E)$. By Proposition 3.5, if $\pi \in P_E$ has $\|\pi\|$ sufficiently small, then there is a unique critical orbit $\alpha$ in every reducible component $\text{Red}(Y, E)$, which is the unique fully reducible orbit if the reducible component has a fully reducible point.

Consider moduli spaces of reducible instantons on the cylinder $\mathbb{R} \times Y$, equipped with the constant perturbation $\pi$ on the cylinder. By Proposition 1.6, the only moduli spaces which can possibly be nonempty are $\overline{M}_{E, 0, \pi}(\alpha, \alpha)$, the spaces of trajectories from $\alpha$ to $\alpha$ in the trivial homotopy class. There is exactly one instanton in each of these: the constant trajectory at $\alpha$. Therefore, the moduli space $\overline{M}_{E, 0, \pi}(\alpha, \alpha) = \emptyset$, by definition.

**Proof.** In other words, the constant trajectory (which is always a solution) remains the only solution. Because each component in $\text{Red}(Y, E)$ lies inside $\hat{B}_{E, 0, k, \pi}(\alpha, \alpha)$ (the zero denoting the trivial homotopy class), and an instanton in this component has energy equal to zero, the trajectory must be constant, as in Corollary 4.3. □

In fact, something similar is true even in the case of cobordisms (or cylinders with nonconstant perturbations).

For the statement of the following proposition, recall the definition of energy of a component of connections with respect to the perturbation $\pi$: if $A$ is a connection in the component $z$ which has positive and negative limits $A_+$ and $A_-$ (connections,
not equivalence classes), respectively, and perturbations $\pi_{\pm}$ on the same ends, then $\mathcal{E}_{\Sigma}^\pm = 2 (\{ (s + f_{\Sigma}) (A_-) - (s + f_{\Sigma}) (A_+) \})$.

**Proposition 4.15.** Let $W$ be an oriented Riemannian 4-manifold equipped with an $SO(3)$-bundle $E$ and with one incoming cylindrical end $(Y_1, E_1)$ and one outgoing cylindrical end $(Y_2, E_2)$. Suppose $b_1(W) = 0$ and $b^+(W) = 0$. Write $\mathcal{P}_\delta^{(4)}$ for the space of perturbations on $W$ so that the perturbations $\pi_i$ on the ends lie in $\mathcal{P}_{E, \delta}$, the subspace of perturbations described at the end of Definition 4.3. If $\beta(w_2(E)) \neq 0$ or one of $(Y_1, E_1)$ is admissible, then $(W, E)$ admits no $\pi$-perturbed reducible instantons in $\tilde{\mathcal{M}}_{E, \pi}$ whatsoever.

Fix $C > 0$. For a dense open subset of $\mathcal{P}_\delta^{(4)} = \mathcal{P}_{E_1, \delta} \oplus \mathcal{P}_{E_1}^{(4)} \oplus \mathcal{P}_{E_2}^{(4)} \oplus \mathcal{P}_{E_2, \delta}$, then for every component in $\text{Red}(W, E)$ with $\mathcal{E}_{\Sigma}^\pi \leq C$, the reducible $\pi$-instantons are a finite set, which are cut out transversely inside the reducible locus so long as they are not $\pi$-flat. If one of the $E_i$ is nontrivial, this is also true for $\pi$-flat connections. In any case, in a neighborhood of 0, even the $\pi$-flat connections are cut out transversely.

Given any compact set $K \subset \mathcal{P}_\delta^{(4)}$, there is a neighborhood $K \subset U_K \subset \mathcal{P}_{L, \delta}^{(4)} = \mathcal{P}_\delta^{(4)} \times V_L$ so that, for a dense open subset of $\pi \in U_K$, the $\pi$-flat instantons form a finite set, all of which are cut out transversely. If $\pi$ is sufficiently small (with respect to some constant $\epsilon_{W, C}$), then for every component in $\text{Red}(W, E)$ with $\mathcal{E}_{\Sigma}^\pi \leq C$, there is further a unique orbit of $\pi$-perturbed reducible instantons in that component; for those components $\text{Red}_{\pi}(W, E)$ containing a fully reducible orbit, that orbit is the unique reducible instanton in that component.

For bundles with $\beta w_2 \neq 0$ or with admissible ends, Proposition 1.7 shows that there are not even reducible connections. Thus there is only something to say for cobordisms with rational homology sphere ends.

Similarly, there is also a version when $b^+(W) > 0$. Our goal is to avoid reducibles in this case, as they cannot be cut out transversely. The statement is slightly more technical than the previous proposition, as we now need to include some finite-dimensional vector space $V_L$ of interior holonomy perturbations to achieve transversality; we will need to pick $L$ dependent on the perturbation $\pi$.

**Proposition 4.16.** Let $W$ be an oriented Riemannian 4-manifold equipped with an $SO(3)$-bundle $E$, with one incoming cylindrical end $(Y_1, E_1)$ and one outgoing cylindrical end $(Y_2, E_2)$. Suppose $b^+(W) > 0$.

If $\beta(w_2(E)) \neq 0$ or one of $(Y_1, E_1)$ is admissible, then $(W, E)$ admits no $\pi$-perturbed reducible instantons in $\tilde{\mathcal{M}}_{E, \pi}$ whatsoever.

If $E$ is nontrivial but neither $E_i$ is admissible, then for each $C > 0$, there exists an open dense set of $\pi \in \mathcal{P}_\delta^{(4)}$ for which there are no reducible $\pi$-ASD connections with $\mathcal{E}^\pi_{\text{an}}(A) \leq C$ which are not $\pi$-flat. If one of the $E_i$ is nontrivial, this is also true for $\pi$-flat connections.

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10 Note here that we do not include the interior holonomy perturbations. That makes this a **stronger** result: in this case, we do not need those to achieve transversality.
If $E$ is nontrivial but both $E_i$ are trivial, then given any compact set $K \subset \mathcal{P}^{(4)}_{\delta}$, one may choose a finite set $\mathcal{L}$ of holonomy perturbations adapted to a collection of thickened loops, so that there is a neighborhood

$$K \subset U_K \subset \mathcal{P}^{(4)}_{\mathcal{L}^\delta} = \mathcal{P}^{(4)}_{\delta} \times V_L$$

with a dense open subset of $\pi \in U_K$ for which there are no $\pi$-flat instantons.

If $E$ is trivial, each fully reducible component always has the full reducible as a solution, no matter the perturbation; these fully reducible solutions are never cut out transversely in the $SO(2)$-fixed locus.

We prove these simultaneously, much like Proposition 3.5: we define a Banach manifold of reducible solutions to the ASD equations, equipped with a map to $\mathcal{P}$, which is proper and a local diffeomorphism at 0. Recall from Proposition 1.6 that the $SO(2)$-fixed subspace is a disjoint union over copies of $\mathcal{B}_\eta(\alpha_{-}, \alpha_{+})$, as $\eta$ varies over certain complex line bundles. Note that this configuration space is unframed, despite being a subset of the framed configuration space of $SO(3)$-connections on $E$. Setting this up takes some small amount of work because the Hilbert manifold the equation is defined on, $\mathcal{B}_\eta(\alpha_{-}, \alpha_{+})$, depends on the limiting orbits, and hence depends on $\pi$ (which determines the reducible critical orbits).

While we discussed such a configuration space in the previous section, we did not show that this (unframed) quotient was a Banach manifold: that is usually not true, but is in the special case of $SO(2)$-bundles.

In the next statement we have fixed some finite set $L$ of holonomy perturbations adapted to a collection of thickened loops.

**Lemma 4.17.** Let $W$ be an oriented Riemannian 4-manifold equipped with an $SO(3)$-bundle $E$ and with one incoming cylindrical end $(Y_1, E_1)$ and one outgoing cylindrical end $(Y_2, E_2)$, both rational homology spheres. Fix a lift of $E$ to a $U(2)$-bundle $\tilde{E}$; if none exist, then $W$ admits no reducible connections whatsoever by Proposition 1.7. Write $\lambda$ for the complex line bundle $\det(\tilde{E})$ and fix a connection $A_0$ on $\lambda$.

There is a Banach manifold $\mathcal{P}_{\lambda, \delta} \mathcal{E}_\eta, k, \delta$, whose objects are pairs $(\pi, A)$ defined as follows. First, $\pi \in \mathcal{P}^{(4)}_{\lambda, \delta}$ is a perturbation so that the perturbations $\pi_i$ on the ends and any $\pi_i$-flat connection $A_i$, the operator $\overline{\text{Hess}}_{A_i, \pi_i}$ has no eigenvalues $c$ with $|c| \leq \delta$. Second, $A$ is an $L^2_{k+1, \delta}$-gauge equivalence class of $L^2_{k, \delta}$ connection on $\tilde{E}$ which respects a splitting $\tilde{E} \cong \eta \oplus (\lambda \otimes \eta^{-1})$ and has $\pi_{\pm}$-flat limits on the ends.

This Banach manifold comes equipped with a smooth submersion $\mathcal{P}_{\lambda, \delta} \mathcal{E}_\eta, k, \delta \to \mathcal{P}^{(4)}_{\lambda, \delta}$. There is a smooth vector bundle $S_{k-1, \delta} \to \mathcal{P}_{\lambda, \delta} \mathcal{E}_\eta, k, \delta$ with fiber isomorphic to $\Omega^2_{k-1, \delta}(i\mathbb{R})$, and a smooth Fredholm section $s(\pi, A) = F^+_{\lambda} + \nabla_{\pi}(A)$.

**Proof.** Denote by $\eta_i$ the restrictions of $\eta$ to the ends. Write $\mathcal{P}_{\lambda, \delta} \mathcal{E}_\eta$ for the set whose elements are triples $(\pi, \alpha_1, \alpha_2)$, where $\pi \in \mathcal{P}^{(4)}_{\lambda, \delta}$ is a perturbation restricting to $\pi_i$ on the ends, and $\alpha_i \in \mathcal{B}_\eta_i$ are equivalence classes of $\pi_i$-flat connections on $\eta_i$.

With minimal change, Lemma 3.3 asserts that the map $\mathcal{P}_{\lambda, \delta} \mathcal{E}_\eta \to \mathcal{P}^{(4)}_{\lambda, \delta}$, sending the parameterized critical set of gauge equivalence classes of connections on $\eta$ to the space of perturbations, is proper. That $\mathcal{P}_{\lambda, \delta} \mathcal{E}$ is a manifold and that the projection above is a covering map follow quickly from the assumption that every $\pi \in \mathcal{P}^{(4)}_{\lambda, \delta}$ is such that $\overline{\text{Hess}}_{A_i, \pi_i}$ never has eigenvalues of absolute value at most $\delta$. Thus, $\mathcal{P}_{\lambda, \delta} \mathcal{E}_\eta$
has charts given by local sections \( \psi \) of the projection to \( \mathcal{P}_{L,\delta}^{(4)} \). Fix such a chart, given by \( U \subset \mathcal{P}_{L,\delta}^{(4)} \).

We may lift the above section \( \psi : U \to \mathcal{P}_{L,\delta} \mathcal{C}_\eta \) to a smooth map \( p : U \to \mathcal{A}_{\eta_1,k} \times \mathcal{A}_{\eta_2,k} \), so that \( p(\pi_1,\cdots,\pi_2) \) is a pair of connections in the same gauge equivalence as \( \psi(\pi_1,\cdots,\pi_2) \). (That is, this is a smooth map picking out critical points for the end-perturbations \( \pi_i \). This entire discussion depends only on the connections over the ends.)

For this chart \( U \), there are two possible situations: it could be that there are no reducible connections over the cobordism restricting to the configuration \( \psi(\pi_1,0,0,\pi_2) \) on the ends, in which case \( \mathcal{P}_{L,\delta} \mathcal{B}_{\eta,k,\delta} \) is empty above this chart and is tautologically a manifold.

More interestingly, if there is some reducible connection extending that configuration, then may choose a smooth map \( e : \mathcal{A}_{\eta_1,k} \times \mathcal{A}_{\eta_2,k} \to \mathcal{A}_{\eta,k,\delta}^{(4)} \) so that \( e(A_1,A_2) \) is a connection \( A \) on \( \eta \) so that \( A \) is constant and equal to \( A_1 \) on the ends. Composing these, \( ep : \mathcal{P}_{L,\delta}^{(4)} \to \mathcal{A}_{\eta,k,\delta}^{(4)} \) is a smooth map choosing a connection \( ep(\pi) \) which is constant and equal to the \( \eta \)-critical connections \( p(\pi) \) on the ends.

We may use this to define

\[
U\mathcal{A}_{\eta,k,\delta}^{(4)} = \{ (\pi,A) \mid \pi \in U, A - ep(\pi) \in \mathcal{L}_{k,\delta}^2(W;i\mathbb{R}) \}.
\]

In particular, this Banach manifold is diffeomorphic to \( U \times \mathcal{L}_{k,\delta}^2(W;i\mathbb{R}) \). Clearly it comes equipped with a smooth projection to \( U \). There is the trivial bundle \( \mathcal{S}_{k-1,\delta} = \mathcal{L}_{k-1,\delta}^2(W;i\mathbb{R}) \) over \( U\mathcal{A}_{\eta,k,\delta}^{(4)} \). Given a connection \( A \) on \( \eta \), the induced connection on \( \eta \oplus (\lambda \otimes \eta^{-1}) \) has curvature equal to \( 2F_A - F_{A_0} \), the non-central part of the curvature of \( \eta \oplus (\lambda \otimes \eta^{-1}) \). Then the section \( s : U\mathcal{A}_{\eta,k,\delta}^{(4)} \to \mathcal{S}_{k-1,\delta} \) is given by

\[
(s,\pi,A) \mapsto 2F_A - F_{A_0} + \nabla_x(A).
\]

This space a smooth action by the Banach Lie group \( \mathcal{G}_{\eta,k+1,\delta} \), preserving the projection map to \( \mathcal{P}_{L,\delta}^{(4)} \) and the section \( s \). While the action is not free, the stabilizer is the same at every point: it is the group of harmonic maps \( W \to S^1 \), which are all constant maps as \( b^i(W) = 0 \). The action of \( \mathcal{G}_{\eta,k+1,\delta} \) factors through its quotient by the subgroup of constant maps, and the action of this quotient group is free. The fact that ‘\( A \)-harmonic gauge transformations’ are the same for all \( A \) is a convenient and unique aspect of the case of \( SO(2) \)-bundles.

Now it is easy to verify that this action is proper, such that each orbit has closed complemented tangent space, giving us a quotient manifold \( \mathcal{U} \mathcal{B}_{\eta,k,\delta} \) with a projection map and a section of a vector bundle \( \mathcal{S}_{k-1,\delta} \) (still the trivial bundle). That the section on \( \mathcal{U} \mathcal{B}_{\eta,k,\delta} \) is Fredholm follows from the same fact fiberwise, on a fixed Banach manifold \( \mathcal{B}_{\eta,k,\delta}(\alpha_1,\alpha_2) \). Patching these together for different charts \( U \) of \( \mathcal{P}_{L,\delta} \mathcal{C}_\eta \) gives us a smooth Banach manifold structure on \( \mathcal{P}_{L,\delta} \mathcal{B}_{\eta,k,\delta} \). □

Note that this Banach manifold is precisely the same as a component of the \( SO(2) \)-fixed point space of \( \mathcal{P}_{L,\delta} \mathcal{B}_{\eta,k,\delta} \), expressed in terms of the \( U(2) \) model. There is similarly no difficulty in showing that \( \mathcal{P}_{\delta} \mathcal{B}_{\eta,k,\delta} \) carries the natural structure of a smooth Banach manifold.

Choose a continuous map \( C_\eta : \mathcal{P}_{L,\delta}^{(4)} \to \mathbb{R}_+ \), choosing for each perturbation \( \pi \) a constant for which there are no reducible instantons of topological type \( \eta \) of energy
This is possible because the map from $\mathcal{P}_{L,\delta}^{(4)}$ to the space of discrete closed subsets of $\mathbb{R}_+$, sending $\pi$ to the possible energies of $\eta$-instantons, is continuous.

One may take $C_\eta$ to be arbitrarily large: given any $C_\eta$ and any $N > 0$ we may find a $C'_\eta$ so that $C'_\eta > C_\eta$ and $C'_\eta > N$. We write $\mathcal{P}_{L,\delta}^{-\delta\mathcal{C}}_{\eta,k,\delta}$ to be those pairs of configurations $(\pi, A)$ so that $\mathcal{E}_\pi(A) \leq C_\eta(\pi)$, and similarly for $< C$; further, write

$$\mathcal{P}_{L,\delta}^{-\delta\mathcal{C}}_{\eta,k,\delta}$$

for the subset $s^{-1}(0)$.

Then by our choice of $C_\eta$ we have an equality of sets

$$\mathcal{P}_{L,\delta}^{-\delta\mathcal{C}}_{\eta,k,\delta} = \mathcal{P}_{L,\delta}^{-\delta\mathcal{C}}_{\eta,k,\delta}.$$

This provides us with the following compactness theorem for reducible trajectories. Note that there are no broken trajectories in the following statement.

**Lemma 4.18.** For $C_\eta$ as above, the map $\mathcal{P}_{\delta}^{-\delta\mathcal{C}}_{\eta,k,\delta} \rightarrow \mathcal{P}_{L,\delta}^{(4)}$ is proper.

**Proof.** This is an application of Proposition 4.10. First ignore the energy constraint. Given a sequence $(\pi_n, A_n)$ of zeroes of $s$ such that $\pi_n \rightarrow \pi$ converges, there is a subsequence of $A_n$ and a broken trajectory $A$ such that $[A_n] \rightarrow [A]$ away from a finite set of points; if $A_n$ was labelled by the component $z$, then the composite trajectory of $A$ is labelled by the component $z - m$, where $m$ is the number of points where energy was lost. However, for each line bundle $\eta$ there is a unique reducible component (and hence all $\eta$-reducibles lie in the same component of connections) by Proposition 1.7. Because a sequence of reducible connections $A_n$ can only converge to a reducible connection, there must therefore be no loss in energy.

So the only noncompactness can arise from breaking of trajectories. We already know by Proposition 4.14 that on the cylinder $\mathbb{R} \times Y$, where $Y$ is equipped with a weakly admissible bundle and regular perturbation, reducible trajectories are constant. So after passing to a subsequence, the configuration $A_n$ converges to an honest reducible connection on $\eta$ with the same limits at $\pm \infty$, as desired.

Now if $(A_n, \pi_n)$ has energy at most $C_\eta(\pi_n)$, then $(A, \pi)$ has energy at most $C_\eta(\pi)$ by continuity. But by definition no reducible $\pi$-instanton has energy $C_\eta(\pi)$, so we may replace these inequalities $\mathcal{E}_{\pi}(A) \leq C_\eta(\pi_n)$ with a strict inequality, as desired.

The following unique continuation lemma is quoted essentially verbatim from [KM07, Lemma 7.1.3]; we will use it frequently.

**Lemma 4.19.** Let $H$ be a real Hilbert space and $I = [t_1, t_2]$ a closed interval, equipped with a family $L(t)$ of unbounded operators $D \rightarrow H$ with common dense domain $D \subset H$, so that $L(t) = L_+(t) + L_-(t)$ for $L_+$ self-adjoint and $L_-(t)$ skew-adjoint and bounded. Further suppose that the time-derivative $L'(t)$ is a well-defined operator $D \rightarrow H$, defined pointwise, which has a bound

$$\|L'(t)x\| \leq C_1(\|L(t)x\| + \|x\|),$$

uniform in $t \in I$ and $x \in D$.

Let $f : I \rightarrow H$ be a continuous map and $z : I \rightarrow D$ be a solution of the equation $z' + Lz = f$, where we have the bound on the inhomogeneous term $\|f(t)\| \leq C_2 \|z(t)\|$, uniform in $t$. Then if $z(t) = 0$ for some $t \in I$, then $z$ is identically zero.
As a warm-up to what follows, we will prove that if the bundle $\mathbf{E}$ is trivial, then fully reducible connections are cut out transversely. Uniformly, we write $\mathcal{P}_{L,\delta}^{(4),'}$ for the subspace of $\mathcal{P}_{L,\delta}^{(4)}$ so that if $\eta$ supports a fully reducible connection, it is cut out transversely in $\mathcal{F}_{E,k,\delta}'$. This notion is vacuous when $\mathbf{E}$ is nontrivial; in that case $\mathcal{P}_{L,\delta}^{(4),'} = \mathcal{P}_{L,\delta}^{(4)}$.

**Lemma 4.20.** When $b^+(W) = 0$ or $\mathbf{E}$ is nontrivial, the subspace $\mathcal{P}_{L,\delta}^{(4),'}$ is an open dense subset of $\mathcal{P}_{L,\delta}^{(4)}$, and similarly for $\mathcal{P}_{\delta}^{(4),'} \subset \mathcal{P}_{\delta}^{(4)}$.

**Proof.** This is proved entirely analogously to Theorem 3.6. Let $\mathbf{A}$ be a fully reducible connection equipped with a perturbation $\pi$ for which $d^+ + DA\nabla_{\pi}$ is not surjective.

First, we argue abstractly: given a Fredholm operator $Q : V \to W$ of Hilbert spaces, use the orthogonal decompositions $V = V_0 \oplus V_1$ and $W = W_0 \oplus W_1$, where $V_0 = \ker(Q)$ and $W_1 = \operatorname{im}(Q)$, if one may find a bounded operator $K : V \to W$ with

$$\Pi_{W_0}K|_{V_0} = K_{00}$$

surjective; then $Q + tK$ is surjective for some sufficiently small $t$. It suffices to prove the dual claim about non-injective maps by passing to the adjoint; then the assumption is instead that $K_{00}$ is injective. The proof is the same as Lemma 3.7.

Now the claim is that for $Q = d^+ + DA\nabla_{\pi}$, we may find such a $K$ as above.

Here the component $K_{00}\ker(Q) \to \ker(Q^*)$ is defined by inclusion, applying $K$, and then applying the $L^2$ projection onto $\ker(Q^*) \subset L^2_{k-1,-\delta}$ (where this subset in fact resides in $L^2_{2,\delta}$ by elliptic regularity).

On the subset $[1,2] \times Y$ of the cylindrical ends, we write $\hat{\nabla}_{\pi} = (dt \wedge \nabla_{\pi(t)})^+$ for some smoothly varying family $\pi(t)$. Then elements of the kernel of $Q$ satisfy the differential equation

$$\sigma'(t) + *d\sigma(t) + DA(t)\nabla_{\pi(t)}\sigma(t) = 0,$$

and similarly for elements of the cokernel, replacing the linear operator $*d + DA(t)\nabla_{\pi(t)}$ with its negative. This is a homogeneous differential equation with the linear operator $L$ being symmetric, and its time-derivative is the time-derivative of $DA(t)\nabla_{\pi(t)}$; this is a $C^1$ family of bounded operators so long as $\mathbf{A}$ is of Sobolev class $L^2_k$ with $k > 4$, and in particular, the time-derivative is a $C^0$ family of operators. For $\mathbf{A}$ a fully reducible connection, we may pick a smooth representative if we like.

Therefore the restriction maps to time-slices $\ker(Q) \to \ker(d^{\ast}_{A(t)}) \hookrightarrow \operatorname{coker}(Q)$ are both injective (though it is not clear that they are jointly injective). Call their images $F_k$ and $F_c$, respectively; these are finite-dimensional $SO(3)$-invariant subspaces of $\ker(d^{\ast}_{A(t)})$. Because $SU(2)$-cylinder functions are dense in $SO(3)$-invariant functions on any compact submanifold of $\mathcal{F}_{E,k}$, we may in particular choose a holonomy perturbation so that $DA(t)\nabla_{\pi(t)}$ restricts arbitrarily close to any chosen $SO(3)$-equivariant self-adjoint map $T : F_k + F_c \to F_k + F_c$.

We want to choose one so that the composite $\pi_*T_{lk} : F_k \to F_c$, given by inclusion, applying $T$, and projecting onto $F_c$, is a surjective equivariant map. We start by separating this into three spaces: $F_{kc} = F_k \cap F_c$ and $F^\perp_k \cong (F_k + F_c)/F_c$ and similarly $F^\perp_c \cong (F_k + F_c)/F_k$.
Because the $SO(3)$-representation $\Omega^1(W; g)$ is isomorphic as $SO(3)$-Hilbert spaces to $\ell^2 \otimes g$, and the only irreducible representations arising here are $g$ itself, we see that each of $F_{kc}, F_{c}^k, F_{c}^s$ are isomorphic to a direct sum of some number of copies of $g$.

Recall Definition 4.3: there is a sequence of inclusions of open sets $\mathcal{P}_{E_i, \delta} \subset U_c \subset \mathcal{P}_{E_i}$, where the set $U_c$ is a connected open set containing 0, for which $D_{A, \pi}$ is surjective for any fully reducible connection $A$ on $E_i$ and $\pi \in U_c$. In particular, this implies via a spectral flow argument that the index of $D_{A, \pi}$ is the same as the index of $D_{A, 0}$, and hence is equal to $3(b^+ - b^1)$.

Therefore, we find that if we have $F_{kc} \cong \mathfrak{g}^a$ and $F_{c}^k \cong \mathfrak{g}^c$, then from the index calculation we must have $F_{c}^k = \mathfrak{g}^{c+b^1-b^*}$. In particular, we may choose the injective, self-adjoint equivariant map on $F_k + F_c$ which is uniquely defined by the properties that it is the identity on $F_{kc}$ and a fixed equivariant surjection $\mathfrak{g}^{c+b^1-b^*} \to \mathfrak{g}^c$ on $F_{c}^k \to F_{c}^2$. In particular, choosing $\pi'$ so that the restriction of $D_{A(t)} \nabla_{\tau'}$ to this locus is sufficiently close to the chosen map, we have constructed an operator so that $K_{0,0}$ is injective on this time-slice.

So suppose that $\sigma$ and $\psi$ are chosen so that $D_{A(3/2)} \nabla_{\tau'} \sigma(3/2)$ projects very nearly to $\psi(3/2)$. Then this remains true for $\sigma(s)$ and $\psi(s)$ for $s \in (3/2 - \epsilon, 3/2 + \epsilon)$ for some $\epsilon$ sufficiently small; in particular, $\langle \beta D_{A(t)} \nabla_{\tau'} \sigma, \psi \rangle_{L^2} > 0$, where $\beta$ is a bump function chosen to have sufficiently small support near $3/2$. We may make this choice of $\beta$ uniformly for all $\psi$ by varying the choice of $\psi$ over the unit sphere (and picking out the appropriate $\sigma_\psi$ continuously), and in particular, so that the map $\Pi_{L^2} D_{A(\psi)} \nabla_{\tau'} : \text{ker}(Q) \to \text{ker}(Q^*)$ is injective. This is precisely what we wanted.

Therefore, for $\epsilon \pi'$ sufficiently small, the operator $Q_{A, \pi + \epsilon \tau'}$ is injective. So the set of points where each fully reducible connections are cut out transversely is a dense open set of $\mathcal{P}^{(4)}_{L, c}$, and because there are only finitely many full reducibles, the same is true for the set where of perturbations where all full reducibles are cut out transversely.

The construction here only used perturbations supported on a compact piece of the collars, so the argument applies perfectly well to the spaces $\mathcal{P}_{L, c}$.

When $b_1(W) = 0$ and $b^+(W) > 0$ but $E$ is trivial, the set $\mathcal{P}^{(4)}_{L, \delta} \setminus \mathcal{P}^{(4)}_{L, \delta}$ is empty: full reducibles are always $\pi$-perturbed ASD connections, regardless of $\pi$, but the index of $D_{A, \pi}$ is always negative, so they cannot be cut out transversely.

Lemma 4.21. Let $(W, E)$ be a Riemannian manifold with two cylindrical ends, both of which are modelled on rational homology spheres. Suppose either that $b^+(W) = 0$ or that $E$ is nontrivial, and fix a component of the $SO(2)$-fixed point space, labeled by a line bundle $\eta$.

If the component $\eta$ does not support any $\pi$-flat instantons which restrict trivially to both ends for any $\pi$, then the zero set

$$\mathcal{P}^{(4)}_{\delta} \mathcal{M}^{<C}_{\eta} \subset \mathcal{P}^{(4)}_{\delta} E^{<C}_{\eta, \delta}$$

is a smooth Banach submanifold. (This holds, for instance, if either of the $E_i$ is nontrivial.)

Otherwise, suppose $b^+(W) = 0$. For any compact set $K \subset \mathcal{P}^{(4)}_{\delta}$, there is a finite set $L$ of holonomy perturbations adapted to a collection of thickened loops so that
the corresponding zero set
\[ \mathcal{P}_{L,\delta} \mathcal{M}_\eta^{<C} \subset \mathcal{P}_{L,\delta} \mathcal{B}_{\eta,k,\delta}^{<C} \]
is a smooth Banach submanifold in a neighborhood of \( K \). That is, there is a neighborhood \( K \subset U_K \subset \mathcal{P}_{L,\delta}^{(4)} \) so that if \( p : \mathcal{P}_{L,\delta} \mathcal{B}_{\eta,k,\delta}^{<C} \to \mathcal{P}_{L,\delta}^{(4)} \) is the projection, then \( p^{-1}(U_K) \cap \mathcal{P}_{L,\delta} \mathcal{M}_\eta^{<C} \) is a smooth Banach submanifold of \( p^{-1}(U_K) \); we will write this Banach submanifold as \( U_K \mathcal{M}_\eta^{<C} \).

**Proof.** By the assumption that fully reducible connections are cut out transversely for all \( \pi \in \mathcal{P}_\delta \), we only need to show this for nontrivial connections on \( \eta \).

We first consider the case for which \( \mathbf{A} \) does not restrict to the trivial connection on both ends (which encompasses the case in which at least one \( E_i \) is nontrivial); consider this as an outgoing end and call this \( (Y,E) \) for uniformity of notation. We write \( \pi(t) \) for the smoothly varying, time-dependent perturbation on the ends for which
\[ \hat{\nabla}_{\pi}(\mathbf{A}) = (dt \wedge \nabla_{\pi(t)}(\mathbf{A}(t)))^+. \]

We know that on the end \([0,\infty) \times Y\), the time-dependent connection \( \mathbf{A}(t) \) satisfies the ODE \( \mathbf{A}'(t) = -* F_{\mathbf{A}(t)} - \nabla_{\pi}(\mathbf{A}(t)) \). Writing \( \alpha = \mathbf{A}(s) \) for some fixed \( s \), and then \( \mathbf{A}(s + t) = \alpha + a(t) \) for sufficiently small \( |t| \) and \( a(t) \in \ker(d^*) \). Because in this abelian setting \( F_{\alpha + a(t)} = F_{\alpha} + d_{\alpha}a(t) \), the ODE reduces to
\[ a'(t) + d_{\alpha}a(t) = -* F_{\alpha} - \nabla_{\pi(t)}(\alpha + a(t)). \]

We first wish to show that if two solutions \( \mathbf{A} \) and \( \mathbf{A}' \) are gauge equivalent at the time-slice \( t_0 = s \), that they are equivalent in a neighborhood of \( s \), and similarly, either the tangent vector \( a'(t) \) is nonzero everywhere on this neighborhood or is zero everywhere. For the first, after applying a gauge transformation to \( \mathbf{A} \) and \( \mathbf{A}' \), we may write \( \alpha + a_0(t) \) and \( \alpha + a_1(t) \) as solutions to the given ODE, where \( a_0(s) = a_1(s) = 0 \).

Write \( f(t) = a_0(t) - a_1(t) \). Then subtracting the ODEs we have
\[ f'(t) + d_{\alpha}f(t) = -\left( \nabla_{\pi(t)}(\alpha + a_1(t)) - \nabla_{\pi(t)}(\alpha + a_2(t)) \right). \]
Writing
\[ g(t) = \nabla_{\pi(t)}(\alpha + v), \]
we know that \( g \) is uniformly continuous on a given compact interval by continuity of \( \pi(t) \) (as a map to the space of perturbations) and \( \nabla \) (as a map from the space of perturbations to the space of smooth maps). Therefore, for \( t \) sufficiently close to \( s \) (where \( a_0(s) = a_1(s) = 0 \)), we have
\[ \| \nabla_{\pi(t)}(\alpha + a_1(t)) - \nabla_{\pi(t)}(\alpha + a_2(t)) \| \leq C \| a_1(t) - a_2(t) \|. \]
This means that the defining ODE for \( f \) satisfies the conditions of the Lemma 4.19 (in particular, the differential operator is time-independent), and therefore \( f(s) = 0 \) implies that \( f(t) = 0 \) for all \( t \) sufficiently close to \( s \). Continuing this way along the entire cylinder, we find that \( \mathbf{A} \) is gauge equivalent to \( \mathbf{A}' \) along the end.

Similarly, the defining ODE for \( a'(t) \) satisfies the unique continuation lemma; when arguing this, one needs to use that \( \pi(t) \) is a \( C^1 \) path for the appropriate uniform continuity claim above, and that \( D_{\mathbf{A}(t)}\nabla_{\pi(t)} \) is a \( C^1 \) family of bounded operators. Therefore either \( \mathbf{A}(t) \) may either be taken to be constant after a gauge transformation or defines an immersed path in \( \mathcal{B}_{\eta,k-1/2} \).
Now consider the adjoint operator, \( \Omega^2_{k,\delta}(W; i\mathbb{R}) \to \Omega^1_{k-1,\delta} \). Rewriting the restriction of \( \Omega^2_{E,\delta}(W; i\mathbb{R}) \) to an an end diffeomorphic to \( I \times Y \) for some interval \( I \), as time-dependent 1-forms on \( Y \) via \( \psi(t) \to (dt \wedge \psi(t))^+ \), the adjoint operator (whose kernel defines the cokernel of our original operator) is given by

\[
\psi(t) \to (d^*\psi(t), \psi'(t) - *d\psi(t) - D_{A(t)}\nabla_{\pi(t)}\psi(t)).
\]

Just as before, zeroes of this operator satisfy the unique continuation lemma. Therefore, if an element \( \psi \) of the cokernel vanishes on some time-slice, it vanishes in a neighborhood thereof; applying [DK90, Corollary 4.3.23], which says that a closed self-dual 2-form that vanishes on a nonempty open set vanishes everywhere, we see that \( \psi = 0 \). In particular the restriction of some nonzero \( \psi \) to any time-slice is nontrivial.

In the case that the path \( A(t) : (0, \infty) \to B_{\eta,k,\delta} \) is embedded, consider the piece contained \([1, 2]\). By the above paragraph, we may consider each cokernel element as a vector field on the curve \( A(t) \).

Lemma 3.2 guarantees that we may choose a holonomy perturbation \( \pi \) approximating any given smooth \( SO(3) \)-invariant function on a compact subset of \( B^E \). The \( SO(2) \)-fixed point set still carries the action of the Weyl group, \( \mathbb{Z}/2 \cong N_{SO(3)}(SO(2))/SO(2) \); denote the nontrivial element as \( i \). Given any compact submanifold \( M \subset B_\eta \), if \( M \cap i(M) = \emptyset \) (and in particular, \( M \) contains no fully reducible critical orbits), we may approximate any given smooth function on \( M \).

Each \( A(t) \) is \( SO(2) \)-reducible and so \( i(A(t)) \neq A(t) \); furthermore, if we had \( i(A(s)) = A(t) \) for any \( s > t \), by the unique continuation property we see that \( A(t) = A(2s - t) \), which contradicts the claim that \( A(t) \) is embedded. So we may choose a holonomy perturbation \( \pi \) so that \( \nabla_{\pi}(A(t)) \) is very close to the given path of tangent vectors over any compact subset of the image of \( A(t) \) in \( B_{\eta,k-1/2} \).

We weight this by some bump function supported in \([1, 2]\) to give an element \( \nabla_{\pi}^{(3)} \) of \( P^1_{E} \), the space of the compactly supported perturbations on this end. Thus \( \int_{[\omega, \nabla_{\pi}^{(3)}A(t)]} > 0 \), as desired.

If \( A(t) \) is constant but not a fully reducible connection, little changes. In this case, \( F_A + \nabla_{\pi}(A) = 0 \) on the end, even as we enter the region in which the perturbation is zero. On that region, this is a closed, self-dual 2-form which vanishes on an open set, and hence is globally zero along the interior of \( W \), and thus this vanishes everywhere. So this is the \( \pi \)-flat case.

Because the \( \nabla_{\pi}(A(t)) \) is constant in \( t \), we cannot approximate the tangent vector across the whole interval as above. However, as a cokernel element \( \psi \) still restricts nontrivially to each time-slice, we may fix some time-slice \( \psi(t_0) \) and pick a perturbation so that \( \nabla_{\pi}(A(t_0)) \) is very close to \( \psi(t) \). For \( \beta \) a bump function with very small support containing \( t_0 \), we take

\[
\nabla_{\pi}(A) = \beta(t) (dt \wedge \nabla_{\pi}(A(s)))^+.
\]

Then we still have

\[
\langle \nabla_{\pi}(A), \psi \rangle > 0,
\]

as desired.

This concludes the argument in the case that there are no \( \pi \)-flat connections that restrict trivially to both ends. In the case that there are, however, the restriction of any perturbation supported in the ends whatsoever is trivial at this \( \pi \)-flat connection: if \( \nabla_{\pi} \) is the gradient of an \( SU(2) \)-holonomy perturbation, then
necessarily $\nabla_\pi$, considered as a vector field on $\tilde{B}_E$, is invariant under the action of $SO(3)$ on framings. Therefore, the action of the Weyl group on $SO(2)$-fixed points leaves the holonomy perturbation invariant; since this action has differential $-1$ at the full reducibles, we see that the gradient vector field must be invariant under negation and hence must be zero.

This implies that the sets of $\pi$-flat connections which restrict trivially to both ends is independent of $\pi$. This in mind, we set $\pi = 0$ and speak of these as flat connections. Furthermore, this space is compact, as is any space of flat connections of fixed topological type.

At a fixed flat connection $A$, choose some loop in the complement of the ends of $W$ along which $A$ restricts nontrivially. Choosing a sufficiently small $\epsilon$ so that the $\epsilon$-neighborhood of this loop is a disc bundle, we work with holonomy perturbations on a neighborhood of $S_{\epsilon/2}(\gamma)$, as in the definition of holonomy perturbations adapted to a thickened loop. Because on this neighborhood $A(t)$ is no longer fully reducible, we may argue as in the above discussion that some finite set $L$ of perturbation adapted to this thickened loop give a surjective map $V_L \to \text{coker}(D_A)$. Now choose a finite collection of disjoint loops so that every flat connection on $W$ (of this topological type) restricts nontrivially to one of these loops, and choose $\epsilon$ so small for each loop that these give a collection of thickened loops (that is, the $\epsilon$-neighborhoods are all disc bundles and all disjoint from one another).

Given a compact set $K \subset \mathcal{P}_\delta^{(4)}$, then we may choose a finite set of perturbations adapted to each of these thickened loops so that, taking $L$ to contain all of these perturbations, the map $V_L \to \text{coker}(D_A)$ is surjective for all flat $A$ and all $\pi \in K$. Here we use the compactness of the space of flat connections in an essential way.

Making such a choice, we have completed the proof.

Proof of Propositions 4.15 and 4.16. We first assume that the component $\eta$ contains no flat connections which restrict trivially to the ends. Then the set $\mathcal{P}_\delta \mathcal{M}_\eta$ is a smooth submanifold of $\mathcal{P}_\delta \mathcal{B}_{\eta,k,\delta}$. For fixed $\pi$, the set of all $\pi$-instantons $A$ are cut out transversely for all in $\mathcal{B}_{\eta,k,\delta}$ if the projection

$$p_\eta : \mathcal{P}_\delta \mathcal{M}_\eta \to \mathcal{P}_\delta^{(4)}$$

has $\pi$ as a regular value. By the Sard-Smale theorem, such $\pi$ are in large supply: they form a dense set.

Furthermore, the projection $p_\eta$ is proper, as a special case of Lemma 4.18. Therefore, the regular values form an open set.

Now suppose that $\eta$ is a component that does support flat connections which are trivial on the ends. By Lemma 4.21, we see that for any compact subset $K \subset \mathcal{P}_\delta^{(4)}$, there is a finite set $L$ of interior holonomy perturbations adapted to a collection of thickened loops, and a neighborhood $U_K \subset \mathcal{P}_\delta^{(4)}$ so that $U_K \mathcal{M}_\eta^{<C} \to \mathcal{P}_\delta^{(4)}$ is a smooth manifold. The projection $U_K \mathcal{M}_\eta^{<C} \to U_K$ is proper, again by Lemma 4.18, and therefore the set of regular values of this projection in $U_K$ form an open dense set, as desired. Together with the stronger statement of Lemma 4.18 that at most finitely many $\eta$ support instantons with energy bounded above by $C$, we may take the intersection of these open sets for the finitely many $\eta$ that are relevant. This gives the open dense set of regular perturbations of Proposition 4.15.
In the case that $b_1(W) = 0$ and $b^+(W) > 0$, this means that $\mathcal{M}_{\eta, \pi} = \emptyset$ for \( \pi \) a regular value, as the index of the operator $D_{A, \pi}$ is negative. In the case that $b_1(W) = b^+(W) = 0$, the moduli space $\mathcal{M}_{\eta, \pi}$ for regular $\pi$ is a finite set.

What remains is to see what the solutions are when $\pi = 0$; then by properness, we immediately understand the solutions for all sufficiently small $\pi$. But for a cobordism $(W, E)$ with $b^+(W) = 0$ but $b_1(W) = n$, the space of reducible ASD connections in each reducible component is the torus $T^n$; this is [Dae15, Lemma 1.6], and again follows from Hodge theory, as the ASD equation is affine in the $SO(2)$-reducible case. See also the related [Frø02, Lemma 2]. In the case we most care about, $b_1(W) = b^+(W) = 0$, this means that there is a unique $\pi$-perturbed reducible ASD connection in each reducible component, so long as $\pi$ is sufficiently small.

and so $A$ is always a $\pi$-perturbed ASD connection; but the index of $D_{A, \pi}$ is the same as the index of $D_{A, 0}$, and hence is negative. So $D_{A, \pi}$ always has nontrivial cokernel.

4.5. **Index calculations.** With a strong grasp on the regularity properties of reducible instantons (internal to the reducible locus), we move on to understanding the index of the ASD operator normal to the reducible locus.

Let $(W, E)$ be a cobordism from $(Y_1, E_1)$ to $(Y_2, E_2)$, thought of as a manifold with cylindrical ends, and let $\pi$ be a perturbation so that the ends $\pi_{\pm}$ are regular perturbations on $Y_i$. Suppose $\delta$ is larger than the least nonzero eigenvalue of the extended Hessian operators of the $\pi_{\pm}$-critical points. Choose $\pi_{\pm}$ sufficiently small that Proposition 3.4 holds.

If $A$ is a $\pi$-perturbed instanton, we would like to compute the index of

$$Q_{A, \pi} : \Omega^1_{k, \delta}(W; \mathfrak{g}E) \to \Omega^2_{k-1, \delta} \oplus \Omega^0_{k-1, \delta}.$$  

In fact, by invariance properties of the index, this only depends on the component $A$ sits inside.

The most useful tool for computing the index of a differential operator on a compact manifold with boundary is the Atiyah-Patodi-Singer index theorem. Let $X$ be a compact manifold with boundary, with metric of product type near the boundary. Let $D$ be an elliptic differential operator acting on sections of bundles $V_1, V_2$ over $W'$ (thought of as a compact manifold with boundary) with specified isomorphisms near the boundary components $[0, \infty) \times Y$ from the given operator to $\frac{\partial}{\partial t} + A$, where $A$ is a self-adjoint elliptic operator whose nonzero eigenvalues $c$ have $|c| > \delta$. Then one may consider $D$ as an operator $D_{APS} : L^2_1(X, V_1; P) \to L^2_2(X, V_2)$, where here $P$ means that we demand that the restriction of $\sigma \in L^2_2(W, V_1; P)$ to the boundary lies in the subspace spanned by the negative eigenvalues; that is, if $P$ is the projection operator corresponding to the eigenspaces with $\lambda \geq 0$, we demand that $P\sigma = 0$. Then even if $A$, the operator at the boundary, has kernel, $D_{APS}$ is a Fredholm operator with a well-defined index. If $X$ is $X \cup_{\partial X} [0, \infty) \times \partial X$ with the appropriate product metric on the ends, we may just as well consider $D$ as an operator on $\tilde{X}$. We write $I(D)$ to mean the index of $D$, computed as a map between weighted Sobolev spaces of constant $\delta > 0$, and $I_{APS}(D)$ to mean the index computed as $D_{APS}$.

**Lemma 4.22.** As long as $\delta$ is less than any nonzero eigenvalue of the boundary operators $A_i$, we have $I_{APS}(D) = I(D)$. 

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Proof. This is essentially [APS75a, Proposition 3.11]. When thought of as a map on weighted Sobolev spaces, \( \text{ker}(D) \) clearly consists of \( L^2 \) solutions of \( Df \) on \( \mathcal{X} \). Conversely, every \( L^2 \) solution has as much smoothness as the operator itself, and in the asymptotic expansion \( \sum a_\lambda e^{-t\lambda} \phi_\lambda \) on the ends, \( a_\lambda \) must be zero when \( \lambda \leq 0 \) if the solution is to be \( L^2 \). So a solution decays exponentially with weight at least \( \delta > 0 \), where \( \delta \) is less than any positive eigenvalue of \( A \).

The kernel of \( D^* \) is computed in weighted Sobolev spaces as a subspace of \( L^2_{k,-\delta} \). To decay exponentially slower than \( e^{-\delta t} \), we must demand that \( a_\lambda = 0 \) for \( \lambda \geq \delta \).

Because positive eigenvalues are at least as large as \( \delta \), we see that a solution is, on the ends, a sum of an exponentially decaying solution and a solution which is constant in time. This is precisely the kernel of \( D^* \) on extended \( L^2 \) sections, as in the computation of \( I(D_{\text{APS}}) \) given in [APS75a, Proposition 3.11]. Because the two kernels and cokernels agree, we have \( I_{\text{APS}}(D) = I(D) \).

So it suffices to compute \( I_{\text{APS}}(Q_{A,r}) \), and for this, we have the Atiyah-Singer index theorem. We will need some simple lemmas first before we can calculate this index.

Lemma 4.23. Suppose \( W \) is a compact manifold with boundary of product type, equipped with an elliptic operator \( D \) that is of product type near the boundary. Suppose further that there is an oriented closed submanifold \( Y \subset W \) with a neighborhood of product type so that the operator may be written as \( d/dt + A \) on this neighborhood.

Write \( D_1 \) for the operator on the compact manifold whose positive boundary contains \( Y \), and \( D_2 \) for the operator on the compact manifold whose negative boundary contains \( Y \); say \( A \) is the operator at \( Y \). Then

\[
I_{\text{APS}}(D) = I_{\text{APS}}(D_1) + I_{\text{APS}}(D_2) + \dim \text{ker}(A).
\]

Proof. This follows immediately from the index theorem itself, [APS75a, Theorem 3.10]. The only term which is not additive is \( -h/2 \), where \( h \) is the kernel of the boundary operator. Because we are contributing two extra copies of \( -\dim \text{ker}(A)/2 \) on the right, we counterbalance that by adding \( \dim \text{ker}(A) \).

Finally, we will need to know how the index of spectral flow is computed.

Lemma 4.24. Let \( D = d/dt + A_t \) be an operator on \([0,1] \times Y\), where \( A_t \) is a time-dependent self-adjoint elliptic operator, possibly with kernel. We define the spectral flow \( \text{sf}(A_t) \) to be the intersection number of the graph of the spectra with the line \( \lambda(A_t) = -\delta \), which \( A_0 \) and \( A_1 \) do not intersect. This is the aggregate number of eigenvalues that go from \( \leq -\delta \) to \( \geq 0 \), counted with sign.

Then \( I_{\text{APS}}(D) = \text{sf}(A_t) - \dim \text{ker}(A_t) \).

Proof. This may be proved using separation of variables. A similar formula is stated below [APS96, Theorem 7.4], only giving the spectral flow term; their argument uses the periodic boundary conditions on \([0,1] \times Y\), which corresponds to the projection to nonnegative eigenvalues at \( t = 0 \) and positive eigenvalues at \( t = 1 \), whereas the APS boundary conditions stated above use the spectral projection to nonnegative eigenvalues at \( t = 1 \). The second operator has smaller domain, of codimension \( \dim \text{ker}(A_t) \), having included the demand that an element \( f \) of the domain projects nontrivially to \( \text{ker}(A_t) \). Because index is additive under composition and the inclusion of this subspace has index \( -\dim \text{ker}(A_t) \), the theorem follows.
Now $W$ may be decomposed as the union of a compact manifold $X$ and the two cylindrical ends. This decomposition will provide the desired calculation. Before stating the result, we recall the definition of one of the terms that will appear.

**Definition 4.10.** Let $\alpha$ be an orthogonal (resp. unitary) flat connection on a real (complex) vector bundle $E$ of dimension $n$ over a closed manifold $Y$. The Atiyah-Patodi-Singer $\rho$-invariant of $\alpha$ is defined to be $\eta_\alpha(0) - n \cdot \eta_\alpha(0)$. Here $\eta_\alpha$ is the Atiyah-Patodi-Singer $\eta$ invariant associated to the $\alpha$-twisted signature operator $\Omega^v(W, E) \to \Omega^v(W, E)$, as is studied in [APS75b]. This constant is denoted $\rho(\alpha)$.

If $E$ has a reduction labelled by $(\zeta_1, \zeta_2)$, where $\zeta_i \in H^2(W; \mathbb{Z})$ and $\zeta_1 + \zeta_2 = \lambda$, we define $\rho((\zeta_1, \zeta_2))$ to be the $\rho$-invariant of the induced $\text{SO}(3)$-flat connection on $E$, given as $\mathbb{R} \oplus (\zeta_1 \otimes \zeta_2^{-1})$.

The following equivalent computations of the $\rho$-invariant given here are listed in the discussion in [HK11, Section 2.9]. We state them without proof.

**Lemma 4.25.** Suppose $E \to Y$ is an $\text{SO}(3)$-bundle, equipped with a reduction $E \cong \mathbb{R} \oplus (\zeta_1 \otimes \zeta_2^{-1})$. Write $\kappa = \zeta_1 \otimes \zeta_2^{-1}$, thought of as a flat complex line bundle. If $z_1 = c_1(\zeta_1)$, then $\rho(z_1)$ coincides with $\rho(\kappa) + \rho(\kappa^{-1})$. Further, if $(W, E) : (Y_1, E_1) \to (Y_2, E_2)$ is a cobordism equipped with a compatible reduction of $E$ as $\mathbb{R} \oplus (\zeta_1 W \otimes (\zeta_2 W)^{-1})$. Again, write $\kappa = \zeta_1 W \otimes (\zeta_2 W)^{-1}$, write $\kappa_i$ for the restrictions to each component, and write $\zeta_i W = c_1(\zeta_i W)$. Then $\rho(\kappa_2) - \rho(\kappa_1) = \text{Sign}(W) - \text{Sign}_\kappa(W)$, where $\text{Sign}_\kappa(W)$ is the index of the twisted signature operator on $W$.

Now we calculate the Atiyah-Patodi-Singer index of the operator $Q_{A, \pi}$, where $A$ is a reducible trajectory on the 4-manifold $W$. By a homotopy, put $A$ in a form so that it is constant sufficiently far on the ends, and constant at the boundary $\{0\} \times Y$ of the ends, constant at the unique unperturbed flat connection on $Y$ in the corresponding component of reducibles. For the statement of the following theorem, recall Definition 3.3 of signature data on a pair $(Y, E)$, and in particular signature data associated to a perturbation.

**Proposition 4.26.** Suppose $(W, E) : (Y_1, E_1, \pi_1) \to (Y_2, E_2, \pi_2)$ is a cobordism (with cylindrical ends) between rational homology spheres, equipped with a perturbation $\pi \in \mathcal{P}_L^4$, which is regular on the ends. Then by Definition 4.3, the perturbations on the ends are sufficiently small that Proposition 3.4 applies.

Let $A$ be a reducible connection on $E$ which, sufficiently far on the ends, is constant and equal to a $\pi$-critical point; suppose $A$ is in the component labeled by $r \in \text{Red}(W, E)$; let $r_i \in \text{Red}(Y_i, E_i)$ be the restrictions to the ends. Write $r = \{z_1, z_2\}$ corresponding to the pair of complex line bundles $\{\eta_1, \eta_2\}$. Because the ends are rational homology spheres, we may write $z_i \in H^2(W; \mathbb{Q})$; the compactly supported cohomology ring has a cup-product with values in $\mathbb{Q}$.

Let $S(r) = 1$ if $r$ is a component of $\text{SO}(2)$-reducibles and $S(r) = 3$ if $r$ contains a full reducible.

Then

$$I_{APS}(Q_{A, \pi}) = -2(z_1 - z_2)^2 + 3(b^1 - b^+) + \frac{\rho(r_2) - \rho(r_1)}{2} + \frac{\sigma_{\pi_2}(r_2) - \sigma_{\pi_1}(r_1)}{2} - \frac{S(r_1) + S(r_2)}{2}.$$

**Proof.** Let $W_N$ be the compact submanifold of $W$ given by including the first $[0, N]$ of each end; for $N$ sufficiently large, the operator $Q_{A}^N$ is of product type near the boundary.
By splitting $W_N$ into three pieces, $[-N,0] \times Y_1 \cup X \cup [0,N] \times Y_2$, we may decompose the operator $Q_{A,\pi}$ into its pieces on these three corresponding ends. Write $Q_{\pm}$ for the pieces of $Q_{A,\pi}$ on the corresponding ends, $Q$ for the piece on $W'$, and $A_{\pm}$ the connections $A$ restricts to on $\{0\} \times Y_i$.

First we calculate $I_{APS}(Q_-)$. Write $A(t)$ for the restriction of $A$ to the negative end, where $t \in [-N,0]$. The path $A(t)$ is homotopic to a path $A_f(t)$ so that $A_f(t)$ is in the unique gauge equivalence class of $\pi(t)$-flat connection in its reducible component: This is the path used in the proof of Proposition 3.9, and our assumption that $\pi_i$ on the ends lie in the sets $P_{E,\delta}$ is to ensure this doesn’t go awry. (See the remark immediately after Definition 4.3.) In particular, because $Q_{A,\pi}$ is the operator $\frac{d}{dt} + \text{Hess}_{A(t),\pi(t)}$, this spectral flow is by definition equal to the function $2N_{\pi_i}(r_1)$ defined in that proposition. Thus by Lemma 4.24,

$$I_{APS}(Q_-) = 2N_{\pi_i}(r_1) - \dim \ker \text{Hess}_{A_1}.$$ 

A similar discussion gives

$$I_{APS}(Q_+) = -S(r_2) - 2N_{\pi_2}(r_2).$$

What remains is to apply the index theorem to $D$ on $X$. If we write $Q_\theta$ to mean the corresponding ASD operator for the trivial connection, this can be read off from [MMR94, Proposition 8.4.1] as giving

$$I_{APS}(D) - 3I_{APS}(D_\theta) = -2p_1(A) + \frac{\rho(r_2) - \rho(r_1)}{2} + 3 - \frac{h_1 + h_2}{2}.$$

Here $h_i = \dim \ker \text{Hess}_{A_i}$. A detailed computation is provided in [HK11, Proposition 2.6], but note that our sign conventions on boundary orientations and the definition of $p_1$ are the negative of theirs. Hodge theory provides the equality

$$I_{APS}(Q_\theta) = -(1 - b^1 + b^+),$$

so we obtain

$$I_{APS}(Q) = -2p_1(A) + 3(b^1 - b^+) + \frac{\rho(r_2) - \rho(r_1)}{2} - \frac{h_1 + h_2}{2}.$$

Summing over these and including boundary kernel terms as in Lemma 4.23, we obtain

$$-2p_1E + 3(b^1 - b^+) + \frac{\rho(r_2) - \rho(r_1)}{2} + 2N_{\pi_1}(r_1) - \frac{h_1}{2} + \frac{h_2}{2} - 2N_{\pi_2}(r_2) - S(r_2).$$

Now $A$ induces a reduction $E \cong \mathbb{R} \oplus (\eta_1 \oplus \eta_2^{-1})$. Pontryagin classes are preserved under stabilization, so we want to compute $p_1(\eta_1 \oplus \eta_2^{-1})$. For a complex line bundle $\zeta$, we have $p_1(\zeta) = c_1(\zeta)^2$, and considering the classes $z_i$ in $H^2_c(W; \mathbb{Q})$ corresponding to $\eta_i$, we obtain $p_1E = (z_1 - z_2)^2$.

We focus now on the last few terms. If we write $D(r_i) = \dim \mathbb{R} H^1(Y_i; \eta_1 \oplus \eta_2^{-1})$, then we have $h_1 = D(r_1) + S(r_1)$ and $h_2 = D(r_2) + S(r_2)$. Because the dimension of a vector space equipped with a nondegenerate symmetric bilinear form is the number of positive eigenvalues plus the number of negative eigenvalues, we see that $D(r) - 4N_{\pi}(r)$ is the number of positive eigenvalues less the number of negative eigenvalues, and hence

$$2N_{\pi_1}(r_1) - \frac{h_1}{2} + \frac{h_2}{2} - 2N_{\pi_2}(r_2) - S(r_2) = \frac{\sigma_{\pi_2}(r_2) - \sigma_{\pi_1}(r_1)}{2} - \frac{S(r_1) + S(r_2)}{2}.$$
The above calculation did not at all depend on the fact that the connection $A$ was reducible. In general, the same formula holds, where if the restriction of $A$ to one of the ends is irreducible, we write $S(r_1) = 0$ for that end; and the first term should be read $-2p_1(E)$, defined as a curvature integral for a connection $A$ on the compact manifold $W$ which restricts to the relevant flat connections on a neighborhood of the boundary.

Note that if we choose a different connection $A'$ that restricts to the same flat connections on the boundary, the only thing that can possibly change in the index formula is $-2p_1(E)$. Fix a base connection $A_0$, equal to the desired flat connections near the boundary. Consider the double of $W$, with $A_0$ on one half and, on the other half, an arbitrary connection $A$ restricting to the desired flat connections on $\partial W$. Clearly $-2p_1$ of this new connection on a closed manifold is $-2p_1(A_0) - 2p_1(A)$. Using the fact that $-2p_1E$ is constant mod 8 on a closed manifold — it reduces to $2w_2E^2 \in 2H^4(X; \mathbb{Z}/8) = 2\mathbb{Z}/8$ (where here we take the Pontryagin square to write $w_2^2 \in \mathbb{Z}/4$) — we immediately have the following corollary.

**Corollary 4.27.** The relative grading $\text{gr}_z(\alpha, \beta) \in \mathbb{Z}$ is independent, modulo 8, of the choice of $z$. Therefore, we may unambiguously write $\text{gr}(\alpha, \beta) \in \mathbb{Z}/8$. 

The above index calculation in mind for reducible connections, we combine the $\rho$ and signature terms into a single function.

**Definition 4.11.** Let $(Y, E)$ be a rational homology sphere equipped with an $SO(3)$-bundle and small regular perturbation $\pi$. We define the perturbed $\rho$-invariant to be $\rho_\pi(r) = \rho(r) + \sigma_\pi(r)$, where $\sigma_\pi$ is the signature datum associated to $\pi$ as in Definition 3.3.

Now if we write $D^\pi_{A, \pi}$ for the normal ASD operator, the linearization of the section defining the moduli spaces in $\mathcal{B}_E(z, k, \delta)(\alpha_1, \alpha_2)$ restricted to the normal space to an orbit, then we see by the discussion at the start of Section 4.2 that $I(D^\pi_{A, \pi}) = I_{\text{APS}}(Q^\alpha_A) + S(r)$, where $S(r) = \dim g_A$ is the dimension of the space of $A$-parallel gauge transformations. Further, at an $SO(2)$-reducible connection $A$, the normal ASD operator splits as a sum of a ‘reducible part’ and an ‘irreducible part’, corresponding to the splitting of $g_E \cong E \cong R \oplus \lambda$; write $D^\pi_{A, \pi} = D^\text{red}_{A, \pi} \oplus D^\text{irred}_{A, \pi}$; one has $I(D^\text{red}) = b^1 - b^+$, This is the component that $S(r) = 1$ contributes to, as the operator $Q^\text{irred}$ is the same as the operator $Q_\theta$ written in the above proof, and hence has index $-1 + b^1 - b^+$.

Correspondingly, we see that at an $SO(2)$-reducible $A$ we have

$$I(D^\text{irred}_{A, \pi}) = -2(z_1 - z_2)^2 + 2(b^1 - b^+) + \frac{\rho_{\pi_2}(r_2) - \rho_{\pi_1}(r_1)}{2} + 1 - \frac{S(r_1) + S(r_2)}{2}.$$ 

**Definition 4.12.** Let $(W, E)$ be a cobordism $(Y_1, E_1) \to (Y_2, E_2)$ equipped with some small perturbation $\pi$, regular at the ends. We say that a reducible $r$ on $(W, E)$ is good if $I(D^\text{irred}_{A, \pi}) \geq 0$, and bad otherwise.

**Remark 4.1.** In fact, $D^\text{irred}_{A, \pi}$ is a complex linear operator, so its index is even.

There is a natural condition on a cobordism-with-perturbation $(W, E, \pi)$ that reduces the class of bad reducibles to a simple, sometimes avoidable, set.

**Definition 4.13.** Let $(W, E, \pi): (Y_1, E_1, \pi_1) \to (Y_2, E_2, \pi_2)$ be a cobordism. For an $SO(2)$-reducible component $r$ on $(W, E)$, we write its restriction to the two ends
as \( r_1 \). We say that \((W, E, \pi)\) is \( \rho \)-monotonic if for every \( SO(2) \)-reducible component \( r \), we have

\[
\rho_{\pi_1}(r_1) \leq \rho_{\pi_2}(r_2).
\]

For a \( \rho \)-monotonic cobordism, one of the most mysterious terms in the index formula is nonnegative, so we may focus on the rest.

**Lemma 4.28.** For a \( \rho \)-monotonic cobordism \((W, E, \pi)\) with \( b_1(W) = b^+(W) = 0 \) and rational homology sphere ends, the only bad reducibles are \( \{z_1, z_2\} \) where \( z_1 - z_2 \) is a torsion class on \( H^2(W; \mathbb{Z}) \) that restricts trivially to the ends. In particular, bad reducibles can only exist if \( H_1Y_1 \oplus H_1Y_2 \to H_1W \) fails to be surjective.

**Proof.** The assumption of \( \rho \)-monotonicity means \( \frac{\rho_{\pi_2}(r_2) - \rho_{\pi_1}(r_1)}{2} \geq 0 \), so

\[
I(D_{\mathbb{A}, \pi}^{\text{red}}) \geq -2p_1A + 1 - \frac{S(r_1) + S(r_2)}{2}.
\]

If both \( r_i \) are \( SO(2) \)-reducible but not fully reducible,

\[
1 - \frac{S(r_1) + S(r_2)}{2} = 0;
\]

if precisely one of the \( r_i \) is fully reducible then that same term is \(-1\), and if both of the \( r_i \) are fully reducible then the final term is \(-2\). Because \( b^+ = 0 \), we have \(-2p_1A = -2(z_1 - z_2)^2 \geq 0 \). Therefore for a \( \rho \)-monotonic cobordism, \( I(D_{\mathbb{A}, \pi}^{\text{red}}) \geq -2 \), with equality if and only if both \( r_i \) are fully reducible and \(-2p_1A = 0 \).

Because \( H^2_2(W; \mathbb{Q}) \cong H^2(W; \mathbb{Q}) \), and the intersection form is nondegenerate definite on \( H^2 \), we see that \(-2p_1A = -2(z_1 - z_2)^2 = 0 \) iff \( z_1 - z_2 = 0 \in H^2(W; \mathbb{Q}) \); this is the same as saying that \( z_1 - z_2 \) is a torsion class. The assumption that the restriction to the ends is fully reducible is precisely the same as saying that \( z_1 - z_2 \) restricts trivially to the ends. Applying the universal coefficient theorem, we obtain a class in \( \text{Ext}(H_1W; \mathbb{Z}) \) which restricts trivially to \( \text{Ext}(H_1Y_1, Z) \oplus \text{Ext}(H_1Y_2, Z) \), and so \( \text{Ext}(H_1W) \to \text{Ext}(H_1Y_1) \oplus \text{Ext}(H_1Y_2) \) is not injective. The natural isomorphism

\[
\text{Ext}(A, Z) \cong \text{Hom}(A, S^1)
\]

for finite abelian groups \( A \) implies that if the map on \( \text{Ext} \) is not injective, then the map \( H_1Y_1 \oplus H_1Y_2 \to H_1W \) fails to be surjective.

Because \( I(D_{\mathbb{A}, \pi}^{\text{red}}) \) is an even integer, if it is larger than \(-2\), it is nonnegative, so \( r \) is a good reducible. \( \blacksquare \)

**Definition 4.14.** Let \((W, E)\) be a cobordism between \( 3 \)-manifolds with signature data \((Y_1, E_1, \sigma_1)\) and \((Y_2, E_2, \sigma_2)\). We say that \((W, E)\) is weakly admissible if one of the following holds.

- The negative end \((Y_1, E_1)\) is admissible, meaning that \( w_2(E_1) \) only lifts to non-torsion classes in \( H^2(Y_1; \mathbb{Z}) \).
- \( \beta w_2(E) \neq 0 \in H^3(W; \mathbb{Z}) \), where \( \beta \) is the integral Bockstein homomorphism.
- \( b_1(W) = b^+(W) = 0 \), for every \( \beta \in \text{Red}(W; E) \) restricting to \( \beta_i \) on the ends,
  
  \[
  \rho(\beta_2) - \rho(\beta_1) + \sigma_{\pi_2}(\beta_2) - \sigma_{\pi_1}(\beta_1) \geq 0,
  \]
  
  and \( H_1(W) \to H_1(Y_1) \oplus H_1(Y_2) \) is surjective. That is, \((W, E)\) is \( \rho \)-monotonic and supports no bad reducibles.
\* \* E is non-trivial, and \( b_1(W) = 0 \) and \( b^+(W) > 0 \).

Later we will need a similar notion for \( U(2) \)-bundles. If \( \tilde{E} \) is a \( U(2) \)-bundle on the cobordism, we say that it is weakly admissible if its reduction to an \( SO(3) \)-bundle \( E \) is weakly admissible.

The weakly admissible \( U(2) \)-bundles correspond to the first, third, and fourth cases above; the second case precisely means that \( E \) admits no lift to a \( U(2) \)-bundle.

We will soon see that every item on this list admits a regular perturbation. It should be noted that in fact we can achieve regular perturbations when \((Y_2, E_2)\) is admissible but the negative boundary component is not. We do not include these in our definition of weakly admissible bundles as they do not glue together well in general. The composite of a cobordism from a rational homology sphere to an admissible bundle, and then back to a rational homology sphere, need not be weakly admissible.

For the definition above, we have the following.

**Lemma 4.29.** The composite of two weakly admissible cobordisms remains weakly admissible.

**Proof.** First, it is clear that the composite of the first type of weakly admissible cobordism with any other remains weakly admissible. That the composite of any cobordism and one with \( \beta W_2 E \neq 0 \) still has \( \beta W_2 E \neq 0 \) follows immediately from the naturality of the Bockstein and the fact that \( w_2 \) is natural under restriction. Suppose we have weakly admissible cobordisms of the third or fourth type that are not of the first or second. Then their boundaries are rational homology spheres, and the composite also has \( b_1(W) = 0 \); the term \( b^+(W) \) is additive under gluing along rational homology spheres, so the composite of any of the third or fourth type with the fourth type is again of the fourth type.

So what remains to check is that the composite of any cobordism of the third type with another of that type remains of that same type. It is clear that the \( \rho \)-monotonicity condition is additive; the interesting thing is to ask about is the homological condition. At first glance, it is mysterious why composites of cobordisms satisfying this condition should still satisfy this condition; it is made more clear by remembering the point of that condition.

Let \( A \) be a reducible on the cobordism \( W \) with \( b_1(W) = 0 \), corresponding to cohomology classes \( \{ z_1, z_2 \} \); write \( r_i \) for the restriction of \( A \) to the corresponding boundary component. The \( \rho \)-monotonicity condition and homological condition, combined, are equivalent to the following: the index

\[
I(D^{\text{irred}}_{A, \pi}) = -2(z_1 - z_2)^2 - 2b^+(W) + \rho_{\pi_2}(r_2) - \rho_{\pi_1}(r_1) + 1 - \frac{S(r_1) + S(r_2)}{2}
\]

is nonnegative for all \((A, \pi)\). (This is the index computed above Definition 4.12.)

Now let \( W_1 \) and \( W_2 \) be weakly admissible cobordisms between rational homology spheres equipped with reducibles \( A; \) write \( A_{12} \) for the reducible on the composite cobordism \( W \), and \( r_1, r_2, r_3 \) for the restrictions of \( s \) to the successive 3-manifolds that serve as boundary components of \( W_1 \) and \( W_2 \). It is easy to see that

\[
I(D^{\text{irred}}_{A_{12}, \pi}) = I(D^{\text{irred}}_{A_1, \pi}) + I(D^{\text{irred}}_{A_2, \pi}) + S(r_2) - 1.
\]

Now the index \( I(D^{\text{irred}}_{A_{12}, \pi}) \) is non-negative for all reducibles; so for the composite, we find \( I(D^{\text{irred}}_{A_{12}, \pi}) \geq S(r_2) - 1 \). But \( r_2 \) is a reducible, and so \( S(r_2) \) is either 1 or 3; thus we obtain \( I(D^{\text{irred}}_{A_{12}, \pi}) \geq 0 \), as desired. \( \blacksquare \)
Consider the situation that $W_1$ is a cobordism from a homology sphere to a rational homology sphere, and $W_2$ is a cobordism from a rational homology sphere back to a homology sphere, so that $H_iW_i$ are both isomorphic to the middle boundary component. Then clearly the composite cobordism fails the homological condition, even though it tautologically satisfies $\rho$-monotonicity (all reducibles are full reducibles, where the $\rho$-invariant and signature data are both zero). So even though they satisfy the homological condition, one of $W_1$ and $W_2$ must fail to be $\rho$-monotonic.

For every weakly admissible cobordism, all reducible configurations that exist for generic perturbation $\pi$ have non-negative normal index. (In the admissible case, we may choose the perturbation $\pi$ so that there are no $\pi$-ASD connections on the cobordism.)

In the first three cases, this remains true when considering paths of perturbations. In the final case, reducibles only appear generically in families of perturbations of dimension $b^+(W)$, but do not necessarily have non-negative index; when they arise, we may not achieve transversality. Strengthening the $\rho$-monotonicity requirement does not solve this problem.

We conclude this section with some remarks on gradings.

Lemma 4.30. Let $(W, E) : (Y_1, E_1, \pi_1) \to (Y_2, E_2, \pi_2)$ be a cobordism between 3-manifolds equipped with weakly admissible bundles and regular perturbations. If $\alpha_i$ and $\beta_i$ are choices of critical orbits, then

$$\text{gr}^W(\alpha_1, \alpha_2) - \text{gr}^W(\beta_1, \beta_2) = \text{gr}(\alpha_1, \beta_1) - \text{gr}(\alpha_2, \beta_2).$$

This follows immediately from the additivity property of the grading and the fact that $\text{gr}(\beta, \beta) = 0$.

Corollary 4.31. Let $(W, E) : (Y_1, \pi_1) \to (Y_2, \pi_2)$ be a cobordism between rational homology spheres equipped with the trivial bundle and small regular perturbations. Write $\theta_i$ for the corresponding trivial connections. We have $w_2(E) \in H^2(W, \partial W; \mathbb{Z}/2)$, and so we may use the Pontryagin square to write $w_2(E)^2 \in H^4(W, \partial W; \mathbb{Z}/4) = \mathbb{Z}/4$, and $2w_2(E)^2 \in 2\mathbb{Z}/8$.

Then

$$\text{gr}^W(\alpha_1, \alpha_2) = -2w_2(E)^2 + 3(b^1 - b^+) + \text{gr}(\theta_2, \alpha_2) - \text{gr}(\theta_1, \alpha_1).$$

Suppose $(W, E, A)$ is a cobordism from $(Y, E, \pi, \alpha)$ to itself, where $Y$ is a rational homology sphere equipped with a regular perturbation $\pi$ and $\pi$-flat connection $\alpha$. Then

$$\text{gr}^W(\alpha, \alpha) = -2w_2(E)^2 + 3(b^1(W) - b^+(W)).$$

Proof. Recall Definition 4.5 that $\text{gr}(\alpha, \beta)$ is given by $\text{ind}(Q_{\alpha, \beta}^\nu)$ for any connection $A$ connecting $\alpha$ and $\beta$ with perturbation $\pi$ limiting to the fixed perturbations on the ends, and

$$\text{gr}(\alpha, \beta) = \text{gr}(\alpha, \beta) + 3 - \dim \alpha.$$

Proposition 4.26 then provides us with

$$\text{gr}^W(\theta_1, \theta_2) = -2w_2(E)^2 - 3(1 - b^1 + b^+) + \text{gr}(\pi),$$

using the fact that $p_1 \equiv w_2^2 \pmod{4}$ and $S(\theta) = 3$, as well as the vanishing of the invariants $\rho(\theta) = \sigma_\pi(\theta) = 0$. Therefore

$$\text{gr}^W(\theta_1, \theta_2) = -2w_2(E)^2 + 3(b^1 - b^+).$$
Then the conclusion is simply a special case of the previous lemma.

The final claim follows from the same index calculation we used to calculate $gr^W(\theta_1, \theta_2)$; the key points are that the $\sigma_\pi(\alpha)$ terms cancel, and that $S(\alpha) = 3 - \dim \alpha$. $\blacksquare$

We may say something about the gradings of reducibles. We begin with the fully reducible connections.

**Proposition 4.32.** Let $Y$ be a rational homology 3-sphere equipped with a trivial bundle and small regular perturbation $\pi$. If $\Theta$ and $\Theta'$ are fully reducible connections, then $gr(\Theta, \Theta') \in 4\mathbb{Z}/8$.

*Proof.* It suffices to prove this when $\pi = 0$; the spectral flow description above, as well as the assumption that $\pi$ is sufficiently small that no eigenvalues cross the weight $\delta$, implies the grading is the same for arbitrary perturbation.

Now recall that there is an action of $G_E/G^+_E = H^1(Y; \mathbb{Z}/2)$ on both the 3-dimensional configuration space $\hat{E}^{e}_*$. The unperturbed Chern-Simons functional is invariant under the full gauge group, and so the critical set is preserved by this action. Therefore, we also have an action on the 4-dimensional configuration space $\hat{E}^{e,*}_{\pi,E,k,\delta}(\alpha, \beta)$; we take the disjoint union here because the action of $x$ takes $\hat{E}^{e,*}_{\pi,E,k,\delta}(\alpha, \beta)$ to $\hat{E}^{e,*}_{\pi,E,k,\delta}(x \cdot \alpha, x \cdot \beta)$.

The identification in Proposition 1.4 implies that $H^1(Y; \mathbb{Z}/2)$ acts freely and transitively on the set of fully reducible critical points (or what is the same, the set of fully reducible points). So there is a unique $x \in H^1(Y; \mathbb{Z}/2)$ with $x \cdot \Theta = \Theta'$. Now pick a connection $A \in \hat{E}^{e,*}_{\pi,E,k,\delta}(\Theta, \Theta')$. Because the action of $H^1(Y; \mathbb{Z}/2)$ preserves the ASD equations, the index of $Q_A$ agrees with the index of $Q_{x \cdot A}$.

Therefore, $gr(\Theta, \Theta') = gr(\Theta', \Theta)$; here we use that $2x = 0$, so $x \cdot \Theta' = (x + x) \cdot \Theta = \Theta$. But we know

$$2gr(\Theta, \Theta') = gr(\Theta, \Theta') + gr(\Theta', \Theta) = gr(\Theta, \Theta) = 0.$$  

Because these take values in $\mathbb{Z}/8$, we see that $gr(\Theta, \Theta')$ is a multiple of 4. $\blacksquare$

We may make a similar observation about general reducibles (both $SO(2)$- and fully reducible connections), with a completely different proof.

**Proposition 4.33.** Let $Y$ be a rational homology 3-sphere equipped with a weakly admissible bundle $E$ and a small regular perturbation $\pi$. If $\alpha$ and $\beta$ are reducible critical orbits, then $gr(\alpha, \beta)$ is even.

*Proof.* First, we show that there exists a cobordism $(W, E) : (Y, E) \to (Y, E)$ so that there is a reducible connection on $(W, E)$ restricting to $\alpha$ and $\beta$ on the corresponding ends. To see this, recall from Proposition 1.6 that components of reducibles are classified by pairs of cohomology classes $\{z_1, z_2\} \subset H^2(W; \mathbb{Z})$ so that $z_1 + z_2$ is a fixed integral lift of $w_2E$ (and, in particular, if there are reducible components such an integral lift exists). We see, therefore, that it suffices to show that every oriented closed 3-manifold equipped with a pair of cohomology classes $(z_1, z_2) \subset H^2(Y; \mathbb{Z})$ is null-bordant through an oriented 4-manifold equipped with a pair of cohomology classes $(z_1^W, z_2^W)$ that restrict to the $z_i$ on the boundary. Because pairs of cohomology classes are classified by maps to $\mathbb{CP}^\infty \times \mathbb{CP}^\infty$, we are asking that $\Omega^w_3(\mathbb{CP}^\infty \times \mathbb{CP}^\infty) = 0$. This follows immediately from the existence of the Atiyah-Hirzebruch spectral sequence, the fact that $\mathbb{CP}^\infty \times \mathbb{CP}^\infty$ has cohomology
only in even degrees, and $\Omega^{\text{ev}}_k = 0$ for $1 \leq i \leq 3$, as this implies the terms $E^2_{k,3-k} = 0$ for all $k$, and hence the same is true of $E^{\infty}$.

Now if $\eta^W_1$ are the associated complex line bundles to $(z^W_1, z^W_2)$, the $SO(3)$-bundle $E$ is $\mathbb{R} \oplus \eta^W_1 \otimes (\eta^W_2)^{-1}$ and the reducible component is labelled by $\{z^W_1, z^W_2\}$; choosing such a bounding manifold for both $(Y, \alpha)$ and $(\nabla, \beta)$, we may simply take the connected sum to obtain the desired cobordism $(Y, E, \alpha) \to (Y, E, \beta)$.

Now recall from Lemma 4.30 that
$$\text{gr}^W (\beta, \alpha) - \text{gr}^W (\beta, \beta) = \text{gr}(\alpha, \beta).$$
To show that $\text{gr}(\alpha, \beta)$ is even, our goal is to show that $\text{gr}^W (\beta, \alpha) \equiv \text{gr}^W (\beta, \beta) \pmod{2}$. Pick a reducible connection $A$, asymptotic to $\beta$ at $-\infty$ and $\alpha$ at $+\infty$; the previous discussion amounts to saying we may do so.

First, we remark that the $A$ enjoys a splitting $\theta \oplus A$, for a connection $A$ on a complex line bundle and $\theta$ the trivial connection on the trivial real line bundle. We may thus write $I(Q_A) = I(Q_\theta) + I(Q_A)$. Because the index of a complex linear operator is even, we see that $I(Q_A') \equiv I(Q_\theta)$ (mod 2) and
$$I(Q_\theta) = b^1(W) - 1 - b^+(W).$$
Therefore, because $\beta$ is reducible and hence $3 - \dim \beta$ is odd, we find that
$$\text{gr}^W (\beta, \alpha) \equiv b^1(W) - b^+(W) \pmod{2}.$$ 

As for $\text{gr}^W (\beta, \beta)$, the final part of Corollary 4.31 says that
$$\text{gr}^W (\beta, \beta) = -2w_2(E)^2 + 3(b^1(W) - b^+(W)).$$
Reducing modulo 2, we find that $\text{gr}^W (\alpha, \alpha) \equiv b^1(W) - b^+(W) \pmod{2}$, and so
$$\text{gr}^W (\alpha, \alpha) \equiv \text{gr}^W (\beta, \alpha) \pmod{2},$$
as desired.

4.6. Transversality for the cylinder and cobordisms.

**Theorem 4.34.** Let $E$ be a weakly admissible bundle over a $3$-manifold $Y$. Suppose a perturbation $\pi_0 \in \mathcal{P}_{E,\delta}$ has been chosen. Then by Definition 4.3 of this space of perturbations, the enumeration of reducibles Proposition 3.4 holds, and the critical set $\mathcal{C}_{\pi_0}$ is a finite set of nondegenerate $SO(3)$-orbits. If $\mathcal{O}$ is a small $SO(3)$-invariant neighborhood of these orbits, and $\mathcal{P}_{E,\mathcal{O},\delta}$ the space of perturbations $\pi \in \mathcal{P}_{E,\delta}$ with $f_\pi|_{\mathcal{O}} = f_{\pi_0}|_{\mathcal{O}}$, then for a residual set of $\pi \in \mathcal{P}_{E,\mathcal{O},\delta}$, the $\mathbb{R}$-reduced moduli spaces of framed instantons $\tilde{M}^0_{E, z, \pi}(\alpha, \beta)$ between any two critical orbits are cut out nondegenerately. Hence these are smooth $SO(3)$-manifolds of dimension $\text{gr}_z(\alpha, \beta) - \dim \alpha - 1$ unless $\alpha = \beta$ and $z$ is trivial, in which case $\tilde{M}^0_{E,\delta, \pi}(\alpha, \alpha) = \alpha$.

**Proof.** First, we remark on the reducibles: by Proposition 4.14, the only reducible solutions are constant. We want to verify that the solutions are nondegenerate at constant trajectories; this amounts to saying that for the operator $D = d/dt + \overleftarrow{\text{Hess}}_{A, \pi}: \Omega^0 \oplus \Omega^1 \to \Omega^0 \oplus \Omega^1$ has no cokernel other than the constant trajectories at $\ker(\Delta_0) \subset \Omega^0(\mathfrak{g}_E)$. This is true by applying separation of variables to the adjoint operator.

Now the only points to worry about are irreducibles, for which this theorem is standard: see [Don02, Section 5.5.1] or [KM11b]. (This is precisely where we
use the perturbations in $P_{E,C}$ which agree with our original perturbations in a neighborhood of the $\pi$-flat connections.)

Before we continue the proof of transversality for cobordisms, we will need the following lemma.

The key assumption we started with was that $I$ is nonnegative, as the following lemma makes clear. The non-equivariant index 0 case is written in [Sal99, Appendix A.3].

**Lemma 4.35.** Fix a compact Lie group $G$ and two separable $G$-Hilbert spaces $X$ and $Y$. The Banach manifold $F_G(X,Y)$ decomposes into open sets $F_G^{I}(X,Y)$, where $I \in \text{RO}(G)$, an element of the Grothendieck group on finite-dimensional $G$-representations, labelling the index of an operator. Each $F_G^{I}(X,Y)$ is stratified by locally closed subsets $F_G^{V,W}(X,Y)$, where $V$ and $W$ are finite-dimensional $G$-representations with $[V] - [W] = I$. These are defined to consist of those operators $T \in F_G(X,Y)$ with $\ker(T) \cong V$ and $\coker(T) \cong W$.

Each $F_G^{V,W}(X,Y)$ is a smooth submanifold of $F_G(X,Y)$; the normal space at an operator $T \in F_G^{V,W}(X,Y)$ is isomorphic to $\text{Hom}_G(\ker T, \coker T)$.

**Proof.** Let $T \in F_G^{V,W}(X,Y)$, and split $X = X_0 \oplus X_1$ and $Y = Y_0 \oplus Y_1$, where $X_0 = \ker(T), X_1 = X_0^\perp$, and $Y_1 = \text{Im}(T)$, while $Y_0 = Y_1^\perp$; then by assumption $T_{11}$ is an isomorphism. A neighborhood of $T$ in $F_G^{V,W}(X,Y)$ consists of equivariant maps $T' = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, written in block-matrix form, where

$$A : X_0 \to X_0, \quad B : X_1 \to X_0, \quad C : X_0 \to X_1, \quad D : X_1 \to X_1,$$

where $D$ is an isomorphism. If $T' \in F_G^{V,W}(X,Y)$ and is sufficiently close to $T$, then the projection map $\ker(T') \to \ker(T) = X_0$ is an isomorphism (and otherwise it cannot possibly be). So suppose this projection map is an isomorphism. Write the equivariant inverse $v : X_0 \to \ker(T')$ and write it in components as $(x, v_1(x))$. The defining property of $v$ is that

$$(Ax + Bv_1(x), Cx + Dv_1(x)) = 0.$$  

Because $D$ is an isomorphism, we may write $v_1(x) = -D^{-1}Cx$, and then we see that $Ax = BD^{-1}Cx$. Conversely, it is easy to see that if $A = BD^{-1}C$, then the projection map $\ker(T') \to X_0$ is an isomorphism. So $A = BD^{-1}C$ is a defining equation for $F_G^{V,W}(X,Y)$ near zero. The map $F_G^{V,W}(X,Y) \to \text{Hom}_G(X_0, Y_0)$ given by $T' \mapsto A - BD^{-1}C$ has derivative $T' \mapsto A$ at zero, which is a surjective linear map, and hence we see that $F_G^{V,W}(X,Y)$ is a smooth manifold near $T$.

**Theorem 4.36.** Let $(W,E,\pi) : (Y_1,E_1,\pi_1) \to (Y_2,E_2,\pi_2)$ be a weakly admissible cobordism, where $\pi \in P_e^{(4)}$; this means, in particular, that the $\pi_i$ are regular perturbations, and the enumeration of reducibles Proposition 3.4 applies. Write, as usual, $P_e^{(4)}$ for the space of perturbations which agree with the $\pi_i$ on the ends.

For any fixed constant $C > 0$, there is a finite set $L$ of holonomy perturbations adapted to thickened loops in the complement of the ends of $W$ and an open subset $U_\pi \subset P_e^{(4)}$ whose closure contains $\pi$, satisfying the following. For any $\pi' \in U_\pi$, every moduli space $\tilde{M}_{R,\pi}(\alpha,\beta)$ with energy at most $C$ is cut out transversely, and hence a smooth $SO(3)$-manifold of dimension $\text{gr} \omega(\alpha,\beta) - \dim \alpha$ if it contains irreducible connections. In particular, arbitrarily small regular perturbations exist.
Proof. As before, we argue inductively on reducibility type. We have already seen in Lemma 4.20 that the set of perturbations for which the fully reducible connections are cut out transversely form an open dense set of $\mathcal{P}_c^{(4)}$, and in Propositions 4.15 and 4.16 that the set of perturbations for which the $SO(2)$-reducible connections are cut out transversely in the reducible locus form a dense open set (restricting to connections with energy at most $C$).\footnote{Though the statement of these lemmas do not fix the perturbations on the ends, the proof applies equally well to $\mathcal{P}_{L,c}$; the only modifications we need to make to achieve transversality were on the compactly supported portions of the ends.} We call this set $\mathcal{P}_c^{(4)}$. A priori, Propositions 4.15 and 4.16 do not say anything about $\pi$-flat reducible connections with trivial limits on the ends, when $\pi \in \mathcal{P}_c^{(4)}$. However, the homological condition of Definition 4.14 guarantees that no such reducible flat connections exist; these correspond precisely to the so-called bad reducibles.

The main difficulty is in achieving transversality normal to the reducible locus. If the weakly admissible cobordism supports no reducible connections (as in the first two types of Definition 4.14), we are already finished with this step. Similarly, in the third type of weakly admissible cobordism, if $\pi \in \mathcal{P}_c^{(4)}$, then there are no reducible $\pi$-instantons whatsoever (reducible instantons form a set of negative expected dimension in the reducible locus, and so to be cut out transversely means they do not appear at all).

So, restricting to $\mathcal{P}_c^{(4)},\pi$ it only remains to achieve transversality along the reducibles for the fourth type of weakly admissible cobordism: this amounts to slightly more than saying that $b^+(W) = b^1(W) = 0$ and the index of the ASD operator normal to the irreducible locus is non-negative.

We focus on a single reducible component at a time; call it $\eta$. Fix a splitting $E \cong \mathbb{R} \oplus (\eta \otimes \lambda^{-1})$ of topological type $\eta$. Further write $\zeta = \eta \otimes \lambda^{-1}$ for convenience of notation. Let $A$ be an $L_{\eta,\delta}$ connection representing a gauge equivalence class of reducible $\pi$-ASD connection respecting this splitting; we need to arrange that the operator $D^{\text{irred}}_{A,\pi}$ has no cokernel.

The space $\mathcal{P}_c^{(4)} \mathcal{M}_{\eta}$ forms a smooth manifold so that the projection to $\mathcal{P}_c^{(4)}$ is a proper local diffeomorphism, and hence a covering map. There are subsets

$$\mathcal{P}_c^{(4)} \mathcal{M}_{\eta}^{(j)}$$

of reducible $\pi$-instantons for which the map $D^{\text{irred}}_{A,\pi}$ has cokernel of dimension $j$. Our claim is that the set $\mathcal{P}_c^{(4)} \mathcal{M}_{\eta}^{(1)}$ is a closed set, which is a locally finite union of smooth manifolds of positive codimension; then their image in $\mathcal{P}_c^{(4)}$ will be as well, and hence its complement will be a dense open set.

Recall that $Q_{A,\pi} : \Omega^+_{k,\delta}(\mathfrak{g}_E) \rightarrow \Omega^+_{k-1,\delta}(\mathfrak{g}_E) \oplus \Omega^0_{k-1,\delta}(\mathfrak{g}_E)$, the extended ASD operator, is defined as $(d^+_A + D_A \tilde{\nabla}, d^1_A)$; because the perturbing term $\tilde{\nabla} \pi A$ is gauge invariant and the connection respects the given splitting, this operator splits as $Q^{\text{red}}_{A,\pi} \oplus Q^{\text{red}}_{A,\pi}$, the former acting on forms valued in trivial line bundle and the latter acting on forms valued in the complex line bundle $\pi$. It is important to remark here that the action of $S^1$ on the spaces of forms valued in $\zeta$ is of weight 2 at every point (it acts on the $\eta$ component of $\eta \otimes \lambda^{-1} = \zeta$), and so the same is true of every $S^1$-invariant subspace. In particular, any finite-dimensional subspace is $\mathbb{C}^n$ where $z \in S^1$ acts by scalar multiplication by $z^2$. 

\[\text{\footnotesize{\textsuperscript{11}}}\]
If \( \mathbf{A} \) is fully reducible, we have already ensured that \( D_{\mathbf{A}, \pi} \) is surjective above. If \( \mathbf{A} \) is \( SO(2) \)-reducible, the line bundle \( \zeta \) admits no nontrivial \( \mathbf{A} \)-parallel sections and so we may identify the cokernel of \( Q_{\mathbf{A}, \pi}^{\text{irred}} \) with the cokernel of \( D_{\mathbf{A}, \pi}^{\text{irred}} \).

We may identify a neighborhood of any \( SO(2) \)-reducible \( (\mathbf{A}, \pi) \in \mathcal{P}_c \mathcal{M}_0 \) with the corresponding neighborhood \( U \subset \mathcal{P}_c^{(d)'} \) of \( \pi \) via the projection, which is a covering map (being a proper local diffeomorphism on this locus). There is a smooth map \( a : U \to \Omega^1_{k, \delta}(W; i\mathbb{R}) \) picking out the unique \( a(\pi') \in \ker(d^A) \) with \( \mathbf{A} + a(\pi') \) an \( SO(2) \)-reducible \( \pi' \)-instanton; precisely, its defining equation is

\[
2d^+ a(\pi') + \hat{\nabla} \pi (\mathbf{A} + a(\pi')) = 0,
\]

where the operator \( d^+ \) is independent of \( \mathbf{A} \) and no wedge products appear in the ASD equation because this occurs in the reducible locus.

Correspondingly, \( \pi \mapsto Q_{\mathbf{A} + a(\pi') \pi'}^{\text{irred}} \) defines a smooth map

\[
Q : U \to \mathcal{F}^I \left( \Omega^1_{k, \delta}(\zeta), \Omega^2_{k-1, \delta}(\zeta) \oplus \Omega^0_{k-1, \delta}(\zeta) \right)
\]
to the space of Fredholm operators of fixed index \( I \), which is non-negative by the assumption that \( \mathbf{A} \) is a good reducible. The subsets \( \mathcal{F}_k^I \) of Fredholm operators with cokernel of complex dimension \( k \) form a smooth manifold by the previous lemma; our goal is to show that \( Q^{-1}(\mathcal{F}_I^1) \) is a closed, nowhere dense set. The closedness is immediate from continuity.

Suppose \( Q_{\mathbf{A}, \pi}^{\text{irred}} \) has nontrivial cokernel. Our goal is to construct a nearby perturbation so that \( Q_{\mathbf{A} + a(\pi'), \pi'}^{\text{irred}} \) is surjective.

Given a path \( \pi_t : \gamma \to U \) with \( \pi_t = \pi + t\pi' \), let us investigate the corresponding change \( Q_{\mathbf{A} + a(\pi_t), \pi_t}^{\text{irred}} \). This operator may be written as

\[
d^A_{\mathbf{A}} + (a(\pi_t) \wedge -)^+ + D_{\mathbf{A} + a(\pi_t)} \hat{\nabla} \pi_t.
\]

Taking the derivative at \( t = 0 \), we obtain the operator

\[
(a'(\pi') \wedge -)^+ + D_{\mathbf{A}} \hat{\nabla} \pi' + (D'_{\mathbf{A}, \pi})(a'(\pi')),
\]

where \( a'(\pi') = \frac{d}{dt} a(\pi_t) \) and \( D'_{\mathbf{A}, \pi}(a'(\pi')) \) is the derivative at \( t = 0 \) of the operator \( D_{\mathbf{A} + a(\pi_t)} \hat{\nabla} \pi_t \).

Now note that in the construction of cylinder functions \( f_\pi \), these perturbations are based on, we may choose the cylinder functions to vanish on the reducible locus (the map \( SU(2)^N \to \mathbb{R} \) used in the construction should be taken to vanish on the reducible locus of the adjoint action of \( SO(3) \) on \( SU(2)^N \).) In this case, \( a(\pi_t) = a(\pi_0) = 0 \), and hence \( a'(\pi') = 0 \). This means that we do not need to pay attention to the terms contributed by \( a(\pi_t) \). So under this assumption, the derivative is simply \( D_{\mathbf{A}} \hat{\nabla} \pi' \).

In the actual definition of the space \( \mathcal{P}_c \) of perturbations, one takes an \( L^1 \) sum of a countable dense set of cylinder functions; either one should take this set to include a countable dense set of cylinder functions that vanish on the reducible locus, or one should simply choose the perturbation to be much smaller on the reducible locus than the normal space to \( \mathbf{A} \). For convenience of discussion, we assume we are in the former setting; but in general one may keep the contribution to the above operator by \( a(\pi_t) \) arbitrarily small while not substantially affecting the other components.

We have by the weak admissibility assumption that \( \mathbf{A} \) is not constant at a full reducible on both ends, as otherwise \( \mathbf{A} \) would be a bad reducible, so fix some
A = A(3/2) on an end for which A is not constant at a full reducible. Then the maps
\[ \ker(Q^\text{irred}_A, \pi) \to \ker(d^*_A)_{k-1/2} \leftarrow \ker((Q^\text{irred}_A)^*) \]
given by restriction to the time-slice at 3/2 are injective by Lemma 4.19. A priori, \( \psi \in \ker(Q^*) \subset L^2_{-k,-\delta} \), but elliptic regularity and control over the ends improve this to \( L^2_{k,\delta,\text{ext}} \).

As in Lemma 4.20, we separate \( \text{Im}(\ker(Q)) = F_k \) and \( \text{Im}(\text{coker}(Q)) = F_c \) into three pieces: \( F_{kc} = F_k \cap F_c \), as well as \( F^\perp_k \cong (F_k + F_c)/F_c \) an \( F^\perp_c \cong (F_k + F_c)/F_k \). These are each subrepresentations of \( \Omega^1(Y; \zeta) \), and hence
\[ F_{kc} \cong C^a, F^\perp_k \cong C^b, F^\perp_c \cong C^c, \]
where \( b-c = I \), and \( S^1 \) acts on each space with weight two. In particular, because \( I \geq 0 \), one may choose an equivariant surjection \( F^\perp_k \to F^\perp_c \); one may define a self-adjoint surjective equivariant map \( f : F_k + F_c \to F_k + F_c \) by taking \( f \) to be the identity on \( F_{kc} \) and the given equivariant surjection \( F^\perp_k \to F^\perp_c \); this uniquely determines \( f \). Choose a perturbation \( \pi' \) so that the associated cylinder function is a quadratic function whose derivative at \( A \) is very close to the above given map; this is possible because \( SU(2) \)-cylinder functions are dense in the set of \( SO(3) \)-invariant functions on any invariant compact submanifold of \( B^E_k \).

Now, again entirely analogous to Lemma 4.20, we construct the corresponding perturbation \( \beta(dt \wedge \nabla_{\pi'})^+, \) where \( \beta \) is a bump function with small support around \( 3/2 \). Choosing the support sufficiently small, one constructs a perturbation for which the associated map \( K_{00} := \Pi D_{A, \pi'} : \ker(Q) \to \text{coker}(Q) \) is surjective; one may choose \( \pi' \) so that the map \( D_{A, \nabla_{\pi'}} : F_k \to F_c \) arbitrarily close to the given map, and \( K_{00}(\pi') \) is surjective, while keeping \( a(\pi_t) = 0 \) for all \( t \).

Now one concludes as in the beginning of Lemma 4.20: whenever \( T \) is a Fredholm operator and \( K \) is an operator for which \( K_{00} : \ker(T) \to \text{coker}(T) \) is surjective, for sufficiently small \( t \), the operator \( Q_A, \pi_t = Q_A, \pi + tD_{\nabla_{\pi'}} \nabla_{\pi'} + \mathcal{O}(t^2) \) is surjective, as desired. In particular, pairs \( (\pi', A') \) achieving transversality are open and dense among all pairs of perturbation and reducible instantons.

In a neighborhood \( U \) of each \( \pi \in \mathcal{P}^{(4)}_c \), we may use the covering projection from \( P_{L,c}^{(4)}M_{(k,\delta)} \) to identify a neighborhood of the fibers as a disjoint union of finitely many copies of \( U \); above we see that in each of these copies of \( U \), the set of \( \pi \) that achieve transversality at the corresponding instanton are open and dense. Taking the intersection of these finitely many open dense sets, we obtain an open dense subset of \( \pi \in \mathcal{P}^{(4)}_c \) so that each \( \pi \) achieves equivariant transversality at every reducible \( \pi \)-instanton, for all four types of weakly admissible cobordism.

What remains is to do the same at the irreducibles. For most reducible connections, we are in standard territory, and may follow the proof as in [Don02] or [KM07]: one uses perturbations on the ends and unique continuations results that assert an element of the cokernel restricts nontrivially to \( \{3/2\} \times Y \), as above; this leaves a residual set of perturbations so that we achieve transversality everywhere except possibly for flat connections which restrict trivially to the ends. This only changes the ASD equations so long as the instanton \( A \) does not restrict to a fully reducible connection on both ends, which is only possible so long as \( A \) is not a \( \pi \)-flat connection which restricts trivially to the ends.
What remains is no different from the proof of Proposition 4.15: for \( \pi \in \mathcal{P}_c \), we may choose a finite set \( L \) of holonomy perturbations associated to some finite collection of thickened loops, and a neighborhood \( U_\pi \subset \mathcal{P}_{L,c}^{(4)} \) so that the parameterized space of flat connections \( U_\pi \mathcal{M}^{\text{flat}} \) defines a smooth manifold with a proper submersion to \( U_\pi \). Restricting to the subset of regular values of this map, which is an open dense set, we see that there is an open dense set of \( U_\pi \) along which we may achieve transversality at the flat connections.

All of this put together, we have the desired result: a residual set of perturbations for which we achieve transversality everywhere. (Up until now, we had demanded only transversality up to a fixed energy bound, but by intersecting over the sets we get for some countable set of \( C_n \rightarrow \infty \) we obtain a residual set achieving transversality for all energy levels.) ■

In the space of perturbations above, we fixed an appropriate finite set of interior holonomy perturbations adapted to a collection of thickened loops. To prove that our invariants are independent of this choice later, we will need the following simple result.

**Lemma 4.37.** Suppose \( (W, E) \) is equipped with an almost-regular perturbation \( \pi \in \mathcal{P}_\delta^{(4)} \), in the sense that all \( \pi \)-instantons are cut out transversely except possibly those in the locus of flat irreducible connections which restrict trivially to the ends. Suppose that \( L_1 \) and \( L_2 \) are finite sets of interior holonomy perturbations, and suppose we are given small interior perturbation \( \psi_i \in V_{L_i} \) so that \( \pi + \psi_i \) are both regular perturbations.

Let \( S_i \) be the union of disjoint copies of \( [-t, t] \times S^1 \times S^2 \) along which the holonomy perturbations in \( L_i \) are defined, each of these being a small neighborhood of a small sphere bundle around a loop in \( W \). Then one may replace each \( L_i \) by another set \( L_i' \) of holonomy perturbations adapted to the same set of thickened loops, but with \( S_i' \) contained arbitrarily close to the loops, and so that there is some small \( \psi_i' \) so that \( \pi + \psi_i' \) are regular perturbations.

After possibly passing from \( L_i \) to \( L_i' \) as above, we may choose a third set \( L_3 \) of holonomy perturbations adapted to a collection of thickened loops so that we have \( S_1 \cap S_3 = \emptyset = S_2 \cap S_3 \), and so that there is a small \( \psi_3 \in V_{L_3} \) so that \( \pi + \psi_3 \) is a regular perturbation.

**Proof.** The assumption that \( \pi \) is almost-regular means we only need to concern ourselves with the locus of flat irreducible connections which restrict trivially to the ends. This is a compact space, because we have assumed there are no \( SO(2) \)-reducible flat connections that restrict trivially to the ends. The trivial connection is cut out transversely as an isolated point because \( b^1 = b^+ = 0 \), or otherwise we have assumed there is no fully reducible connection on the cobordism.

The property we used of the collection of loops was that the map \( \mathcal{M}^{\text{flat}} \to SU(2)^N/SO(3) \) is an embedding for the set of \( N \) loops; from there, we are able to apply the unique continuation lemma to see that some finite set of holonomy perturbations adapted to each loop assembles into a surjective map \( V_L \to \text{coker}(Q_A) \) for all \( A \). In particular, choosing the neighborhoods of the thickened loops to be smaller does not affect this property, and so (perhaps after choosing many more holonomy perturbations along a single \( S^1 \times S^2 \)) we may pass from \( L_i \) to \( L_i' \) with no difficulty.
What remains to see for the last claim is that \((W \setminus (S_1^r \cup S_2^r)) \to W\) is still surjective at the level of free homotopy classes of loops; then every irreducible flat connection restricts nontrivially to some loop in \(W \setminus (S_1^r \cup S_2^r)\) and we may construct the set \(L_3\) exactly as before. This is clear: take the loops of \(L_3\) to be the same as those of \(L_1\), and then push them slightly off from the loops in both \(L_1\) and \(L_2\); as long as the \(\epsilon\)-neighborhoods of these loops in \(L_1\) and \(L_2\) are taken small enough, they will be disjoint from this new collection of loops; we choose the neighborhoods of the new loops in \(L_3\) small enough so that the \(\epsilon\)-neighborhoods of \(L_1\) and \(L_2\) are each disjoint from the neighborhoods of the loops in \(L_3\).

4.7. Gluing. We follow the approach to gluing given in [KM07, Chapter 19], and in particular we need to briefly discuss weighted Sobolev spaces and the perturbed ASD equations on compact cylinders.

We write \(Z^T = [-T, T] \times Y\) and \(Z^\infty = [0, \infty) \times Y \cup (-\infty, 0] \times Y\); we view \(Z^\infty\) as a limit of the \(Z^T\), stretching until what used to be 0 becomes the point at \(\infty\). We uniformly have \(\partial Z^T = Y \cup Y\). The function spaces of interest to us are the Sobolev spaces of sections of vector bundles over \(Z^T\). The \(L^2_k\) Sobolev space is the completion of the space of compactly supported smooth sections on \(Z^T\); notice that there are no boundary conditions on these sections. As a remark before continuing, the restriction map to the boundary takes value in the \(L^2_{k-1/2}\) Sobolev space.

We define the weighted Sobolev spaces on \(Z^T\) as in [Lin18, Page 72]: let \(\sigma_\delta : \mathbb{R} \to [-\delta, \delta]\) be an odd smooth function such that \(\sigma_\delta(t) = -\delta\) for \(t \geq 1\) and \(\sigma_\delta(t) = \delta\) for \(t \leq -1\). For each finite \(T \geq 2\) we let \(g_{T,\delta}\) be the positive smooth even function on \([-T, T]\) which is equal to 1 on the boundary and \(\sigma_\delta = \log(g_{T,\delta})\). For \(t \in [1, T]\), we have \(g_{T,\delta} = e^{-\delta(t-T)}\), an exponentially decreasing function with final value 1. Then for sections on the finite cylinder \(Z^T\) we set \(\|f\|_{L^2_{k,\delta}} = \|g_{T,\delta} \cdot f\|_{L^2_k}\). For \(T = \infty\) then as before we will use \(g_{\infty,\delta} = e^{t|\delta|}\). For finite \(T\), this is equivalent to the usual \(L^2_k\) norm, with implicit constants growing exponentially in \(T\).

Now as before we may introduce the moduli space of instantons on the finite cylinder \(Z^T\) in the same component as the constant solution \(\gamma_A\) in the usual way: we consider the configuration space \(\gamma_A + \Omega_{k,\delta}(Z^T; \mathfrak{g}_E)\) and quotient by the space \(\mathcal{G}_{E, k+1, \delta}\) of gauge transformations. In the case \(T = \infty\) these gauge transformations should be asymptotic on the noncompact end to elements of \(\Gamma_A\), possibly different on each component. These moduli spaces with no boundary conditions are infinite-dimensional (though not manifolds near reducible solutions) and depend on the Sobolev index \(k\), as the restriction of an \(L^2_k\) ASD connection to the boundary may only be of Sobolev class \(L^2_{k-1/2}\), in any \(L^2_{k+1/2}\) gauge equivalence class.

When defining framed moduli spaces on the finite cylinder \(Z^T\), we take the quotient of \(\mathcal{A}_{A,k}(Z^T) \times E_{(-T),b}\) by the gauge group defined above (that is, we choose the framing to be at the left boundary component). We write this as \(\widehat{M}_{A,k}\); here \(A\) denotes that we are based at the constant trajectory \(\gamma_A\). Because these moduli spaces depend on the Sobolev index \(k\), we include it in our notation.

In the infinite case \(Z^\infty\), there should be two framings, one for each component, which have the same value in the orbit through \(A\) in \(\mathcal{B}_E\) when we take the holonomy to \(\pm \infty\). To mirror the case of finite cylinders, we write this as \(\mathcal{A}_{A,k,\delta}(Z^\infty) \times E_{(0^+,b)} \times E_{(0^-,b)}\); the notation \((0^+, b)\) signifies that this lies in the component \([0, \infty) \times Y\).
Write $\tilde{\mathcal{M}}_{A,k,\delta}(Z^\infty)$ for the quotient. Now we may take $A$-holonomy to $(\infty, b)$ or $(-\infty, b)$, respectively, and project the framing factors to $E_b/\Gamma_\alpha$; and we write $\tilde{\mathcal{M}}_{A,k,\delta}(Z^\infty)$ for the subset on which the two framings project to the same element of $E_b/\Gamma_\alpha$.

For all $T$, there is a restriction map

$$R^T : \tilde{\mathcal{M}}_{A,k,\delta}(Z^T) \to \mathcal{B}_{E,k-1/2}(Y \sqcup \overline{Y}).$$

When $T = \infty$, an element of $\tilde{\mathcal{M}}_{A,k,\delta}(Z^\infty)$ is an equivalence class of connection with a framing at each basepoint on the two boundary components. Therefore, $R^\infty$ may be given by literally restricting the connection and framing to the boundary (up to gauge equivalence). For finite $T$ this is a little more complicated. On the left boundary component $Y$, this is the usual restriction map (the framed basepoint is on $Y$), but on $\overline{Y}$, it is given by restriction in the connection coordinate and parallel transport from $-T$ to $+T$ along $\mathbb{R} \times \{b\}$ in the framing coordinate.

Now we turn to slices for the gauge group action so that we may compute neighborhoods of $\tilde{\mathcal{M}}_{A,k,\delta}$ as an equation on a linear space.

As before, we may consider the Coulomb slice $\ker(d_{A,k,\delta}^*) \subset \Omega^1_{k,\delta}(Z^T; \mathfrak{g}_E)$ where $\gamma_A$ denotes the constant trajectory at $A$; in particular

$$d_{A,k,\delta}^*(dt \wedge \sigma(t) + \omega(t)) = -\sigma'(t) + d_A^*\omega(t).$$

The Coulomb slice is no longer a slice for the gauge group action: if we try to solve the equation $d_{A,k,\delta}^*(d_{A,k,\delta}^* + \omega) = 0$, to write an arbitrary element of $\Omega^1_t$ as a sum of an element of $\text{Im}(d_{A,k,\delta}^*)$ and an element of $\ker(d_{A,k,\delta}^*)$, we find that $d_{A,k,\delta}^* d_{A,k,\delta}^* = -d_{A,k,\delta}^* \omega$ has a unique solution $d_{A,k,\delta}^* \sigma$ for each $\omega$ and fixed boundary values $d_A^*\sigma|_{Y \sqcup \overline{Y}}$.

We obtain the Coulomb-Neumann slice around $\gamma_A$, written $CN_{A,k,\delta}$: the subset of $\omega \in \Omega^1_{k,\delta}(Z^T; \mathfrak{g}_E)$ on which $d_{A,k,\delta}^* \omega = 0$ and, writing $\omega = \phi + dt \wedge \psi$, where $\psi$ is a time-dependent 0-form, we demand that $\psi|_{Y \sqcup \overline{Y}} = 0$. Every connection on $Z^T$ sufficiently close to $\gamma_A$ is gauge equivalent to one on the Coulomb-Neumann slice, and the only remaining ambiguity is that $CN_{A,k,\delta}$ carries the action of the stabilizer $\Gamma_{\gamma_A}$ in the gauge group, and $\gamma_A + a$ is gauge equivalent to $\gamma_A + u(A)$. In particular, observe that $CN_{A,k,\delta} \times_{\Gamma_{\gamma_A}} SO(3)$ gives an $SO(3)$-invariant neighborhood of $\gamma_A$ in $\mathcal{B}(Z^T)$.

Before moving on, we recall Lin’s abstract Morse-Bott gluing theorem. Let $E$ be a vector bundle over a closed oriented 3-manifold $Y$, and let $L$ be a self-adjoint elliptic operator acting on $L^2_0(Y; E)$ with kernel $H_0$. In what follows, we consider the operator $D := d/dt + L$ acting on finite cylinders $Z^T = [-T, T] \times Y$ and infinite cylinders $Z^\infty = [0, \infty) \times Y \sqcup (-\infty, 0] \times Y$.

On the infinite cylinder, we write $L^2_{k,\delta,\text{ext}}(Z^\infty; \pi^*E)$ for the space $H_0 + L^2_{k,\delta}(Z^\infty; \pi^*E)$, where $H_0$ indicates sections $s_{h_0}$ in $\ker(D)$ on $Z^\infty$ which are constant in time at some element $h_0$ in $H_0 = \ker(L)$. In fact, we write

$$\mathcal{E}^T_{\delta} = L^2_{k,\delta}(Z^T; \pi^*E), \quad \mathcal{E}^\infty_{\delta} = L^2_{k,\delta,\text{ext}}(Z^\infty, \pi^*E)$$

and

$$\mathcal{F}^T_{\delta} = L^2_{k-1,\delta}(Z^T; \pi^*E), \quad \mathcal{F}^\infty_{\delta} = L^2_{k-1,\delta}(Z^\infty, \pi^*E).$$

We have a projection map $\Pi^\infty_{\delta} : \mathcal{E}^\infty_{\delta} \to H_0$ given by taking the asymptotic value at $\pm \infty$ and projection maps $\Pi^T_{\delta} : \mathcal{E}^T_{\delta} \to H_0$ given by projection onto the subspace of constant sections in the $L^2_{0,\delta}$ norm (equivalently, in the $L^2_{0,\delta}$ norm).
Suppose we are given a bounded linear operator $\Pi : L^2_{k-1/2}(Y \cup \overline{Y}; E) \to H$ for some Hilbert space $H$; by restriction to the boundary this induces maps $\Pi : L^2_{k,\delta}(Z^T, \pi^*E) \to H$ and $\Pi : L^2_{k,\delta,\text{ext}}(Z^\infty, \pi^*E) \to H$.

We now assume, crucially, that the map

$$(D, \Pi_0^\infty, \Pi) : \mathcal{E}^\infty_\delta \to \mathcal{F}^T_\delta \oplus H_0 \oplus H$$

is an isomorphism, which implies the same for $\delta'$ sufficiently close to $\delta$. Now suppose we are given a non-linear map

$$\alpha : C^\infty(Z^T, \pi^*E) \to L^2_{k,\text{loc}}(Z^T, \pi^*E),$$

obtained by restriction to slices from a map $\alpha_0 : C^\infty(Y; \pi^*E) \to L^2_{k,\text{loc}}(Y; \pi^*E)$.

We assume that $\alpha$ extends to a smooth map $L^2_k([-1,1] \times Y; \pi^*E) \to L^2_{k-1}([-1,1] \times Y; \pi^*E)$ with $\alpha(h_0)$ for every $h_0 \in H_0$ and we assume that $\alpha$ is purely nonlinear, in the sense that its derivative at $0 \in L^2_k([-1,1] \times Y; \pi^*E)$ is zero. This implies that $\alpha$ defines smooth maps $\mathcal{E}^T_\delta \to \mathcal{F}^T_\delta$ with the same property.

Now write

$$F^T = D + \alpha : \mathcal{E}^T_\delta \to \mathcal{F}^T_\delta$$

and

$$M(T) = (F^T)^{-1}(0) \subset \mathcal{E}^T_\delta.$$

The following is [Lin18, Proposition 3.5.15].

**Proposition 4.38.** For $T \in [T_0, \infty]$, the sets $M(T)$ are Hilbert submanifolds of $\mathcal{E}^\infty_\delta$ in a neighborhood of 0. There exist $\eta > 0$ and smooth maps

$$u(T, -) : B_0(H_0 \oplus H) \to M(T)$$

which are diffeomorphisms onto their image and have

$$(\Pi_0, \Pi)u(T, (h_0, h)) = (h_0, h) \text{ and } u(T, (h_0, 0)) = s_{h_0}.$$

For $T \in [T_0, \infty]$, we have a map

$$\mu_T : B_0(H_0 \oplus H) \to L^2_{k-1/2}(Y \cup \overline{Y}; E)$$

obtained as a composition of $u(T, -)$ with restriction to the boundary. For each $T$, $\mu_T$ is a smooth embedding; $\mu_T$ is smooth as a function on $[T_0, \infty) \times B_0(H_0 \oplus H)$, and $\mu_T \to \mu_\infty$ as $T \to \infty$ in the $C^\infty_{\text{loc}}$ topology on maps $B_0(H_0 \oplus H) \to L^2_{k-1/2}(Y \cup \overline{Y}; E)$. Finally, there is an $\eta' > 0$ independent of $T$ so that the images of the maps $u(T, -)$ contains all solutions $u \in M(T)$ with $\|u\|_{L^2_{k,\delta}} \leq \eta'$.

We will now state our main gluing lemma. In what follows, we will use Lin’s abstract gluing theorem to prove our lemma in three cases, depending on the reducibility type of the orbit $\alpha$, along with some mild changes (the flavor of the changes depending on how reducible $\alpha$ is).

**Lemma 4.39.** Let $(Y, E, \pi)$ be a closed oriented 3-manifold equipped with a weakly admissible SO(3)-bundle and regular perturbation $\pi$. We write $\alpha$ for a $\pi$-critical orbit; if $A$ is a connection in the gauge equivalence class of $\alpha$, we write the corresponding Coulomb slice $K_\alpha = \ker(d^*_A) \cap \Omega^1_{k-1/2}(Y; \mathfrak{g}_E)$, and $B(K_\alpha)$ for its unit ball. The Hilbert space $K_\alpha$ carries the action of the stabilizer of $A$ in the gauge group; thinking of $K_\alpha$ as the normal space to a point in the orbit $\alpha$ in $\mathbf{B}^c_{k-1/2}(Y)$, this is the same as the action of $\Gamma_\alpha$, the stabilizer of a point of $\alpha$ in SO(3).
We may thus extend $\mathcal{K}_\alpha$ to a vector bundle $SO(3) \times_H \mathcal{K}_\alpha$ over the orbit $\alpha$, equipped with an $SO(3)$ action (acting on the factor of $SO(3)$ on the left). We write this vector bundle as $\tilde{\mathcal{K}}_\alpha$ and the associated unit disc bundle as $B(\tilde{\mathcal{K}}_\alpha)$.

There is a $T_0$ so that for all $T \in [T_0, \infty]$, we may find smooth, $SO(3)$-equivariant maps

$$\tilde{u}(T, -) : B(\tilde{\mathcal{K}}_\alpha) \to \tilde{\mathcal{M}}_{\alpha,k,\delta}(Z^T)$$

which are diffeomorphisms onto neighborhoods of the constant solution $\gamma_\alpha$, and such that the map

$$\tilde{\mu}_T : B(\tilde{\mathcal{K}}_\alpha) \to \tilde{\mathcal{B}}_{E,k-1/2}^c(Y \sqcup \overline{Y}),$$

given by composing $u(T, -)$ with the restriction map

$$R^T : \tilde{\mathcal{M}}_{\alpha,k,\delta}(Z^T) \to \tilde{\mathcal{B}}_{E,k-1/2}^c(Y \sqcup \overline{Y})$$

described above using holonomy in the framing coordinate for finite $T$, has the following properties.

First, $\tilde{\mu}_T$, being the composition of equivariant maps, is $SO(3)$-equivariant; $\tilde{\mu}_T$ is a smooth embedding of $B(\tilde{\mathcal{K}}_\alpha)$ for all $T \in [T_0, \infty]$, and $\tilde{\mu}_T$ is smooth as a function of $[T, \infty) \times B(\tilde{\mathcal{K}}_\alpha)$. Though not smooth as a function on $[T_0, \infty]$, we at least have $\tilde{\mu}_T \to \tilde{\mu}_T$ in the $C_{loc}^\infty$ topology on $B(\tilde{\mathcal{K}}_\alpha)$ as $T \to \infty$.

Finally, there is an $\eta > 0$, independent of $T$, so that the images of $\tilde{u}(T, -)$ contain all solutions $[\gamma] \in \tilde{\mathcal{M}}_{\alpha,k,\delta}(Z^T)$ such that

$$\|\gamma - \gamma_\alpha\|_{L^2_{k,\delta}(Z^T)} \leq \eta.$$

The vector space $\mathcal{K}_\alpha$ has an eigenspace decomposition $\mathcal{K}_\alpha^- \oplus \mathcal{K}_\alpha^+ = \mathcal{K}_\alpha$; correspondingly we have an $SO(3)$-equivariant fiber product decomposition

$$B(\tilde{\mathcal{K}}_\alpha) \cong B(\tilde{\mathcal{K}}_\alpha^-) \times_{\alpha} B(\tilde{\mathcal{K}}_\alpha^+).$$

We also have the decomposition

$$\tilde{\mathcal{M}}_{\alpha,k,\delta}(Z^T) = \tilde{\mathcal{M}}_{\alpha,k,\delta}(\mathbb{R}^{>0} \times Y) \times_{\alpha} \tilde{\mathcal{M}}_{\alpha,k,\delta}(\mathbb{R}^{<0} \times Y).$$

The map $\tilde{u}(\infty, -)$ respects this decomposition: it is a fiber bundle map over $\alpha$, and on each fiber it respects the product structure.

The easiest case is when $\alpha$ is irreducible, as then we may solve the gluing problem in a Morse (not Morse-Bott) setting.

Proof of Lemma 4.39 in the irreducible case. We will be studying the ASD operator with a gauge fixing condition; the linear operator $L$ on $Y$ is the perturbed operator $\text{Hess}_{\alpha,y}$. In the irreducible case, this operator has no kernel, and Lin’s gluing theorem reduces to the abstract gluing theorem [KM07, Theorem 18.3.5].

First we will apply the abstract gluing lemma to see the corresponding statement about unframed moduli spaces: we will see that there are maps $u(T, -) : B(\mathcal{K}_\alpha) \to \mathcal{M}_{\alpha,k,\delta}(Z^T)$ parameterizing solutions of the perturbed ASD equations in a neighborhood of $\gamma_\alpha$ satisfying the same conditions (without the equivariance); the corresponding restriction maps $R^T$ are given by the actual restriction of connections to the boundary (as opposed to before, where it involved a global quantity, the holonomy).

The argument in [KM07, Section 18.4] applying the abstract gluing theorem to the Seiberg-Witten equations readily applies all the same to the ASD equations.
We sketch the argument, as we will use essentially the same argument in the reducible case. Fix a connection \( A \) on \( Y \) in the gauge equivalence class of \( \alpha \). We identify a neighborhood of \( \alpha \) in \( B_{E,k-1/2} \) as \( \ker(d_A^*) \) via the Coulomb slice.

On the cylinder \( Z^T \), we identify the spaces

\[
\Omega^1_{k,\delta}(Z^T; g_E) \quad \text{and} \quad (\Omega^0 \oplus \Omega^2^+)_{k,\delta}(Z^T; g_E)
\]

with

\[
L^2_{k,\delta}(Z^T; \pi^*(\mathbb{R} \oplus T^*Y) \otimes g_E);
\]

the first by writing every 1-form as \( \psi + dt \wedge \sigma \) for \( \psi \) a time-dependent 1-form on \( Y \), and the second by writing every self-dual 2-form as \( (dt \wedge \psi)^+ \).

Now note from this isomorphism that when we have \( a \in \Omega^1_{k,\delta}(Z^T; g_E) \), we may keep track of more information at the boundary than just \( a|_{Y \cup \bar{Y}} \); this restriction kills any term of the form \( dt \wedge \sigma \). So we write

\[
r : \Omega^1_{k,\delta}(Z^T; g_E) \to \Omega^1_{k-1/2}(Y \cup \bar{Y}; g_E) \oplus \Omega^0_{k-1/2}(Y \cup \bar{Y}; g_E),
\]

to record both the restriction of \( a \) and its normal value \( \sigma(\pm T, b) \) (or \( \sigma(0^\pm, b) \) in the case of the infinite cylinder).

On \( \Omega^1_{k-1/2}(Y; g_E) \), we may project to \( \ker(d_A^*) \), which has an eigenspace decomposition \( K^- \oplus K^+ \) for the action of \( D_{A,\pi} \). (The positive and negative eigenspaces swap upon orientation-reversal.) Write

\[
\Pi : \Omega^1_{k-1/2}(Y \cup \bar{Y}; g_E) \oplus \Omega^0_{k-1/2}(Y \cup \bar{Y}; g_E) \to K^- \oplus \Omega^0_{k-1/2}(Y; g_E) \oplus K^+ \oplus \Omega^0_{k-1/2}(\bar{Y}; g_E);
\]

this is the spectral projection on the \( \ker(d_A^*) \) term (projecting to negative eigenvalues on \( Y \) and positive eigenvalues on \( \bar{Y} \)), and records the data of \( \sigma|_{\bar{Z}^T} \).

Now, on the linear space \( \Omega^1_{k,\delta}(Z^T; g_E) \), consider the equations

\[
d_{\gamma_A}^+ a + (a \wedge a)^+ + (dt \wedge \nabla_\pi(\gamma_A + a))^+ = 0 \\
d_{\gamma_A}^- a = 0 \\
\Pi(a|_{\bar{Z}^T}) = c
\]

Here

\[
c \in K^- \oplus \Omega^0_{k-1/2}(Y; g_E) \oplus K^+ \oplus \Omega^0_{k-1/2}(\bar{Y}; g_E);
\]

we henceforth write this space as \( H = H^- \oplus H^+ \) to simplify notation. The above equations are simply the ASD equations with boundary conditions corresponding to the spectral projection of the \( \Omega^1(Y) \) component and restriction of the \( \Omega^0(Y) \) component.

The linearization of these equations is

\[
Q_{\gamma_A,\pi} a = 0 \\
\Pi(a|_{\bar{Z}^T}) = 0,
\]

so the most important thing to check is that \textit{these linearized equations determine an isomorphism} \( \Omega^1_k(Z^T; g_E) \to (\Omega^2^+ \oplus \Omega^0)_{k-1}(Z^T; g_E) \oplus H \); equivalently it suffices to show that

\[
\Omega^1_k([0, \infty) \times Y; g_E) \to (\Omega^2^+ \oplus \Omega^0)_{k-1}([0, \infty) \times Y; g_E) \oplus H^-
\]
is an isomorphism. To do this, we write $Q_{\gamma_A, \pi} = \frac{d}{dt} + \text{Hess}_{A, \pi}$, and split
$$\Omega^0(Y) \oplus \Omega^1(Y) = \Omega^0 + \text{Im}(d_A) \oplus \ker(d_A^*);$$
rewriting $\text{Im}(d_A) \cong \Omega^0_{k+1/2}(Y; g_E)$, the operator $\text{Hess}_{A, \pi}$ takes the form
$$\begin{pmatrix}
0 & -\Delta_A \\
-1 & 0
\end{pmatrix},$$
which is invertible because $A$ is irreducible. On the other component, our operator takes the form $D_{A, \pi}$, which is invertible because $\pi$ is a regular perturbation.

First let us see that $(Q_{\gamma_A, \pi}, \Pi)$ is surjective. We may write an arbitrary element of the domain as
$$(\sigma_s, \psi_s, h, c) \in (\Omega^2 \oplus \Omega^0_{k-1})([0, \infty) \times Y; g_E) \oplus H^\perp,$$
where $\sigma_s$ is a time-dependent 0-form and $\psi_s$ is a time-dependent 1-form, thought of as the self-dual 2-form via $(dt \wedge \psi_s)^+; h \in K^{-\alpha}_c$ and $c \in \Omega^0_{k-1/2}(Y; g_E)$.

We may decompose $\Omega^0(Y)$ into the eigenspaces of $\Delta_A$ and $\ker(d_A^*)$ into the eigenspaces of $D_{A, \pi}$. For $\phi \in \Omega^1([0, \infty) \times Y; g_E)$ we write $\phi_s = \omega_s + d_A \eta_s$, where $\omega_s \in \ker(d_A^*)_{k-1/2}$. We will explicitly write out $(Q_{\gamma_A, \pi}, \Pi)$ in terms of these.

Using the eigenspace decompositions, we may write $\sigma_s = \sum b_\alpha(s) \sigma_\lambda$, and $\eta_s = \sum c_\alpha(s) \sigma_\lambda$. We also have $\psi_s = \sum d_\alpha(s) \omega_\lambda$, where the coefficients range over a basis of eigenfunctions.

Now we find that these must satisfy $\sigma'(s) = \Delta_A \eta(s)$ and $\psi'(s) = \sigma(s)$. The second equation gives us for each eigenspace $b_\alpha = c_\alpha, \lambda$, but we also have $b_\alpha(s) = \lambda c_\alpha(s)$. Combining these we solve to find $c_\alpha(t) = c_\alpha(0) \cdot e^{\pm \sqrt{\lambda}}$. However, under the assumption that $\phi \in L^2_k$ and hence $\eta \in L^2_k$, the sign must be negative. By specifying $c$ above, we specify the values of $c_\alpha(0)$ for all $\lambda$, and hence both $b$ and $c$ for all time. Because $\sum c_\alpha(0)$ defines an $L^2_k$ function on $Y$ and the sum $\sum e^{-\sqrt{\lambda}} c_\alpha(0)$ satisfies an elliptic equation, regularity implies that this defines an $L^2_k$ function on $[0, \infty) \times Y$ (and similarly for $b$).

Now for $\psi$, we get the formula $\omega_\lambda(s) = \lambda \omega_\lambda(s)$. This can only contribute to the solution if $\lambda < 0$, as otherwise $\omega$ is not $L^2$. We specified the values of $\omega_\lambda(0)$ for $\lambda < 0$ in the choice of $h \in H^\perp$, and thus have specified $\omega$ for all time; above it has the desired regularity. We have constructed a solution to the equations of the desired regularity and with desired boundary values, so $(Q, \Pi)$ is surjective.

To see that $(Q, \Pi)$ is injective, we remark that the above argument always produced unique solutions.

This in hand, the abstract gluing theorem gives us a map
$$u(T, -) : B_{\eta} \left( K^-_\alpha \oplus K^+_\alpha \oplus \Omega^0_{k-1/2}(Y \sqcup Y) \right) \to \Omega^1_{k, \delta}(Z^T; g_E)$$
parameterizing solutions in a neighborhood of 0. To use this to get a parameterization of $\gamma_A$ in $\mathcal{M}_{A, k, \delta}(Z^T)$, we observe that a neighborhood of $\gamma_A$ in the latter is given by the Coulomb-Neumann slice $CN_{A, k, \delta}$, and this parameterization is compatible with the restriction maps; so to obtain the desired parameterizations, we simply need to restrict the domain of $u$ to $B_{\eta}(K^-_\alpha \oplus K^+_\alpha)$, imposing the Neumann gauge condition that the $dt \wedge \sigma$ component is zero on the boundary.

Recall that projection $\tilde{\mathcal{M}}(Z^T) \to \mathcal{M}(Z^T)$ forms a principal $SO(3)$-bundle over the irreducible instantons. Our goal is to choose a section of this over $B(K^-_\alpha)$, sitting inside $\tilde{\mathcal{M}}(Z^T)$ via $u(T, -)$, which will automatically give us an equivariant map $SO(3) \times B(K^-_\alpha) \to \tilde{\mathcal{M}}(Z^T)$, as desired. But we should be careful in how we choose this lift so that the maps $\mu_T$ have the desired properties.
We identified a neighborhood of $\gamma_A$ in $\mathcal{M}_{A,k,\delta}(Z^T)$ with a subset of $\mathcal{CN}_{A,k,\delta}(Z^T)$. We may thus define the section above $\mathcal{CN}_{A,k,\delta}(Z^T)$; choose the section to be $\mathcal{CN}_{A,k,\delta}(Z^T) \to SO(3) \times \mathcal{CN}_{A,k,\delta}(Z^T)$, given by $A \to (p, A)$ for $p$ a fixed framing.

Then writing $B(\mathcal{K}_\alpha) = SO(3) \times \mathcal{K}_\alpha$, our parameterization

$$\tilde{u}(T,-) : SO(3) \times B(\mathcal{K}_\alpha) \to \tilde{\mathcal{M}}(Z^T)$$

is given as $(\tilde{u})(g, h) = \tilde{s}(u(h)) \cdot g$ in the ‘framed Coulomb-Neumann slice’ $SO(3) \times \mathcal{CN}_{A,k,\delta}$. That this is a smooth embedding for all $T$ follows from the corresponding fact for unframed moduli spaces, and because $\mu_T$ is assumed smooth in finite $T$, so is $\tilde{\mu}_T$; similarly $\tilde{\mu}_T \to \tilde{\mu}_\infty$ in the $C^\infty_{\text{loc}}$ topology.

Finally, because the unframed map $u(\infty, -)$ respects the product structure and we defined $\tilde{u}$ to be a fiber bundle map, the last statement is true.

We quickly discuss linear models for the configuration space $\tilde{B}_{A,k,\delta}(Z^T)$ when $A$ is fully reducible. When $T$ is finite, choosing a base framing $p$, a neighborhood of 0 in the Coulomb slice

$$\mathcal{CN}_{A,k,\delta}(Z^T) \subset A_{A,k,\delta}(Z^T) \times \{p\} \subset \tilde{A}_{A,k,\delta}(Z^T)$$

projects to a chart in $B_{*\pi E,k,\delta}(Z^T)$ around $\gamma_A$, so we may effectively study neighborhoods of $\gamma_A$ in the moduli space on finite cylinders by studying a subset of the equations in Coulomb-Neumann gauge. We will consider the Coulomb-Neumann slice as a subset of $\Omega^1_{k,\delta}(Z^T)$ and work there. In this case, the space $H_0 = \ker(L)$ above is the Lie algebra $\mathfrak{g}_0$; when considered as a subset of $\Omega^1(Z^T)$, these consist of the 1-forms $dt \wedge \sigma$, where $\sigma$ is a fixed $A$-parallel section of $\mathfrak{g}_0$.

The discussion of the Coulomb-Neumann slice remains correct on the infinite cylinder: everything in a neighborhood of $\gamma_A$ in $\tilde{B}_{A,k,\delta}(Z^\infty)$ is gauge equivalent to an asymptotically decaying connection in Coulomb-Neumann gauge. Recall here that our definition of $\tilde{B}_{A,k,\delta}(Z^\infty)$ involves two framings: one on each boundary component. Applying $A$-parallel gauge transformations on each component, we may change those framings arbitrarily while obtaining another connection in Coulomb-Neumann gauge.

In fact, while we will consider the space $\mathcal{E}_\delta^\infty = \Omega^1_{k,\delta,\text{ext}}(Z^\infty; \mathfrak{g}_E)$ of 1-forms which asymptotically decay to $dt \wedge \sigma$, in the case of the infinite cylinder the extra kernel arising as $dt \wedge \sigma$ is somehow illusory: applying a gauge transformation we may change any connection in this extended space to a connection in Coulomb-Neumann gauge.

Similarly, for any $a \in \mathcal{CN}_{A,k,\delta}$, we have by definition of Coulomb-Neumann gauge the integration by parts formula $0 = \langle d^*_{\gamma A} a, b \rangle_{L^2} = \langle a, d_{\gamma A} b \rangle_{L^2}$. Above, the projection $\Pi_0$ was projection onto the constant sections $dt \wedge \sigma$ in the $L^2_{0,\delta}$ inner product; then this is defined to be the same as $\langle a, g_{T,\delta}^2 dt \wedge \sigma \rangle_{L^2}$, where $g_{T,\delta}$ is the weight function in the definition of Sobolev spaces on finite cylinders.

But we may choose an antiderivative $\frac{d}{dt} G_{T,\delta} = g_{T,\delta}^2$, and then the right-hand side of this inner product is $g_{A,\delta}(G_{T,\delta} \sigma)$, because $\sigma$ is $A$-parallel. Therefore, for any solution in Coulomb-Neumann gauge, $\Pi_0^T a = 0$. With this, we begin the proof.
Proof of Lemma 4.39 in the fully reducible case. The equations we look to study with Lin’s abstract gluing theorem are nearly the same as last time: now they are:

\[ d^+_\gamma a + (a \wedge a)^+ + (dt \wedge \nabla_\pi (\gamma_A + a))^+ = 0 \]

\[ d^\#_\gamma a = 0 \]

\[ \Pi(a|_{\mathcal{Z}^T}) = c\Pi_0^\gamma a = h_0 \]

That our operator \((\mathcal{D}, \Pi_0, \Pi)\) satisfies the invertibility assumption follows by the same separation of variables argument as in the irreducible case. Lin’s theorem, therefore, gives us a parameterization

\[ u(T, h_0, h) : B_\eta(H_0 \oplus H) \to \Omega^1_k,\delta(Z^T) \]

of solutions to the above equations on \(Z^T\) with \((h_0, h) \in B_\eta(H_0 \oplus H)\); recall that here \(H_0\) consists of 1-forms of the form \(dt \wedge \sigma\) for \(\sigma \in \ker(\Delta_A)\) and

\[ H = \mathcal{K}_-^A \oplus \Omega^0(Y; \mathfrak{g}_E) \oplus \mathcal{K}_+^A \oplus \Omega^0(\overline{Y}; \mathfrak{g}_E). \]

For his application, Lin restricts to the subdomain \(H_0 \oplus \mathcal{K}_-^A \oplus \mathcal{K}_+^A\); based on the discussion above, we see that in our case we should actually restrict to the subdomain \(\mathcal{K}_-^A \oplus \mathcal{K}_+^A\) and ignore the terms coming from \(H_0\). We abuse notation and write \(u(T, h) = u(T, 0, h)\). As long as \(T\) is finite, all solutions thus obtained are in Coulomb-Neumann gauge, and this defines a parameterization of a small neighborhood of \(\gamma_A\) in \(\mathcal{M}_{A,k}(Z^T)\).

Before going on, we should explain why this map \(u\) is equivariant; this follows as a consequence of uniqueness of solutions. That is, if \(g\) is a gauge transformation preserving \(\gamma_A\), then \(u(gh_0, gh)\) is the unique solution \(a\) to the equations

\[ d^+_\gamma a + (a \wedge a)^+ + (dt \wedge \nabla_\pi (\gamma_A + a))^+ = 0 \]

\[ d^\#_\gamma a = 0 \]

\[ \Pi(a|_{\mathcal{Z}^T}) = gc\Pi_0^\gamma a = 0 \]

Here \(c = (h, 0, 0)\) and \(\Pi\) is the projection operator to \(\mathcal{K}_-^A \oplus \mathcal{K}_+^A \oplus \Omega^0(Y \cup \overline{Y}; \mathfrak{g}_E) \oplus H_0\). But this equation is also satisfied by \(gu(T, h)\), because the projections are equivariant and the ASD equations are invariant under gauge transformations.

So \(u(T, h)\) defines a map

\[ B_\eta(K_\alpha) \to \mathcal{C}N_{A,k,\delta}(Z^T) \times \{p\} \subset \widetilde{\mathcal{A}}_{A,k,\delta}(Z^T) \]

for \(T\) finite, where \(p\) is a fixed framing; projecting to \(\mathcal{B}_{A,k,\delta}(Z^T)\), this gives a neighborhood of \(\gamma_A\) in \(\widetilde{\mathcal{M}}_{A,k,\delta}(Z^T)\).

So we have our parameterization of a neighborhood of \(\gamma_A\) for all \(T \in [T_0, \infty)\); we should check the desired properties of the restriction maps. The abstract gluing theorem guarantees certain properties of the maps \(\mu_T^+: B_\eta(H) \to \ker(d^\#_A)_{k-1/2}\) obtained as the restriction of \(\tilde{u}(T, h)\) to the corresponding boundary component. However, our gluing lemma asks for properties of a restriction map whose definition includes the framing, written \(\tilde{\mu}_T\). In the case of restriction to the leftmost boundary component, there is no change, as the framing is already fixed on the left boundary component; there is no need to perform any holonomy. But the right endpoint map is more complicated.
Let’s be precise. Let \((A, p)\) be a framed connection on \(Z_T\), where the framing is chosen at \((-T, b)\). Then the definition of the right endpoint map is \[
(A, p) \mapsto \left( A|_{\gamma^b T}, \text{Hol}_A^{(-T,b)\rightarrow (T,b)} p \right);
\]
we then pass to the quotient by the gauge group action (as this is an equivariant map).

Now, we identified a neighborhood of \(A\) in \(B_{E,k,\gamma}^{k-1/2}(\mathcal{Y})\) with the projection of the Coulomb slice \(\ker(d_A^* \times \{p\} \subset \mathcal{A}_{E,k,\gamma}^{k-1/2}(\mathcal{Y})\), where \(p\) is the same fixed framing as above. Therefore, identifying a neighborhood of \(\gamma_A\) in \(\mathcal{M}_{A,k,\delta}(Z_T)\) with the corresponding subset of the Coulomb-Neumann slice (and a neighborhood of \(A\) with the corresponding Coulomb slices), the restriction \(\mu_T\) to the left boundary component is still given by the same \(\mu_T : \mathcal{CN}_{A,k,\delta}(Z_T) \rightarrow \ker(d_A^* \times \{p\})\) as in the abstract gluing theorem, because our framing is chosen fixed at the left boundary component to begin with. However, the restriction to the right boundary component is now given by \(\tilde{\mu}_T(a) = \sigma^T(a) \cdot \mu_T(a) \in \ker(d_A^* \times \{p\}^{k-1/2}(\mathcal{Y}))\), where we have identified \(\sigma^T(a) \in SO(3)\) with an \(A\)-parallel gauge transformation.

Precisely, the map \(\sigma^T\) is given by \[
\sigma^T(a) = \left( \text{Hol}_{\gamma,\mu}^{(-T,b)\rightarrow (T,b)} \right)^{-1},
\]
using the same framing \(p\) at \(\pm T\) to identify this isomorphism \(E_{(-T,b)} \rightarrow E_{(T,b)}\) with an element of \(SO(3)\). We have \(\sigma(0) = 1\): the connection \(\gamma_A\) is in temporal gauge and so there is no holonomy across \(\mathbb{R} \times \{b\}\).

What we need to do first is see that \(\sigma^T(u(T, h))\) converges in the \(C^\infty_{loc}\) topology. We will then use this to define the map \(\tilde{u}(x, -)\), as an appropriate modification of \(u(x, 0, -)\).

First we recall how holonomy is defined. First, given \(a \in \gamma_A + \Omega^{1}_{k,\delta}(Z_T)\), there is a natural restriction map to \[
dt d\Omega^{1}_{k-3/2,\delta}([-T, T], \mathfrak{g}_b) = d\dt + d\Omega^{0}_{k-3/2,\delta}([-T, T], \mathfrak{g}_b)
\]
to the line \(\mathbb{R} \times \{b\}\). The holonomy along the path only depends on the values of \(a\) on this path, and more precisely, if we decompose \(a = \dt \wedge \eta(t) + \psi(t)\), it only depends on \(\eta(b, t)\). Precisely, let \(\gamma(t) : [-T, T] \rightarrow SO(3)\) be the unique solution to the differential equation \(\gamma'(t)\gamma^{-1}(t) = \eta(b, t)\) with \(\gamma(-T) = 1\), guaranteed as long as \(k - 3/2 \geq 1\), so that \(\eta\) is \(C^1\). Solving this differential equation gives a smooth map \(\text{Lift} : L_{k-3/2,\delta}^2([-T, T], \mathfrak{g}_b) \rightarrow L_{k-5/2,\delta}^2([-T, T], SO(3))\), and then evaluation at \(T\) is clearly smooth.

If we write \(u_+(h)\) as the component of \(u(x, h)\) on the cylinder \([0, \infty) \times \mathcal{Y}\), and similarly for \(u_-(h)\), we write \(U(T, h) = \tau_+^s u_+(h) + \tau_-^s u_-(h)\). We see from [Lin18, Lemma 3.5.19] that \(U(T, h) - \tilde{u}(T, h)\) goes to 0 in the \(C^\infty_{loc}\) topology. So it suffices to check that \(\sigma^T(U(T, h))\) has a limit in \(C^\infty_{loc}\).

Now we may be very explicit. By translation, consider instead the interval \([0, 2T]\); we may see that \(\sigma^T(U(T, h)))^{-1}\) is the solution to the differential equation \(\gamma'(t)\gamma^{-1}(t) = u_+(h)(t, b) + u_-(h)(t - 2T, b)\) with \(\gamma(0) = 1\), evaluated at \(2T\). We may instead consider this as the product \(g_2 g_1\) of two elements of \(SO(3)\): first, \(g_1\) is the time-\(T\) value of that same equation; second, \(g_2\) is the time-\(T\) value of the solution to the differential equation \(\gamma'(t)\gamma^{-1}(t) = u_-(h)(-t, b) + u_+(h)(2T - t, b)\) with \(\gamma(0) = 1\). (That \(g = g_2 g_1\) is just the statement that the holonomy of two paths,
traversed in succession, is the product of their holonomies and that the holonomy of a path traversed in reverse is the inverse of the holonomy.}

We examine $q_1$; the analysis of the other term is very similar. Notice that the stated differential equation only needs the values of the given functions on $[0, T]$, so we introduce a cutoff function $\beta$ which has $\beta(t) = 1$ for $t \leq 0$ and $\beta(t) = 0$ for $t \geq 1/2$, and $\beta_T(t) = \beta(t - T)$; write

$$ q_T(h) = u_+(h)(t, b) + \beta_T(t) (u_-(h)(t - 2T, b)). $$

It is clear that $U(T, h)$ and $q_T(h)$ give the same time-$T$ holonomy map, as they agree on $[0, T]$. Our first claim, to be proved below, is that the time-dependent map $B_\eta(\mathcal{K}_\alpha) \to L_{k-3/2, \delta}^2([0, \infty), \mathcal{g}_\delta)$ given by sending $h \to \beta_T(t) u_-(h)(t - 2T, b)$ converges to 0 in $C^\infty_{\text{loc}}$; given that this is true, our goal becomes finding the $C^\infty_{\text{loc}}$ limit as $T \to \infty$ of

$$ \text{ev}_T \text{Lift} (u_+(h)(t, b)). $$

Now there is nothing left to do here, as $u_+$ is time-independent! We see that the output converges in the $C^\infty_{\text{loc}}$ topology to

$$ \text{Hol}^{(0^+, b) \to (\infty, b)}_{\gamma_A + u_+(h)}. $$

Similarly we may identify the limit of $g_2$ as the inverse of the corresponding $\text{Hol}^{(0^-, b) \to (\infty, b)}_{u_-(h)}$.

We conclude, therefore, that $\sigma^T(u(T, h))$ converges in $C^\infty_{\text{loc}}$ to $\sigma^\infty(h)$, which is defined to be the inverse of

$$ \left( \text{Hol}^{(0^-, b) \to (\infty, b)}_{u_-(h)} \right) \left( \text{Hol}^{(0^+, b) \to (\infty, b)}_{\gamma_A + u_+(h)} \right). $$

This makes some intuitive sense; the appearance of $\sigma^T$ corresponds to the twist in framing as we move the left endpoint framing to the right endpoint. So on the infinite cylinder, we should be forced to take holonomy from left endpoint to right endpoint.

Based on the above, we define the map

$$ \tilde{u}(\infty, -) : B_\eta(\mathcal{K}_\alpha) \to \mathcal{C}N_{A, k, \delta}(Z^\infty) \times SO(3) \times SO(3) \subset \tilde{A}_{A, k, \delta}(Z^\infty) $$

to be

$$ \tilde{u}(\infty, h) = (u(\infty, h), p, \sigma^\infty(h) \cdot p); $$

we project from this framed Coulomb slice to the configuration space $\tilde{\mathcal{M}}(Z^\infty)$. Then the above discussion shows that indeed, $\tilde{\mu}_T \to \tilde{\mu}_\infty$ as $T \to \infty$.

Now to prove the claim that $\beta_T(t) (u_+(h)(t - 2T, b))$ converges to 0 in $C^\infty_{\text{loc}}$. The point is that the application of the inverse function theorem defining $u_+$ can be factored through a Sobolev space with smaller exponent, so that one may find pointwise bounds on all of the derivatives of $u_+$:

$$ \| D_m u_+ (h)(t - 2T, b) \| \leq C_m (h) e^{-\delta(t - 2T)}, $$

where the $C_m$ are continuous functions of $h$ depending on $m$; the same is true when adding in the bump function (whose derivatives all have compact support).

Because the support of $\beta_T u_+(t - 2T)$ is contained in $[0, T + 1]$, we obtain bounds by $C_m (h) e^{-\delta(1 - T)}$; taking $T \to \infty$ we see that this goes to 0 in $C^\infty_{\text{loc}}$. □
What remains is the case of $SO(2)$-reducible critical orbits. We use a mix of the above techniques; we work in the Coulomb-Neumann slice as a model for the normal space to $\gamma_A$, and then extend this to an $SO(3)$-invariant parameterization.

**Proof of Lemma 4.39 in the $SO(2)$-reducible case.** We study the same equations as above, where $\gamma_A$ is an $SO(2)$-reducible constant trajectory. We start by choosing $p \in S^2 = E_{0,b}/\Gamma_A$, a choice of framing modulo the action of $\Gamma_A$; restrict to the subspace $\mathcal{M}'(Z^T)$ given by those pairs $(A,q)$ with framings that project to $p$ in the quotient. (This is the space that has a neighborhood modelled by the solutions to the equations in the Coulomb-Neumann slice.) We find just as above that Lin’s abstract gluing theorem again provides us with an $S^1$-equivariant parameterization $B_q(K_0) \to \mathcal{M}(Z^T)$. The issue is in verifying that the restriction maps to the right end, which depend on holonomy, converge as $T \to \infty$ in $C^\infty_{\text{loc}}$, as before.

We may choose an arbitrary lift of $p$ to an actual framing $\tilde{p}$. Note that because $\gamma_A$ is in temporal gauge, the holonomy from $-T$ to $T$ is the identity. Now choosing the domain of the parameterization $u$ small enough, we may demand that $\operatorname{Hol}_{\gamma_A + u(T,0,h)}(T,b) - \tilde{p}$ lies in an $S^1$-invariant neighborhood of $\Gamma_A \cdot \tilde{p}$. Exponentiating the normal bundle, we may choose an $S^1$-equivariant diffeomorphism $U \cong S^1 \times D^2$.

This gives us an $S^1$-equivariant projection map $P: U \to S^1$. We write

$$\overline{\operatorname{Hol}}_{(-T,b)\to(T,b)} = \operatorname{PHol}_{\gamma_A + u(T,0,h)}(T,b)^{-1} \tilde{p},$$

this is precisely the amount of holonomy in the ‘$S^1$-direction’, and the analogue of $\sigma^T(u)^{-1}$ for the $SO(2)$-reducible case (that is, the inverse of this term is precisely the rotation appearing in the right-most restriction map.)

We saw above that $\operatorname{Hol}_{(-T,b)\to(T,b)}$ does converge in $C^\infty_{\text{loc}}$ as $T \to \infty$, and to

$$\left(\operatorname{Hol}_{u(x,0,h)}(-T,b)\to(-\infty,b)\right)^{-1} \operatorname{Hol}_{u(x,0,h)}(0^+,b)\to(x,b).$$

Because the projection $P: U \to S^1$ is smooth, the same is true of $\overline{\operatorname{Hol}}_{u(T,0,h)}(T,b)$; we use this to define the map $u(x,0,h)$ on $B_q(K_0) \to \mathcal{M}'_{A,k,d}(Z^C)$ as before, and to see that it is $SO(2)$-equivariant.

All that is left is to extend this to an $SO(3)$-equivariant parameterization of the whole of $\mathcal{M}_{A,k,d}(Z^T)$. We write the neighborhood of $(\gamma_A,\tilde{p})$ as $S^1 \times_{S^1} \mathcal{CN}_{\gamma_A,k,d}(Z^T)$. (That is, we allow ourselves to vary over all framings that lie above the fixed $p \in S^2$, and then quotient by the natural action of $\Gamma_A$.) Then there is an obvious $S^1$-equivariant map $S^1 \times_{S^1} \mathcal{CN}_{\gamma_A,k,d}(Z^T) \to SO(3) \times_{S^3} \mathcal{CN}_{\gamma_A,k,d}(Z^T)$. This is our desired section. We thus extend our parameterization to an $SO(3)$-equivariant parameterization $SO(3) \times_{S^3} B_q(K_0) \to \mathcal{M}(Z^T)$ of a neighborhood of the orbit through $\gamma_A$, as desired.

With this, the proof of the gluing theorem follows essentially the same lines as in [KM07, Chapters 19 and 24.7]. We outline the procedure for the sake of completeness.

First, the moduli spaces $\tilde{\mathcal{M}}_{E,z,\pi,k}(Z^T)$ of $\pi$-perturbed instantons on the cylinder, finite or infinite, are cut out transversely inside $B_{E,z,\pi,k}(Z^T)$, and therefore form a smooth Hilbert manifold no matter the perturbation $\pi$. The proof of this is no different than [KM07, Theorem 17.3.1]; the point is that we have a unique
continuation result for zeroes of the adjoint operator $Q_{A,\pi}^*$, as the equations are of gradient-flow type.

Fix a compact manifold $W$ with boundary $Y_1 \sqcup Y_2$, with cylindrical metric near the boundary, and $SO(3)$-bundle $E$ which restricts to bundles $E_i$ over the respective ends $Y_i$. Suppose we have fixed small perturbations $\pi_i$ on $(Y_i, E_i)$ so that there are finitely many nondegenerate critical orbits of $cs_{Y_i} + f_{\pi_i}$. Further, we assume $(W, E)$ is weakly admissible, so it admits some perturbation achieving transversality on all moduli spaces of bounded dimension on the infinite cobordism $\hat{W}$.

On the cylindrical ends of the infinite manifold $\hat{W}$, we fix the constant perturbation $(dt \wedge \nabla_{\pi_i}(A))^+$. This is dampened by a cutoff function in the collar $[0, \infty) \times (Y_1 \sqcup Y_2)$, equal to 0 near 0 and equal to 1 for $x \geq 1$. Further, we add to this a perturbation of the form $(dt \wedge \nabla_{\pi_i(t)}(A))^+$, where $\pi_i(t) : (1, 2) \to \mathcal{P}_{E_i}$ is compactly supported. If $W^L$ is the complement of $(L, \infty) \times (Y_1 \sqcup Y_2)$, then the moduli space of instantons $\hat{\mathcal{M}}_{E,\pi,k}^r(W^L)$ on the compact manifold $W^L$ is a smooth Hilbert manifold; this is a matter of showing that any nonzero element of the kernel of $Q_{A,\pi}^*$ on $W^L$ must restrict nontrivially to the boundary. This is a unique continuation result, and follows as in [KM07, Proposition 24.3.1]: first show that if a kernel element restricted trivially to the boundary, it would restrict trivially to the whole of $W^L$. On the cylindrical ends of the infinite manifold $\hat{W}$, we fix the constant perturbation $(dt \wedge \nabla_{\pi_i}(A))^+$.

From here, we may describe moduli spaces on the manifold $W$ with infinite cylindrical ends as a fiber product. Let $I_j^+$ and $I_j^-$ be a finite sequence of intervals $(0 \leq j \leq n, 0 \leq \ell \leq m)$, where the first negative interval is $I_0^- = (-\infty, 0]$, the first positive interval is $I_0^+ = [0, \infty)$, and all other intervals are finite. Write $\hat{B}^{\pm} = \hat{B}_{E,k-\ell/2}(Y^{\pm})$. There are evaluation maps

$$\text{ev}_\pm : \hat{\mathcal{M}}_{E,\pi} \to \hat{B}^- \times \hat{B}^+,$$

and similarly for $\hat{\mathcal{M}}_{E,\pi,k}(Z^{I_j^+}) \to \hat{B}^+ \times \hat{B}^+$. For the infinite ends, we only have one evaluation map. We assemble all of this into a map

$$(R_-, R_+) : \hat{\mathcal{M}}_{E,\pi} \times_{j=0}^n \hat{\mathcal{M}}_{E,\pi,k}(Z^{I_j^+}) \times_{\ell=0}^m \hat{\mathcal{M}}_{E,\pi,k}(Z^{I_\ell^-}) \to (\times_{j=1}^n \hat{B}^+ \times \times_{\ell=1}^m \hat{B}^-)^2.$$

**Lemma 4.40.** Let $W$ be the cobordism with cylindrical ends attached. There is a natural map from the fiber product $\text{Fib}(R_-, R_+) \to \hat{\mathcal{M}}_{E,\pi,k}(W)$, which is a homeomorphism. An instanton $A$ in the image is cut out transversely if and only if for
the corresponding element of $\tilde{\mathcal{M}}_{E,\epsilon} \times \prod \tilde{\mathcal{M}}(Z^I)$, the map $R_- \times R_+$ is transverse to the diagonal.

We do not repeat the proof, which follows essentially as in [KM07, Theorem 19.1.4].

We now want a canonical way to cut up the real line (or multiple copies of the real line, in the case of broken trajectories) so that every instanton is written as a fiber product, as above, where most intervals are of a fixed large length $2L$ (where most of the energy is supported), and otherwise are of large variable length $T \geq T_0$. The mechanism for this is given in [KM07, Section 19.2]. For $A \in \mathcal{E}_{E_i, k-1/2}(Y_i)$, we write $c_i(A) = \|\nabla A(\mathbf{s}) + f_x\|_{L^2}$. Fix $\epsilon > 0$ so that any nontrivial instanton on $R \times Y_i$ has $c_i(A(t)) \leq \epsilon$ for some $t \in R$. Let $\beta$ be a cutoff function equal to 1 for $x \geq \epsilon$ and 0 for $x \leq \epsilon/2$.

**Definition 4.15.** Let $I = [-L, L]$. An instanton on $I \times Y$ is said to be centered if

1. $c_i(A(t)) \leq \epsilon/2$ for $t \in [-L, -L + 1] \cup [L - 1, L]$,
2. $c_i(A(t)) \geq \epsilon$ for some $t \in [-L, L]$, and
3. The center of mass
   $$\int t\beta_c(A(t))dt/\int \beta_c(A(t))dt$$
   is zero.

The space of centered instantons forms a smooth Hilbert manifold. Then we have the following analogue of [KM07, Proposition 24.7.3], with the same proof.

**Lemma 4.41.** For any compact subset

$$K \subset \tilde{M}_{E_1, k, \delta}(\alpha_0, \alpha_1) \times_{\alpha_1} \cdots \times_{\alpha_n} \tilde{M}_{E_1, k, \delta}(\alpha_{n-1}, \alpha_n) \times_{\alpha_n} \tilde{M}_{E_2, k, \delta}(\beta_0, \beta_1) \times_{\beta_1} \cdots \times_{\beta_m} \tilde{M}_{E_2, k, \delta}(\beta_{m-1}, \beta_m),$$

there is an $L_0$ so that for all $\infty > L \geq L_0$, we have a neighborhood $V(K, L) \subset \mathcal{M}_{E, \epsilon}^{W}$, for which

1. For any $A \in V(K, L)$, the restriction of $A$ to $\partial W^L$ has $c(A|_{\partial W^L}) \leq \epsilon/2$,
2. Any $A \in V(K, L)$ admits a unique collection of cylinders of length $2L$ in the complement of $W^L$ so that $A|_I$ is a centered instanton, while
3. The complement of both $W^L$ and these intervals consists of cylinders of length at least $T_0$, where $T_0$ is larger than the least $T$ for which Lemma 4.39 applies to each of $Y_1$ and $Y_2$.

As a corollary, we have a map $d : V(K, L) \to (0, \infty)^{n+m}$ measuring the distance between the centers of successive centered intervals; for the ends closest to $W$, we consider $\partial W$ to be the location of that corresponding center. It is easy to see that the values of $d$ do not depend on $K$ or $L$; in particular, taking the union over all $V(K, L)$, we obtain a map $V \to (0, \infty)^{n+m}$ from a neighborhood of the stratum in $\tilde{M}$, the neighborhood $V$ consisting of all instantons that may be partitioned as above.
At the same time, we have a map

$$\text{ev}_T : (0, \infty)^{n+m} \times \mathcal{M}([-\infty, 0] \times \{Y_1(\alpha_0) \times \prod_{i=1}^n \mathcal{M}_{\text{cen}}(Z^i_T) \times \mathcal{M}_\pi(W^L)}$$

$$\times B_\eta(\mathcal{K}_{\beta_j}) \times \prod_{i=1}^n \mathcal{M}_{\text{cen}}(Z^i T^2) \times \prod_{i=1}^n \mathcal{M}_{\text{cen}}(Z^i T^2) \times \left( \times \prod_{i=1}^n B_{E_i,k-1/2} \times \prod_{i=1}^n \bar{B}_{E_2,k-1/2} \right)^2,$$

the map in each case given by restriction; perhaps most notably here, for the $B_\eta(\mathcal{K}_{\cdot})$ factors we are using the maps $\tilde{\eta}(T, -)$ of Lemma 4.39 to parameterize a neighborhood of $\gamma_A$ in $\mathcal{M}_{\gamma_A, k, \delta}(Z^T)$, where $T \in [T_0, \infty]$, and then restricting to the boundary; Lemma 4.39 included the fact that these restriction maps converge in $C^0_{\text{loc}}$ as $T \to \infty$.

We call the domain $(0, \infty)^{n+m} \times \mathcal{M}$ and the codomain $\mathcal{N}$. This map is smooth on each stratum of $(0, \infty)^{n+m}$, and for a convergent sequence $T_n \in (0, \infty)^{n+m}$, the maps $\text{ev}_{T_n}$ converge to the appropriate limit in the $C^0_{\text{loc}}$ topology. We may identify $\text{ev}_{T_n}^{-1}(\Delta)$ in the above fiber product as the neighborhood $V(L)$ in $\mathcal{M}_{W}^{E, k, \delta}(\alpha, \beta, m)$. Now we may apply [KM07, Lemma 19.3.3].

**Theorem 4.42.** Let $W$ be a compact 4-manifold equipped with two boundary components, cylindrical metric near the boundary, an $SO(3)$-bundle $E$ restricting to the pullback of fixed bundles over the collars, and a curve $\gamma$ which is cylindrical at the basepoints on the ends. Further let $W$ be equipped with a perturbation $\pi$ so that $\mathcal{M}_{E, \pi, k}(\alpha, \beta)$ is cut out transversely whenever the expected dimension is less than 10.

Then for any open stratum $\sigma \subset \mathcal{M}_{E, \pi, k}(\alpha, \beta)$, we may find a neighborhood $V(\sigma)$ and an $SO(3)$-invariant map $V(\sigma) \to (0, \infty)^{n+m}$ which is stratum-preserving, a submersion on each stratum, and a topologically trivial fiber bundle for the inverse image of a neighborhood of $\infty$. In particular, $\mathcal{M}_{E, \pi, k}(\alpha, \beta)$ is a compact topological manifold with corners and a smooth structure on each stratum.

**Proof.** Above we constructed a map $\text{ev}_T : (0, \infty)^{n+m} \times \mathcal{M} \to \mathcal{R}$, where $\mathcal{N}$ and $\mathcal{R}$ are Hilbert manifolds; here $\mathcal{N}$ and $\mathcal{R}$ are Hilbert manifolds; here $\eta \in (0, \infty)^{n+m}$. For fixed $T$, the map $\text{ev}_T$ is smooth, and $T \to \text{ev}_T$ is convergent in the $C^0_{\text{loc}}$ topology because the same is true of the restriction maps $\mu_T$ on $B(\mathcal{K}_{\cdot})$ and $B(\mathcal{K}_{\cdot})$. Thus the aforementioned lemma tells us that $\text{ev}^{-1}(\Delta) \to (0, \infty)^{n+m}$ is a topological submersion, which means precisely that $V(\sigma) \to (0, \infty)^{n+m}$ is a trivial fiber bundle near $\infty$. That the projection is smooth and a submersion on each stratum follows because $\mu_T$ is actually smooth in $T$ on each stratum.

We summarize the results of this section as follows.

**Proposition 4.43.** For a regular perturbation $\pi$, and $\text{gr}_z(\alpha, \beta) \leq 7$ as in Corollary 4.13, $\mathcal{M}_{E, \pi, k}(\alpha, \beta)$ can be given the natural structure of a compact topological $SO(3)$-manifold with corners with a smooth structure on each open stratum. We have the following decomposition of the boundary:

$$\mathcal{c} \mathcal{M}_{E, \pi, k}(\alpha, \beta) = \bigcup_{\gamma \in \gamma_{Z_1 \# Z_2}} \mathcal{M}_{E, \pi, k}(\alpha, \gamma) \times_{\gamma} \mathcal{M}_{E, \pi, k}(\gamma, \beta).$$
The same is true on a cobordism \( W \) as long as \( \text{gr}_\alpha^W(\alpha, \beta) \leq 10 - \dim \alpha \). In that case, we have the decomposition
\[
\partial \mathcal{M}^W_{E, z, \pi}(\alpha, \beta) = \bigcup_{\gamma \in \mathcal{C}_{z, 1, \pi} \# z_2 = z} \mathcal{M}^W_{E, 1, z_1, \pi_1}(\alpha, \gamma) \times _\gamma \mathcal{M}^W_{E, 2, z_2, \pi_2}(\gamma, \beta),
\]
\[
\bigcup_{\zeta \in \mathcal{C}_{z_2, 1, \pi} \# z_2 = z} \mathcal{M}^W_{E, 2, z_1, \pi}(\alpha, \zeta) \times _\zeta \mathcal{M}^W_{E, 2, z_2, \pi_2}(\zeta, \beta).
\]

4.8. Families of metrics and perturbations. Let \( S \) be a compact smooth manifold with corners. Let \((W, E)\) be a cylindrical end cobordism \((Y_1, E_1) \rightarrow (Y_2, E_2)\), and suppose each \((Y_i, E_i)\) is equipped with a regular metric and perturbation \((g_i, \pi_i)\). A family of metrics and perturbations parameterized by \( S \), written \( \pi_S \), is the data of:
- a smooth metric on the bundle \( p^\alpha_W, TW \) on \( S \times W \), restricting to the product metric \( dt^2 + g_i \) on \( \mathbb{R} \times Y_i \) on each end, and
- a smooth map \( S \rightarrow \mathcal{P}_c^{(4)} \), the latter being the space of perturbations on the ends of \( W \) which agree with \( \pi_i \) on the corresponding ends. We sometimes allow the codomain to be the space \( \mathcal{P}_c^{(4)} \) instead, adding in addition a finite-dimensional space of perturbations supported on the interior.

We may then define the parameterized moduli spaces \( \mathcal{M}_{E, z, S}(\alpha, \beta) \) of pairs \((s, A)\), where \( A \) is a \( \pi_s \)-perturbed \( L^2_{k, \delta} \) instanton going between critical orbits \( \alpha \) and \( \beta \) in a fixed component of trajectories \( z \). Fiberwise compactifying by ideal instantons and broken trajectories we obtain \( \mathcal{M}_{E, z, S}(\alpha, \beta) \); the result is compact by a version of Proposition 4.10 which allows for variations of metric on the interior (which requires no change in the argument).

There is a bundle over \( S \times \mathcal{B}_{E, z, k, \delta}(\alpha, \beta) \), written \( V^+_S \), whose fiber above \((s, A)\) is \( \Omega^2_{k-1, \delta}(W; g_E) \); the notation \(+, s\) indicates that we are taking the self-dual 2-forms with respect to the metric \( g_s \).

**Definition 4.16.** At each perturbed instanton \((s, A)\) \( \in \mathcal{M}_{E, z, S}(\alpha, \beta) \), taking the derivative of the \( \text{SO}(3) \)-equivariant map \( \sigma : S \times \mathcal{B}_{E, z, k, \delta}(\alpha, \beta) \rightarrow V^+ \) defined by the instanton equation induces a map on the normal space to each orbit,
\[
(\sigma_A)_s : T_s S \times \ker(\alpha^{k,s}_{A, k, \delta}) \rightarrow \Omega^2_{k-1, \delta}(W; g_E).
\]

We say the parameterized moduli spaces \( \mathcal{M}_{E, z, S}(\alpha, \beta) \) are cut out regularly if \( (\sigma_A)_s \) is surjective for all perturbed instantons \((s, A)\) in the homotopy class \( z \). In this case the parameterized moduli space \( \mathcal{M}_{E, z, S}(\alpha, \beta) \) is a smooth \( \text{SO}(3) \)-manifold of dimension \( \text{gr}_z(\alpha, \beta) + \dim S \).

We say the compactified parameterized moduli spaces \( \overline{\mathcal{M}}_{E, z, S}(\alpha, \beta) \) are cut out regularly if every moduli space \( \overline{\mathcal{M}}_{E, w, S}(\gamma, \gamma') \) appearing in the compactification is cut out regularly.

If \( \overline{\mathcal{M}}_{E, z, S}(\alpha, \beta) \) is cut out regularly for all \((\alpha, \beta, z)\) with \( \text{gr}_z(\alpha, \beta) + \dim S \leq 8 \), we say that \( \pi_S \) is regular.

Recall Definition 4.14 of weakly admissible cobordism. In the first two cases, no reducibles arise for any perturbation, and still do not arise in families. In the third case, there is a unique reducible of each type for every perturbation, and in particular for the moduli spaces parameterized by a family of perturbations there is a \( \dim S \)-dimensional space of reducibles of each type.
In the final case, we have $b^+(W) > 0$ and $E$ is nontrivial. In this case, for a generic perturbation, there are no reducibles; the expected dimension of the space of reducibles is $-b^+(W)$. This means that they do arise in families. Something else needs to be asserted to guarantee that in families, reducibles are still cut out transversely. Instead of adding a strong additional homological condition, we will simply avoid them.

Recall that we say $(A, π)$ is regular if the map $D^*_A,π : \ker(d_A^*) → Ω^{2,+}$ is surjective. This is equivalent to asking that the usual ASD operator $Q^{b^+}_{A,x} : Ω^1 → (Ω^{2,+} ⊕ Ω^0_0)$ is surjective, where $Ω^0_0$ is defined to be the $L^2$ orthogonal complement to the space $g_A$ of $A$-parallel sections of the bundle $g_E$.

Equivalently, at any $A$, we see that this is asking whether the map $Q_{A,π} + r_A : Ω^1 ⊕ g_A → Ω^{2,+} ⊕ Ω^0$ is surjective, where the map $r_A : g_A → Ω^0$ is such that $⟨r_A, σ⟩ = ∥σ∥^2$.

Fix a fully reducible connection $A$. Observe that because the map $P_{δ}^{(4)} → F(Ω^1, Ω^{2,+} ⊕ Ω^0)$ given by sending

$$π → (d^+, d^*) + D_A V_π + r_A = Q_{A,π} + r_A$$

is continuous, the set of $π$ so that $(A, π)$ is regular forms an open set containing $0$. Let $U_A$ be the largest open ball in $P_{δ}^{(4)}$ containing $0$ so that $(A, π)$ is regular for all $π ∈ U_A$. If $π ∈ U_A$ for all fully reducible connections $A$ over $E$, we say that $π$ is $E$-small; this condition is vacuous when $E$ is nontrivial, as it supports no fully reducible connections.

**Proposition 4.44.** Suppose $(W, E)$ is equipped with a weakly admissible bundle. First suppose that $b^+(W) = 0$. Let $π_0, π_1 ∈ P_{L',c}^{(4)}$ be $E$-small regular perturbations on $W$ with the same values on the ends.

Then there is a finite set $L'$ of holonomy perturbations adapted to a collection of thickened loops, with $L ⊂ L'$, and a regular family $π_t$ of metrics and perturbations $π_t : [0, 1] → P_{L',c}^{(4)}$, all with the same values on the ends, so that all of the $π_t$ are $E$-small. If one instead begins with a path $π_t$ of perturbations so that $(A, π_t)$ is regular for all fully reducible $A$ and any $t$, with $π_0$ and $π_1$ regular at all $A$, then one may make an arbitrarily small perturbation to $π_t$ on the interior of $[0, 1]$ to make it a regular family.

If $b^+(W) = 1$, let $π_t$ be a path of metrics and perturbations, constant on the ends, with $π_0$ and $π_1$ regular perturbations; suppose further that no $π_t$ supports a reducible instanton. Then we may modify $π_t$ by an arbitrarily small perturbation on the interior so that $π_t$ forms a regular family of metrics and perturbations.

In the proof of this statement, we need a weaker version of the usual transversality theorem.

**Lemma 4.45.** Let $f : M → N$ be a smooth map of separable metrizable Banach manifolds, and let $S ⊂ N$ be a submanifold of codimension $n$, closed as a subset of $N$. For every $x ∈ M$ such that $f(x) ∈ S$, suppose that $\text{codim(Im(df_x) + T_{f(x)}S)} ≤ k$. Then $f^{-1}(S)$ is contained in a countable union of submanifolds of codimension at least $n - k$. If $S$ is closed and $M$ is finite-dimensional, this union may be taken to be locally finite; if $M$ is compact, it may be taken to be finite.

**Proof.** For each $x ∈ f^{-1}(S)$, suppose $\text{codim(Im(df_x) + T_{f(x)}S)} = j ≤ k$. Choose a neighborhood $U_x$ of $x$ modelled on the Banach space $T_x M$ and a neighborhood...
of $f(x) \in (N, S)$ modelled on the pair of Banach spaces $(T_{f(x)} N, T_{f(x)} S)$, so that in this model $f(0) = 0$. Choose a linear map $g : \mathbb{R}^J \to T_{f(x)} N$ so that $(df_{x}, i, g) : T_{x} M \oplus T_{x} S \oplus \mathbb{R}^J \to T_{f(x)} N$ is surjective, and consider $f + g : U_x \oplus \mathbb{R}^J \to T_{f(x)} N$. By assumption this is transverse to $T_{f(x)} S$ at 0, and hence $(f + g)^{-1}(S)$ is a smooth manifold of codimension $n - j$ in a neighborhood of 0; call this manifold $P_x$. The map ker$(f + g) \to T_{x} M$ is injective, because ker$(g)$ is zero by a dimension-count. Thus in a sufficiently small neighborhood of zero, the projection map $\pi : P_x \to U_x$ is injective; replace $U_x$ by a smaller open set $U'_x$ so that $\pi^{-1}(U'_x) \to U'_x$ is injective.

Doing this for each $x$, we obtain an open cover of $f^{-1}(S)$; because $M$ is a separable metrizable space and hence so is any subspace, $f^{-1}(S)$ is Lindelöf: every open cover has a countable subcover. In particular, we may cover $f^{-1}(S)$ by countably many of these $U'_x$, enumerate them as $U'_i$. Because Im$P'_i$ is a submanifold of $U'_i$, and $f^{-1}(S) \cap U'_i \subset$ Im$P'_i$, we find that $f^{-1}(S)$ is contained in a countable union of submanifolds of codimension $n - j \geq n - k$.

If $S$ is closed, so is $f^{-1}(S)$, and if $M$ is in addition finite-dimensional then $f^{-1}(S)$ is paracompact, so we may choose a locally finite subcover. If $M$ is furthermore compact, then $f^{-1}(S)$ is compact, and so we may choose a finite subcover.

Proof of Proposition 4.44. As before, we start at the reducible locus and induct upwards. Recall from Definition 4.14 that there are four types of weakly admissible bundles. If either $(Y_1, L_1)$ is admissible or $\partial w_2(E) \neq 0$, then the bundle supports no reducible instantons for any small perturbation $\pi$, so we may ignore these first two cases for now.

Suppose now that $(W, E)$ is of the third type. Because $U_E$ is a ball around 0 in $P^{(4)}$, its intersection with a hyperplane (such as the space of perturbations with fixed ends) is either a ball or empty. If $\pi_0$ and $\pi_1$ lie in one such nonempty slice, then we may clearly connect them by a path in that same slice. In whatever case, we assume we have chosen a path $\pi_t$ from $\pi_0$ to $\pi_1$ so that all full reducibles are cut out transversely, for all $\pi_t$.

We focus attention now on $SO(2)$-reducibles which are not fully reducible; these correspond to connections respecting some splitting $\mathbb{R} \oplus \zeta \cong g_E$, where $\zeta \cong \eta^{\delta(2)} \lambda^{-1}$ is a complex line bundle; the $SO(2)$ of parallel gauge transformations acts by weight one on $\eta$, and hence by weight two on $\zeta$.

The content of Lemma 4.17 is that there is a smooth submanifold $\mathcal{P}_{L,c}\mathcal{M}_0$ of the Banach manifold $\mathcal{P}_{L,c}B_{\pi,k,\delta}$ of pairs $(\pi, A)$, where $\pi$ has fixed ends and $A$ is an equivalence class of $\delta$-exponentially decaying connection on the line bundle $\zeta$; the submanifold is those pairs so that $A$ is a $\pi$-instanton. Then Lemma 4.18 shows that the projection of this parameterized moduli space to $\mathcal{P}_{L,c}^{(4)}$ is proper. (One needs to include an energy bound if one wants to view this as varying over all line bundles $\eta$, which we will want to when we conclude.)

Modify $\pi_t$ now to be transverse to this projection; by assumption this is true at $\pi_0$ and $\pi_1$. Hodge theory implies that the derivative of this projection has index $b_1(W) - b^+(W) = 0$ (in particular, because the perturbations on the ends are small enough that the index is the same as that at the trivial perturbation).

We should further perturb to ensure that $\pi_S$ is regular normally to the reducibles, not just inside the reducible locus.
What Theorem 4.36 began with was that there is a smooth manifold $Q^{\text{irred}} : P_{L,c}M_\eta \to \mathcal{F}_j \left( \Omega^1, \Omega^0((\zeta) \oplus \Omega^{2,\pm}(\zeta)) \right).

The spaces $\mathcal{F}_j$ of operators of index $I$ whose cokernel is isomorphic to $C^I$ are smooth submanifolds, and the normal space to an operator $A$ is $\text{Hom}_S(\ker(A), \text{coker}(A))$ by Lemma 4.35. In particular, for us $I \geq 0$, so this space is $\text{Hom}_S(\mathcal{C}^{j+I}, C^I) = C^{j^2+jI}$, a $2j^2 + 2jI$-dimensional normal space.

(We work locally to choose lifts of elements of $M$ to actual connections. The sets $P_{L,c}M_\eta^{(s)}$, consisting of $(\pi, A)$ that map to an operator whose cokernel is isomorphic to $C^I$, are independent of the choice of lift.)

Ideally, we would have shown that the map $Q^{\text{irred}}$ is transverse to this family of submanifolds; then $P_{L,c}M_\eta^{(s)}$ would all have codimension at least 2, and in particular a generic map from an interval would be transverse to all of these. Unfortunately, this is not what Theorem 4.36 showed. Rather, because the perturbations were defined using a construction on a 3-manifold, the maps we constructed arose from embeddings $\ker(Q) \to V \leftarrow \text{coker}(Q)$ for some finite-dimensional space $V$, and constructing arbitrary self-adjoint maps $V \to V$. This self-adjointness conditions means that if the images of $\ker(Q)$ and $\text{coker}(Q)$ ever intersected, there would be restrictions on the set of maps $\ker(Q) \to \text{extcoker}(Q)$ we could construct this way. But we see that at least the image of $P_{L,c}^{(4)} \to \text{Hom}(\ker(Q^{\text{irred}}), \text{coker}(Q^{\text{irred}})) \cong \text{Hom}(C^I, C^I)$ is of dimension at least as large as the space of symmetric equivariant maps $C^I \to C^I$; that is, the image has dimension at least $i(i+1)$. In particular, applying Lemma 4.45, the subset $P_{L,c}M_\eta^{(s)}$ is contained in a countable union of submanifolds $P_t \subset P_{L,c}M_\eta^{(s)}$ of codimension at least $2i^2 - i(i-1) = i^2 + i \geq 2$. In particular, one may perturb any map $[0, 1] \to P_{L,c}^{(4)}$ by an arbitrarily small amount and without changing the endpoints to be transverse to these projections $P_t \to P_{L,c}^{(4)}$.

(One should first choose a neighborhood $U$ in $P_{L,c}^{(4)}$ around each $\pi_0$ and $\pi_1$ so that if $\pi_t \in U$, then $\pi_t$ is regular, and delete the portions of the $P_t$ lying above each $\bar{U}$, so as to ensure that $\pi_0$ and $\pi_1$ are not in the image of the $P_t$.)

In any case, we have constructed a path $\pi_t$ so that the reducibles are cut out transversely with respect to the family $\pi_t$.

In the remaining case $b^+(W) > 0$, we must assume that $\pi_t$ supports no reducible instantons for any $t$. When $b^+(W) > 1$, there is no difficulty in assuring this by a small homotopy of the path.

What remains is to show that we may achieve family transversality for the irreducible instantons. The usual approach is to say that we have a projection from a parameterized manifold $P_{L,c}M \to P_{L,c}^{(4)}$, and we want to modify the arc to be transverse to this map. The proof is exactly the same as Theorem 4.36, except that some additional care needs to be taken at the set of $\pi_t$-flat instantons. This is still a compact space, but the perturbations of $L$ may not be enough to achieve transversality at all $\pi_t$-flat connections. Instead we must add further interior holonomy perturbations to achieve transversality at all of these, but there is no difficulty in following the same arguments as before. The essential condition was that every $\pi_t$-flat instanton has nontrivial holonomy along some loop in the family. Every free homotopy class of loop in $W$ may be homotoped to lie outside the
manifolds $[-t, t] \times S^1 \times S^2$ along which we apply interior holonomy perturbations, and so this remains true after demanding that the thickened new loops do not overlap with the previous thickened loops. Thus proceeding exactly as before, we may produce a larger set $L'$ of interior holonomy perturbations adapted to thickened loops so that there is a small map $\pi_t : [0, 1] \to \mathcal{P}_{L',c}$, agreeing with our original path near the ends.

\[
\hat{\nabla}_{s'}(\pi) A : T_s S \to \text{coker}(Q_{\text{irred}}),
\]

is an equivariant map. The former space has trivial circle action, and the latter space has circle action of weight 2: the only equivariant map is zero.

When $E$ admits no reducible connections, there is no difficulty applying the usual ideas: there is a smooth map $\pi : \mathcal{P}M \to P$, and we choose a map $S \to P$ transverse to $\pi$ extending a given map $B : S \to P$ transverse to $\pi$.

Then we have smooth moduli spaces $\tilde{\mathcal{M}}_{E,z,S}(\alpha, \beta)$ with smooth projection maps $\tilde{\mathcal{M}}_{E,z,S} \to S$. The space $\tilde{\mathcal{M}}_{E,z,S}$ is not compact but the fiberwise Uhlenbeck compactification $\overline{\mathcal{M}}_{E,z,S}$ is. As long as $\text{gr}_W^z(\alpha, \beta) + \dim \alpha + \dim S \leq 10$ (this being at least the dimension of $\overline{\mathcal{M}}_S/\text{SO}(3)$), there is no Uhlenbeck bubbling arising in this compactification. Note that here it is possible for $\overline{\mathcal{M}}_{E,z,S}$ to be nonempty even if $\text{gr}_z(\alpha, \beta)$ is negative. In this case, the set of $s \in S$ so that $\pi_s$ is nonempty is a compact submanifold with corners of codimension $-\text{gr}_z^W(\alpha, \beta)$.

We then have the following analogue of Theorem 4.42.

**Proposition 4.46.** If $\pi_S$ is a regular family of metrics and perturbations indexed by $S$, and $\text{gr}_z^W(\alpha, \beta) + \dim \alpha + \dim S \leq 7$, then $\overline{\mathcal{M}}_{E,z,S}$ has the natural structure of a compact topological $\text{SO}(3)$-manifold with corners and a smooth structure on each stratum. Its boundary has the decomposition

\[
\partial \overline{\mathcal{M}}_{E,z,S}(\alpha, \beta) = \bigcup_{\gamma \in \mathcal{C}_1} \mathcal{M}_{E_1,z_1,\pi}(\alpha, \gamma) \times_{\gamma} \overline{\mathcal{M}}_{E_2,z_2,\pi}(\gamma, \beta)
\]

\[
\bigcup_{\zeta \in \mathcal{C}_2} \mathcal{M}_{E_1,z_1,\pi}(\alpha, \zeta) \times_{\zeta} \overline{\mathcal{M}}_{E_2,z_2,\pi}(\zeta, \beta) \cup \overline{\mathcal{M}}_{E,z,S}(\alpha, \beta).
\]

In particular, suppose $S = [0, 1]$, and write $W$ as the composite of cobordisms with cylindrical ends $W_1$ and $W_2$, obtained by splitting together the positive end of $W_1$ with the negative end of $W_2$. Write the broken perturbation $\pi(1)$ as $\pi_-$ on each of the $W_i$, where $\pi_-$ is the perturbation on the negative end of $W_1$ and $\pi_+$ is the perturbation on the positive end of $W_2$; write the common perturbation in the center
as $\pi_b$. So long as $g_0^W(\alpha, \beta) \leq 9 - \dim \alpha$, we have a decomposition
\[
\bar{\mathcal{M}}_{E,S,z}(\alpha, \beta) = \bigcup_{\gamma \in C_\pi} \mathcal{M}_{E-,z_1,\pi_-(\alpha, \gamma)} \times_\gamma \mathcal{M}_{E,z_2,S}(\gamma, \beta)
\]
\[
\bigcup_{\eta \in C_{\pi_b}} \mathcal{M}_{E_1,z_1,\pi_1(\alpha, \eta)} \times_\eta \mathcal{M}_{E_2,z_2,\pi_2(\eta, \beta)}
\]
\[
\bigcup_{\zeta \in C_{\pi_c}, z_1 \# z_2 = z} \mathcal{M}_{E_{+},z_1,S(\alpha, \zeta)} \times_\zeta \mathcal{M}_{E_{+},z_2,S(\zeta, \beta)}
\]
\[
\cup \mathcal{M}_{E_2,\pi(0)}(\alpha, \beta).
\]

Proof. Fix an open stratum $\sigma \subset \mathcal{M}_{E,z,S}(\alpha, \beta)$ corresponding to $n$ breakings along $Y_1$ and $m$ breakings along $Y_2$. The result will follow essentially as in Theorem 4.42. There we described a gluing map
\[
ev_T : (0, \infty)^{n+m} \times \mathcal{N} \to \mathcal{R}
\]
between Hilbert manifolds. The manifold $\mathcal{N}$ consisted of products of spaces parameterizing $n$ long (length $L$) broken pieces of framed instantons on $[-L, L] \times Y_1$, and similarly $m$ on $(Y_2, E_2)$, a space parameterizing instantons on cylinders $[T, T] \times Y_i$ for all $T_0 \leq T \leq \infty$ sufficiently large, and a space parameterizing instantons on the compact manifold $W^L$ (which is $W$ with the cylindrical ends $[0, \infty) \times Y$ truncated to $[0, L] \times Y$). The codomain $\mathcal{R}$ consisted of restrictions of these to $2n$ copies of the configuration space of framed connections on $Y_1$ and $2m$ copies of the configuration space of framed connections on $Y_2$. This is smooth on each stratum of $\mathcal{N}$, and the individual maps $ev_T$ converge in $C^\infty_{loc}$ as $T_0 \to T \in (0, \infty)^{n+m}$. The moduli space of instantons on $W$, compactified by broken trajectories, is identified with $ev^{-1}(\Delta)$, the subset of instantons on each piece so that the restrictions to corresponding boundary pieces agree.

When we allow the metric and perturbation to vary in some family $S$, this variation occurs in a compact part of the manifold with cylindrical ends $W$. As long as $L$ above is taken large enough, this variation only takes place in $W^L$, and hence only affects that term of the product; we may replace the term $\mathcal{M}_{E,S}(W^L)$ instead with a parameterized version $\mathcal{M}_{E,S}(W^L)$. The modified evaluation map above is written $(0, \infty)^{n+m} \times \mathcal{N}_S \to \mathcal{R}$, and is again smooth on each stratum of $(0, \infty)^{n+m}$, with convergent sequences of maps $ev_{T_k}$ giving sequences of maps $ev_{T_k}$ which converge in $C^\infty_{loc}$. Again we have that $\mathcal{M}_{E,z,S} = ev^{-1}(\Delta)$.

Therefore we may apply [KM07, Lemma 19.3.3], which asserts that $ev^{-1}(\Delta) \to (0, \infty)^{n+m}$ is a topological submersion, meaning that it is a topologically trivial fiber bundle near $\infty$. In particular, this gives a chart for a neighborhood of $\sigma \subset \mathcal{M}_{S,z}$, an open embedding $(0, \infty)^{n+m} \times \sigma \to \mathcal{M}_{S,z}$ that is the identity on $\sigma$, is stratum-preserving, and is a diffeomorphism on each open stratum.

We will need a version of these results for more general families of metrics and perturbations termed ‘families of broken metrics’ (see [Dae15] and [KM11a], where this notion was used to study spectral sequences from Khovanov homology).

Suppose are given a 4-manifold $W$ with cylindrical ends $(-\infty, 0] \times Y_1$ and $[0, \infty) \times Y_2$, and a sequence of Riemannian 3-manifolds $M_1, \cdots, M_k$, equipped with
an isometric embedding
\[ \cup \varphi_i : (-1 - \epsilon, 1 + \epsilon) \times M_i \to W. \]

Each interval \([-1, 1] \times M_i\) will be equipped with a perturbation \(\nabla_{\pi_i}\), which restricts to some neighborhood of \((-c_i, c_i) \times M_i\) as \((dt \wedge \nabla_{\pi_i})^+\) for some regular perturbation \(\nabla_{\pi_i}\) on the pair \((M_i, E_i)\) (that is, so that the critical orbits are cut out nondegenerately and all moduli spaces up to some sufficiently large dimension are also cut out nondegenerately). We say that the perturbation is adapted to the cut.

To define the restriction map for the framed instanton moduli spaces, we need to assume that the path \(\gamma : \mathbb{R} \to W\) is adapted to the cuts, in the sense that there is a sequence \(t_1, \ldots, t_\ell\) of real numbers with \(t_i + 2 < t_{i+1}\) so that for the interval \(t \in [t_i - 1, t_i + 1]\), we have \(\gamma(t + t) = \varphi_i(t - t, x_i)\) for some fixed \(x_i \in M_i\). In particular, \(\gamma\) passes through the \(M_i\) in the order listed.

The canonical family of broken metrics on \(W\) is parameterized by \([0, \infty]^\ell\); for a fixed element \((T_1, \ldots, T_\ell)\), one replaces the isometric copies of \([-1, 1] \times M_i\) with isometric copies of \([-1 - T_i, 1 + T_i] \times M_i\) if \(T_i < \infty\), or with \([-1, \infty) \cup (-\infty, 1]\) \times M_i\) if \(T_i = \infty\), and replaces the image of \(\gamma\) appropriately. The perturbation \(\nabla_{\pi_i}\) associated to \((T_1, \ldots, T_\ell)\) is defined by assuming that its value on the interval \((-c_i - T_i, c_i + T_i)\) is equal to \((dt \wedge \nabla_{\pi_i})^+\) as above; this remains true in the infinite case. We say that the metric parameterized by \((\infty, \ldots, \infty)\) is an \(\ell\)-times broken metric on \(W\), with cuts along \(M_1, \ldots, M_\ell\).

An family of broken metrics and perturbations is given by a compact smooth manifold with corners \(S\) equipped with a family of metrics and perturbations on \(W\), possibly containing broken metrics and perturbations. If \(s \in S\) corresponds to one of these broken metrics, cut along \(M_1, \ldots, M_\ell\), then we demand there is a chart \(\varphi : (T, \infty]^{\ell} \times U \to S\), where \(U\) is some topological manifold with corners and \(\varphi\) is a stratum-preserving open embedding, so that \(\varphi(T_1, \ldots, T_\ell, x)\) is the element of the canonical family of broken metrics and perturbations associated to \((T_1, \ldots, T_\ell)\) described above. The variance in metric and perturbation parameterized by \(x \in U\) occurs in the complement of the canonical intervals \(I \times M_i\). The union over the images of \(\varphi(U)\) in \(S\), as the \(\varphi\) vary over charts for neighborhoods of a specific sequence of cuts \(M_1, \ldots, M_\ell\), is called the cut stratum for \(M_1, \ldots, M_\ell\).

If \(S\) parameterizes a family of broken metrics and perturbations, we define the parameterized moduli space \(\hat{\mathcal{M}}_{\ell, S}\) as follows. For a fixed broken metric and perturbation \(\pi\), cut along \(M_1, \ldots, M_\ell\), we are left with a sequence \(W_0, \ldots, W_\ell\) of manifolds with two cylindrical ends; other than the outer two, we have \(W_i\) a manifold with incoming cylindrical end \((-\infty, 0] \times M_i\) and outgoing cylindrical end \([0, \infty) \times M_{i+1}\). Each of these has a corresponding collection of moduli spaces

\[
\bigcup_{\alpha_{i+1} \in E_{\pi_i}} \hat{\mathcal{M}}_{\pi}^{W_i} \bigl(\alpha_i, \alpha_{i+1}\bigr); \quad z_i \in \pi_i(\partial \hat{E}_{\pi_i}(\alpha_i, \alpha_{i+1}))
\]

we say that an element of \(\hat{\mathcal{M}}_{\pi}^{W_i}\) is a sequence of framed instantons \(A \in \hat{\mathcal{M}}_{\pi}^{W_i} \bigl(\alpha_i, \alpha_{i+1}\bigr)\) so that the concatenation \(z_1 \ast \cdots \ast z_\ell\) is \(z\). That is, writing the space given by the disjoint union of \(\pi_{\ell}\)-critical orbits on \(M_i\) as \(\tilde{R}_{\pi_i}(M_i)\), we have

\[
\hat{\mathcal{M}}_{\pi}^{W_i} := \hat{\mathcal{M}}_{\pi}^{W_0} \times \tilde{R}_{\pi_1}(M_1) \ast \cdots \ast \tilde{R}_{\pi_\ell}(M_\ell) \hat{\mathcal{M}}_{\pi}^{W_\ell},
\]

which then may be decomposed as a disjoint union according to the paths \((z_1, \ldots, z_\ell)\).
The moduli space of the family is then defined to be
\[ \tilde{M}_{E,z,S}(\alpha_0, \alpha_n) = \bigcup_{s \in S} \tilde{M}_{E,z,\pi_s}(\alpha_0, \alpha_n), \]
topologized so that \((s_n, A_n)\) converges to \((s, A)\) if \(s_n \to s\) and \(A_n \to A\) in \(L^p\) on compact sets.

We say that the individual moduli space \(\tilde{M}_{E,z,\pi}(\alpha_0, \alpha_n)\) is regular if each component moduli space \(\tilde{M}_{E,z,\pi}(\alpha_1, \alpha_{\ell+1})\) is cut out regularly in dimensions at most 10.

We say that a family of broken metrics and perturbations parameterized by \(S\) is regular if, for each cut stratum \(\sigma\) of \(S\) parameterizing \(\ell\)-broken metrics with cuts along \(M_1, \ldots, M_\ell\), but varies metric and perturbation elsewhere on \(W\), we have that each of the component moduli spaces \(\tilde{M}_{E,z,\pi}(\alpha_1, \alpha_{\ell+1})\) is regular. For \(\ell = 0\), this just says that the open submanifold of \(S\) parameterizing unbroken metrics should be regular in the sense of Definition 4.16.

One may then define the fiberwise Uhlenbeck compactification \(\overline{M}_S\) as before, by setting \(\overline{M}_s\) to be the fiber products
\[ \overline{M}_{E_0,\pi_0} \times \overline{R}_{\pi_1}(M_1) \times \cdots \times \overline{R}_{\pi_\ell}(M_\ell) \overline{M}_{E,\pi_\ell} \]
of Uhlenbeck compactifications (by ideal instantons and broken trajectories); \(\overline{M}_{E,\pi,S}\) is the union of these over \(s \in S\), topologized as above. One easily verifies that \(\overline{M}_S\) is compact by the same argument as the case of families of unbroken metrics; now if \(s_n \to s\) is a sequence that limits to a metric and perturbation with more breakings, energy slides off the newly-infinite ends. If \(\overline{M}_S\) is regular and of dimension at most 10, then this compactification only includes broken trajectories, not ideal instantons.

**Proposition 4.47.** Let \(S\) parameterize a family of broken metrics and perturbations on \((W, E)\) so that \(\partial S\) is a regular family. Then the family of broken metrics on \(S\) is a regular family in a neighborhood of the boundary, and on the interior only consists of unbroken metrics and perturbations, and so by Proposition 4.44, if \(E\) is weakly admissible we may modify \(S\) on the interior to make it a regular family of metrics and perturbations.

Now suppose that \(S\) parameterizes a regular family of broken metrics and perturbations on \((W, E)\). Let \(\sigma \subset S\) be a cut stratum of codimension \(\ell\), so parameterizing metrics with a sequence of cuts along \(M_1, \ldots, M_\ell\). For any stratum \(\tilde{\sigma} \subset \overline{M}_{E,\pi_\ell}(\alpha, \beta)\) of codimension \(n + k_1 + \cdots + k_\ell + m\), parameterizing broken \(s\)-perturbed instantons for \(s \in \sigma\) with \(n\) breakings along the incoming end \(Y_1\), with \(k_1\) breakings along the internal end \(M_1\), and with \(m\) breakings along the outgoing end \(Y_2\). Then there is a neighborhood
\[ V(\sigma) \subset \overline{M}_{E,z,S}(\alpha, \beta) \]
and an \(SO(3)\)-invariant map \(V(\sigma) \to (0, \infty)^{n+k_1+\cdots+k_\ell+m}\) which is stratum-preserving, a submersion on each stratum, and a topologically trivial fiber bundle for the inverse image of a neighborhood of \(\infty\). In particular, \(\overline{M}_{E,S,z}(\alpha, \beta)\) is a compact topological manifold with corners and a smooth structure on each stratum, with the same decomposition as in Proposition 4.46.

**Proof.** This follows the same lines as the proof of the gluing theorem: first, we may use Lemma 4.39 to build charts \(\mu_T : B_n(H_1) \to \overline{M}_{\alpha,\delta}(Z^T)\) near the constant
solution \( \gamma_0 \), where \( Z^T_t = [-T, T] \times M_t \) for finite \( T \) and 
\[
(0, \infty) \cup (-\infty, 0] \times M_0
\]
for \( T = \infty \). For uniformity of notation, set \( M_0 = Y_1 \) and \( M_{t+1} = Y_2 \). Then as in the proof of Lemma 4.41, for sufficiently large fixed \( L \) may assemble a map with domain the product 
\[
(0, \infty)^{n+k_1+\cdots+k_t+m} \times \tilde{M}_{E_1, k, \delta}(\alpha_0) \times^n B_{\eta_0}(\hat{H}_0) \times^n \tilde{M}_{cen}(Z^L_0) \\
\times \tilde{M}_{\sigma, E_1, k} \times^k B_{\eta_1}(\hat{H}_1) \times^k \tilde{M}_{cen}(Z^L_1) \\
\cdots \\
\times \sigma \tilde{M}_{\sigma, E_1, k} \times^k B_{\eta_{t+1}}(\hat{H}_{t+1}) \times^k \tilde{M}_{cen}(Z^L_{t+1}) \\
\times^m B_{\eta_{t+1}}(\hat{H}_{t+1}) \times^m \tilde{M}_{cen}(Z^L_{t+1}) \times \tilde{M}_{E_2, k, \delta}(\beta).
\]

Here the \( \times_{\sigma} \) indicates that the moduli spaces 
\[
\tilde{M}_{W_0} \times \tilde{M}_{W_1} \times \cdots \times \tilde{M}_{W_\ell}
\]
should then be considered as those moduli spaces altogether parameterized by \( \sigma \), but not that we parameterize each by \( S \) separately and take the product. Secondly, here we mean by \( W_i \) the compact manifolds given by truncating the cylindrical ends of \( W_i \), written \( \text{End}(W_i) = [0, \infty) \times \partial W_i \), at some large finite \( [0, L] \).

The codomain of the map is restriction to the boundary of each piece (this uses parallel transport along the path \( \gamma \)):
\[
\left( \times^n \bar{B}_{Y_1, k-1/2} \times^k \bar{B}_{M_1, k-1/2} \times \cdots \times^k \bar{B}_{M_\ell, k-1/2} \times^m \bar{B}_{Y_{t+1}, k-1/2} \right)^2.
\]

Call the domain \( (0, \infty)^{n+k_1+\cdots+k_t+m} \times \mathcal{N} \) and the codomain \( \mathcal{R} \). Mapping to an element of the diagonal in the codomain means that the corresponding instantons on each piece may be glued to a (possibly broken) instanton on the sequence of manifolds with cylindrical ends \( W_0, \ldots, W_\ell \). The assumption that \( \partial S \) is regular means, in particular, that \( \sigma \) is a regular family of metrics and perturbations, and so the map \( ev_{\infty} : \mathcal{N} \to \mathcal{R} \) is transverse to the diagonal. By convergence in \( C^1_{loc} \), the same is true for other nearby \( ev_T \), and in particular the families \( (T, \sigma) \) are regular families of metrics and perturbations for \( T \) sufficiently large. This is the first part of the proposition.

Now [KM07, Lemma 19.3.3] gives the second part: the fact that this map 
\[
(0, \infty)^{n+k_1+\cdots+k_t+m} \times \mathcal{N} \to \mathcal{R}
\]
is transverse to the diagonal at \( \infty \) means that the projection 
\[
ev^{-1}(\Delta) \to (0, \infty)^{n+k_1+\cdots+k_t+m}
\]
is a topological submersion at \( \infty \), which is smooth on each stratum, and hence we may obtain a corresponding chart for a neighborhood of \( ev^{-1}(\Delta) \subset ev^{-1}(\Delta) \), which is smooth on each stratum. This space \( ev_{\infty}^{-1}(\Delta) \) contains a neighborhood \( V(\sigma) \) of the stratum of \( \bar{M}_S \) described in the statement, and so we may conclude.

\( \blacksquare \)
4.9. Orientations. We begin by briefly recalling some generalities on determinant lines. Let \( H_1 \) and \( H_2 \) be Hilbert spaces, and let \( \mathcal{F}(H_1, H_2) \) be the space of Fredholm maps. We would like to claim that there is a natural line bundle over \( \mathcal{F}(H_1, H_2) \), called the determinant line bundle, whose fiber over \( T \) is isomorphic to \( \Lambda^*(\ker(T)) \otimes \Lambda^*(\coker(T))^* \); we intuit this as being the determinant line of the virtual vector space \( \ker(T) - \coker(T) \).

If one attempts to do this naively (with literally those fibers), the result does not have continuous transition functions due to jumps in the kernel. Instead, given any Fredholm operator \( T_0 \), choose a finite-dimensional subspace \( J \subset H_2 \) so that \( T_0 \otimes \text{Id}_J : H_1 \oplus J \to H_2 \) is surjective, and instead define the determinant line bundle to be \( \Lambda^*(\ker(T) \oplus \text{Id}_J) \otimes \Lambda^*(J^*) \) for \( T \) near \( T_0 \). Because \( T + \text{Id}_J \) is surjective for \( T \) close enough to \( T_0 \), this gives a well-defined line bundle over this chart; one should argue that the line bundles are identical for different choices of \( J \), and that doing this on an open cover defines a legitimate line bundle on \( \mathcal{F}(H_1, H_2) \).

The idea of the determinant line bundle originates in \[\text{Qui}85\] in the context of Riemann surfaces; the operators \( \tilde{T}_J \) define a family of Fredholm operators over the moduli space of all Riemann surfaces. Our application is similar, as with most applications of this construction. Also see \[\text{KM}07\], Section 20.2, in which the authors describe the construction of the determinant line bundle, and explain how to construct a canonical associative isomorphism

\[
q : \det(\mathcal{F}(H_1, H_2)) \otimes \det(\mathcal{F}(K_1, K_2)) \to \det(\mathcal{F}(H_1 \oplus K_1, H_2 \oplus K_2)),
\]

its definition involves signs that depend on the choice of \( J \) in a local description as above. All in all, we have smooth real line bundles over each \( \mathcal{F}(H_1, H_2) \) which have canonical direct-sum isomorphisms.

Now let \((W, E)\) be a Riemannian 4-manifold equipped with weakly admissible \( SO(3)\)-bundle and regular perturbation \( \pi \), with two cylindrical ends: one incoming \((Y_1, E_1, \pi_1)\) and one outgoing \((Y_2, E_2, \pi_2)\). If \( \alpha \) is a \( \pi_1 \)-flat connection on \( Y_1 \), and similarly \( \beta \) is \( \pi_2 \)-flat, then the space of connections \( \mathcal{A}_{E,k,\delta}(\alpha, \beta) \) admits a smooth map

\[
\mathcal{A}_{E,k,\delta}(\alpha, \beta) \to \mathcal{F}\left(\Omega^{1,\delta}_k, \Omega^{2,+}_k \oplus \Omega^0_{k-1,\delta}\right),
\]

sending a connection \( A \) to the perturbed ASD operator

\[
Q_{A,\pi} = (d_A^1 + \hat{D}_A \hat{\nabla}_\pi, d_A^0);
\]

pulling back the determinant line bundle on the space of Fredholm operators, we find that we have a line bundle over \( \mathcal{A}_{E,k,\delta}^{(4)} \). Furthermore, because \( Q_A \) is invariant under the gauge group, we see that \( Q \) descends to an \( SO(3) \)-invariant map \( \mathcal{B}_{E,k,\delta}^\mathcal{F} \to \mathcal{F} \), and hence that there is an \( SO(3) \)-equivariant line bundle \( \det(Q) \) over this space.

**Lemma 4.48.** The line bundle \( \det(Q) \) over \( \mathcal{B}_{E,k,\delta}^\mathcal{F} \) is trivializable.

**Proof.** For convenience of notation we drop the super/subscripts on \( E \); everything here is modulo even gauge.

We begin with Donaldson’s stabilization trick, adapted for \( SO(3) \)-bundles (as opposed to \( SU(2) \)-bundles). There is a natural map \( \mathcal{A}_{E,k,\delta} \to \mathcal{A}_{E \mathcal{S}C,k,\delta} \) given by taking the direct sum with the trivial connection; this is equivariant for the even gauge group (where for an \( SO(n) \)-bundle, ‘even’ means that the gauge transformation lifts to a section of \( \text{Aut}(E) \times_{SO(n)} \text{Spin}(n) \), the action by conjugation).

Therefore, this map descends to a map on the quotients \( \mathcal{B}_{E}^\mathcal{F} \to \mathcal{B}_{E \mathcal{S}C}^\mathcal{F} \). Write \( \det(Q') \)
for the determinant line bundle for connections over $E \oplus \mathbb{C}$; for a connection of the form $A \oplus \theta$, the operator $Q'$ splits as a direct sum of the ASD deformation complex for the three bundles $\mathfrak{g}, \mathbb{C}^* \otimes E$, and $\mathbb{R}$, where on the first term we use the connection induced by $A$, on the second term the connection $\theta \otimes A$, and on the final term the trivial connection. Note that the second term is a complex linear operator, and hence has canonically trivial determinant.

Thus this stabilization map pulls back $\det(Q')$ to $\det(Q) \oplus \det(Q_0)$, where the final term is a trivial line bundle. In particular, if $\det(Q')$ is trivializable, then $\det(Q)$ is as well. Write $\mathcal{B}(\ell)$ for the configuration space of connections on $E \oplus \mathbb{C}^\ell$, modulo even gauge, and write $\tilde{\mathcal{B}}(\ell)$ for the smooth $SO(2\ell + 3)$-manifold of framed configurations on $E \oplus \mathbb{C}^\ell$, whose quotient is $\tilde{\mathcal{B}}(\ell)$.

Next, because the reducible subspace is of infinite codimension, transversality implies that the inclusion of the irreducible subspace $\tilde{\mathcal{B}}^*_\ell(\ell) \to \tilde{\mathcal{B}}(\ell)$ is a weak homotopy equivalence. In particular the restriction map

$$[\tilde{\mathcal{B}}(\ell), \mathbb{R}P^{2\ell}] \to [\tilde{\mathcal{B}}^*_\ell(\ell), \mathbb{R}P^{2\ell}]$$

is a bijection, so any line bundle which is trivializable over $\tilde{\mathcal{B}}^*_\ell(\ell)$ is necessarily globally trivializable. Now our bundle is crucially an $SO(3)$-equivariant line bundle over $\tilde{\mathcal{B}}^*_\ell(\ell)$, so it is the pullback of some line bundle over $\mathcal{B}^*_\ell(\ell)$, the space of irreducible connections modulo gauge (with no framing). Our goal now is to show that this line bundle is trivial, which will start from a calculation of $\pi_1\mathcal{B}^*_\ell(\ell)$. By definition, this space is the quotient of $A^*$ by the even gauge group $G_{E,S^1G}$; because $A^*$ is again the complement of a union of infinite codimension submanifolds, we see that it too is contractible, and so $\pi_1\mathcal{B}^*_\ell(\ell) = \pi_0G_{E,S^1G}$. Next, we must calculate this group of components. This is where stabilization is first useful.

Fix a principal $G$-bundle $P$ over a CW complex $Y$. It is not difficult to show that the pointed mapping space $\text{Map}_*(Y, BG)$ gives a model for $BG(P)$, where the superscript indicates we are only interested in the component which classifies the bundle $P$. Now, because $BSO$ is an $H$-space (with product structure given by direct sum of vector bundles), in fact every component of $\text{Map}_*(Y, BSO)$ is homotopy equivalent to any another. For $n$ sufficiently large, $\pi_1\text{Map}_*(Y, BSO) \cong \text{Map}_n(Y, BSO(n))$ (this is true for $i = 1$ already as soon as $n = 7$). To compute the fundamental group of the quotient by all gauge transformations, it suffices to compute the fundamental group of the identity component of $\text{Map}_*(Y, BSO)$. As before, because we are only interested in the quotient by even gauge transformations, we actually need only compute $\pi_1\text{Map}_*(Y, BSO)$. Because there is a fibration $F \to B\text{Spin} \to K(\mathbb{Z}, 4)$ with $F$ 6-connected, we see that $\pi_1\text{Map}_*(Y, BSO) \cong \pi_1\text{Map}_*(Y, K(\mathbb{Z}, 4))$. Now, it is a theorem of Thom [Tho57] that we have a splitting

$$\text{Map}_*(Y, K(G, n)) = \prod_{i=1}^n K(H^{n-i}(Y; G), i),$$

and in particular, $\pi_1\text{Map}_*(Y, K(\mathbb{Z}, 4)) \cong H^4(Y; \mathbb{Z})$.

In our case, the gauge group $G^{e,h}$ consists of even gauge transformations which are asymptotically parallel; this fits into an exact sequence $G^e \to G^{e,h} \to G_\alpha \times G_\beta$, where the groups $\Gamma$ are the stabilizers of the corresponding connections and $G^e$ is the gauge group of asymptotically trivial gauge transformations. In particular, we see that $\pi_0G^e \to \pi_0G^{e,h}$ is a surjection. The homotopy type of the group of gauge
transformations which are asymptotically trivial is the same as the relative gauge group over \((W, \partial W)\); the discussion of the above paragraph goes through without difficulty to this relative setting, replacing cohomology with relative cohomology.

We conclude that \(\pi_0 \mathcal{G}^e = H^3(W, \partial W; \mathbb{Z}) \cong H_1(W; \mathbb{Z})\), by Poincaré duality, and in particular \(H_1(W; \mathbb{Z}) \to \pi_1 \tilde{\mathbb{E}}_{BCV}\) is surjective.

Now that we know what the loops in this space are, the goal is to show that \(w_1(\det(Q'))\) pairs trivially against them. Now we are in standard territory, and the proof follows exactly as in [D87, Lemma 2.23] (modifying the Poincaré duality argument to account for the boundary): if one chooses a base connection \(A\), a concentrated charge-one instanton \(J\) on \(S^4\), and a loop \([\gamma] \in H_1(W)\) in the interior of \(W\), then one performs a family connected-sum construction along the loop to construct a loop of connections. One may identify both that the corresponding loop in \(\pi_1 \tilde{\mathbb{E}}_{BCV}\) is the image of \([\gamma]\), and that the determinant line bundle over this loop of connections is trivial (if one so desires, this can be carried out using the index-gluing constructions along the boundary of a 4-manifold, which will be described later in this section).

Therefore, the pairing of \(w_1(\det(Q'))\) against \(H_1(W; \mathbb{Z})\) is identically zero, and so \(\det(Q')\) is trivializable. Being the pullback of the sum of this bundle and a trivial line bundle, the same is true of \(\det(Q)\), as desired. ■

Now choose a framed \(\pi\)-instanton \([A, p] \in \tilde{\mathcal{M}}_{E, z, \pi}\), where \(\pi\) is a regular perturbation. The tangent space \(T_{[A, p]} \tilde{\mathcal{M}}_{E, z, \pi}\) is identified with the first cohomology of the deformation complex

\[
\Omega^0_{k+1, \delta}(g_E) \xrightarrow{\delta_{A, \text{ev}_b}} \Omega^1_{k, \delta}(g_E) \oplus g \xrightarrow{D_{A, \pi}} \Omega^{2, +}_{k-1, \delta}(g_E),
\]

where \(g\) means the fiber of \(g_E\) above \(b\), and \(\text{ev}_b\sigma = \sigma(b)\). We denote this complex \(C_A\). Here we require \(k\) to be sufficiently large that every \(L^2_{k+1}\) function is continuous, so that evaluation is a continuous operation. Write \(g_A\) to denote the subspace of \(g\) that extends to \(A\)-parallel gauge transformations (so that \(g_A\) is the Lie algebra of \(\Gamma_A\)), and write \(e : g_A \to \Omega^0_{k+1, \delta, \text{ext}}\) for the operator that takes a point-value and sends it to the corresponding parallel section.

Take a map \(p_A : g_A \to \Omega^0(g_E)\) which has \(\langle p_A(h), e(h) \rangle > 0\) for all \(h \neq 0 \in g_A\). We write the operator \(\tilde{Q}_{A, \pi} : \Omega^0_{k, \delta} \oplus g \to \Omega^2_{k-1, \delta} \oplus \Omega^0_{k-1, \delta}\) for \(\tilde{Q} = Q + p_A\) (suppressing the \(p_A\) from notation; as we shall see shortly, this is not so depraved).

**Lemma 4.49.** The composite map \(\ker(\tilde{Q}) \to \ker(D_{A, \pi}) \to H^1(C_A)\) is an isomorphism if \(p_A\) is sufficiently small.

**Proof.** By assumption that \(\pi\) is a regular perturbation, the complex \(C_A\) has cohomology concentrated in degree one, and \(\text{coker}(Q_{A, \pi}) = g_A\). By assumption that \(p_A(h)\) pairs nontrivially with \(e(h)\) for any \(h \in g_A\), we see that \(\tilde{Q}_{A, \pi}\) is surjective. Because the index of the operator \(\tilde{Q}\) agrees with the index of the complex \(C\), we see that \(\ker(\tilde{Q})\) and \(H^1(C_A)\) have the same dimension; so it suffices to show surjectivity.

So suppose \(D_{A, \pi} \omega = 0\); our goal is to show there is some \(\sigma \in \Omega^0_{k, \delta}\) so that \(\Delta_A \sigma + p_A(\sigma(b)) = -d_A^* \omega\). The map \(\Delta_A : \Omega^0_{k+1, \delta, \text{ext}} \to \Omega^0_{k-1, \delta}\) is a self-adjoint operator with kernel \(g_A\).

In particular, because \(p_A : \ker(\Delta_A) \to \text{coker}(\Delta_A)\) is assumed to be an isomorphism, the map \(\Delta_A + t p_A\) is surjective for sufficiently small \(t\). So (replacing \(p_A\)
with \( tp_A \) if necessary, \( t \) small) we see that \( \Delta^0_{k+1, \delta, \text{ext}} \rightarrow \Omega^0_{k, \delta, \text{ext}} \) is an isomorphism. Therefore we may indeed solve the desired equation, so that we see the map \( \ker(\overline{Q}) \rightarrow H^1(C_A) \) is surjective (and hence an isomorphism), as desired. 

In particular, we have a canonical isomorphism in a neighborhood of \([A, p]\) between \( \det(T\overline{\mathcal{M}}) \) and \( \det(\overline{Q}) \). The sign of this isomorphism is independent of the auxiliary choice of \( p_A \). What we look to investigate is, then, the latter operator.

Recall now that the index of Fredholm operators is invariant under homotopy through Fredholm operators; because the map \( Q_{A, \pi} + \alpha p_A \), for \( t \in [0, 1] \), gives a homotopy through Fredholm operators (as \( g \) is finite-dimensional), we see that \( I(Q_{A, \pi}) \cong I((Q_{A, \pi}) \oplus g) \), and in particular \( \det(Q_{A, \pi}) \cong \det(Q_{A, \pi}) \det(g) \). Therefore, a choice of orientation of the Lie group \( SO(3) \) and an orientation of \( Q \) canonically gives us an orientation of \( \overline{Q} \). This is completely independent of the choice of \( r \), and we may as well remove \( r \) from the discussion of the index theory, and instead study \( Q_{A, \pi} \) itself.

In particular, a choice of orientation of the line bundle \( \det(Q_{A, \pi}) \) for \( A \) in some component of \( \mathcal{B}^E_{\mathcal{E}} \), and a fixed orientation of \( \text{so}(3) \), therefore gives a global orientation of the framed moduli space \( \overline{\mathcal{M}} \) sitting inside that component.

If \( A \in \mathcal{B}^E_{\mathcal{E}, z, k, \delta}(\alpha, \beta) \), where here \( z \in \pi_0 \mathcal{B}^E_{\mathcal{E}, k, \delta} \) labels a path-component, we write \( \Lambda^W_{\mathcal{E}}(\alpha, \beta) \) for the two-element set of orientations of \( \det(Q_A) \otimes \det(g) \). As above, a choice of element of \( \Lambda^W_{\mathcal{E}}(\alpha, \beta) \) induces an orientation on the moduli space \( \overline{\mathcal{M}}_{\mathcal{E}, z, \pi}(\alpha, \beta) \).

Let \( (W_1, E_1) \) be two cobordisms (considered as manifolds with two cylindrical ends), so that the positive end of \( (W_1, E_1) \) is isometric to the negative end of \( (W_2, E_2) \). Here we include the possibility that either or both of the \( W_i \) are cylinders; write \( (W_{12}, E_{12}) \) for the composite cobordism, glued by cutting off the positive end of \( W_1 \) (say, \( [0, \infty) \times Y_2 \)) at \( \{T\} \times Y_2 \) and similarly for the negative end of \( W_2 \). If it is not necessary for clarity, we drop the \( T \) from notation. If \( z_1 \) and \( z_2 \) are components of \( \mathcal{B}^E_{\mathcal{E}, z, k, \delta}(\alpha, \beta) \) and \( \mathcal{B}^E_{\mathcal{E}, z, k, \delta}(\beta, \gamma) \), respectively, then denote the apparent ‘composite component’ as \( z_{12} \). What we would like is to take an element of each of \( \Lambda^W_{\mathcal{E}}(z_1, \alpha, \beta) \) and \( \Lambda^W_{\mathcal{E}}(z_2, \beta, \gamma) \), and output an element of \( \Lambda^W_{\mathcal{E}}(z, \alpha, \gamma) \) (possibly depending on some extra input coming from \( \beta \) alone).

To do this, we will use the mechanism of gluing indices along boundary components. We begin by setting up some notation.

Let \( X_1 \) and \( X_2 \) be 4-manifolds equipped with \( SO(3) \)-bundles \( E_i \), possibly with cylindrical ends, so that \( \partial X_i \) is nonempty and the metric is of product type near the boundary. We decompose each \( \partial X_i \) into a union \( \partial_- X_i \cup \partial_+ X_i \) of connected components (either term possibly empty) which we call ‘positive’ or ‘negative’, and orient them so that a neighborhood of \( \partial_+ X_i \) is isometric to \( (-t, 0] \times \partial_+ X_i \) and a neighborhood of \( \partial_- X_i \) is isometric to \([0, t) \times \partial_- X_i \). Suppose furthermore that \( (\partial_+ X_1, E_1) \) is oriented isometric to \( (\partial_- X_2, E_2) \). Write

\[
Y_1 = \partial_- X_1, \quad Y_2 = \partial_+ X_1 = \partial_- X_2, \quad Y_3 = \partial_+ X_2,
\]

and write \( Y = Y_1 \cup Y_2 \cup Y_3 \).

Then we may form the composite \( X_{12} = X_1 \cup Y_3 X_2 \), equipped with \( SO(3) \)-bundle \( E_{12} \).

Suppose each \( (X_i, E_i) \) is equipped with a choice of 4-manifold perturbation, regular on the ends, which is also constant in time near the boundary. Suppose
We are interested in studying the perturbed ASD operator on these compact manifolds with boundary. We should introduce a restriction on the boundary values to make this a Fredholm operator, as in Section 4.7. However, the choice there (the Coulomb-Neumann gauge) is not appropriate for us here; that boundary value problem is not continuously varying as we change the reducibility type of $A$. Instead, we use the standard Atiyah-Patodi-Singer spectral boundary value problem. On the composite $X_{12}$, the ASD operator takes the form $d/dt + L_{A,\pi}$ on $(-t, t) \times Y$ for the signature operator $L$ to the compact piece lying between these.

We define the

$$H_{\lambda} = \text{det}(\lambda - A^*)\Lambda_{A,\pi},$$

where $D_{A,\pi} = *d_A + D_A \nabla_\pi$. Henceforth we write $\Omega^0(Y) \oplus \Omega^1(Y)$ as $H = H_1 \oplus H_2 \oplus H_3$, only writing the Sobolev indices if necessary.

Suppose $\lambda \in \mathbb{R}$ is not an eigenvalue of $L_{A,\pi}$ for any $A$ arising as the restriction of $A$ to a boundary component. Then we may split $H$ as $H^{>\lambda} \oplus H^{<\lambda}$ as the closure of the linear span of eigenvectors with eigenvalue greater than, or less than, $\lambda$. Then we define the $\lambda$-weighted ASD operator on $X_1$ to be

$$Q_{\lambda, \pi} = (D_{A,\pi}, d_A^*, \Pi_1^{>\lambda}, \Pi_2^{<\lambda}), \Omega^1_{k,\delta}(X_1) \to \Omega^2_{k,\delta}(X_1) \oplus \Omega^1_{k-3/2}(X_1) \oplus H_1^{<\lambda} \oplus H_2^{>\lambda},$$

where $r$ is a projection to the appropriate boundary components and the operators $\Pi$ are the $L^2$-orthogonal projections onto the corresponding eigenspaces. The operator written on $X_2$ is the same, with the boundary value projections changed to $\Pi_2^{>\lambda}$ and $\Pi_3^{<\lambda}$, and similarly with $X_{12}$.

With this notation in hand, the following is essentially the content of the beginning of [KM07, Section 20.3], especially pages 383-384.

**Lemma 4.50.** Let $K$ be any compact family of connections $A_{12}$ and perturbations as above, where $\lambda$ is never an eigenvalue for any $A_{12}$ for any $A$ given by restriction of $A \in K$ to the boundary. Then the operators $Q_{\lambda, \pi}$ form a family of Fredholm operators, and there is a canonical isomorphism of line bundles over $K$

$$\det(Q_{\lambda, \pi}) \otimes \det(Q_{\lambda, \pi}) \to \det(Q_{\lambda, \pi}),$$

given by a homotopy between the operators $Q_{\lambda, \pi} \oplus Q_{\lambda, \pi}$ and an operator whose kernel and cokernel are canonically identified with that of $Q_{\lambda, \pi}$.

What we really care about are connections $A \in \hat{B}_{E, z, k, \delta}(\alpha, \beta)$ on a manifold $W$ with two cylindrical ends and no boundary components. To make use of the above lemma, we should break these into pieces. For $T$ large, write $N$ for the restriction of $A$ to $(-\infty, -T] \times Y_1$, $P$ for the restriction to $[T, \infty) \times Y_2$, and $C$ for the restriction to the compact piece lying between these.

We fix a small constant $\epsilon > 0$. What we demand of our choice of $T$, writing $N = p^*a + a$ for $p^*\alpha$ the pullback connection $d/dt + d_\alpha$ and $a \in \Omega^1_{k,\delta}((-\infty, -T] \times Y_1)$, is the following. First, that the signature operator $L_{\alpha + \alpha(-T)}$ never has an eigenvalue...
of absolute value $\epsilon$ for any $t \in [0, 1]$. Second, that the only eigenvalue of $L_{\alpha, \pi}$ in $[-\epsilon, \epsilon]$ is 0, which corresponds to $\ker(L_{\alpha, \pi}) = g_\alpha$, the space of $\alpha$-parallel sections of $\mathfrak{g}_E$; this is the entire kernel by the assumption that $\alpha$ is a nondegenerate critical orbit. We demand similarly for $P$. This is a smallness condition on $N$ and $P$, or rather the size of the differences between those and constant connections. In particular, the operators $Q_{N, \pi}^\epsilon$ and $Q_{P, \pi}^\epsilon$ are both Fredholm operators, and there are canonical isomorphisms
\[ \det(Q_{N, \pi}^\epsilon) \cong \det(Q_{P, \pi}^\epsilon), \quad \det(Q_{P, \pi}^\epsilon) \cong \det(Q_{\beta, \pi}^\epsilon). \]
This remains true for families of such connections, so long as $T$ is chosen to satisfy the given properties for all connections in the family; this is always possible if the family is compact.

Then by Lemma 4.50, we have an isomorphism
\[ \det(Q_{P^* \alpha, \pi}^\epsilon) \otimes \det(Q_{C, \pi}^\epsilon) \cong \det(Q_{A, \pi}). \]
To reduce this to studying $C$ alone, we should understand explicitly the operators corresponding to each end.

**Lemma 4.51.** Write $Z^- = (-\infty, 0] \times Y_1$ and $Z^+ = [0, \infty) \times Y_2$.

The operator
\[ Q_{P^* \alpha, \pi}^\epsilon : \Omega^{1, \pi}_k(Z^-) \to \Omega^{2, \pi}_k(Z^-) \oplus \Omega^0_{k-1, \pi}(Z^-) \oplus H_1^{\pi, \epsilon}_\alpha \]
is an isomorphism, and so there is a canonical trivialization $\det(Q_{P^* \alpha, \pi}^{-\epsilon}) \cong \mathbb{R}$. The operator
\[ Q_{P^* \beta, \pi}^\epsilon : \Omega^{1, \pi}_k(Z^+) \to \Omega^{2, \pi}_k(Z^+) \oplus \Omega^0_{k-1, \pi}(Z^+) \oplus H_1^{\pi, \epsilon}_\beta \]
is injective with cokernel canonically identified with $g_\beta$, the space of $\beta$-parallel sections of $\mathfrak{g}_E$, and so we have a canonical isomorphism $\det(Q_{P^* \beta, \pi}^{-\epsilon}) \cong \det(g_\beta)^*$, giving a trivialization as soon as we orient the vector space $g_\beta$.

**Proof.** Because the operator
\[ (D_{P^* \alpha, \pi}, d_{P^* \alpha}) = (d_{P^* \alpha}^\pi + D_{\alpha} \nabla_{\pi}, d_{P^* \alpha}) \]
takes the form $d/dt + L_{\alpha, \pi}$, and identically with $\beta$, we may solve the equation $(D_{P^* \alpha, \pi}, d_{P^* \alpha})\omega = 0$ by separation of variables, decomposing $\Omega^0(Y_1) \oplus \Omega^0(Y_2)$ into the eigenspaces of $L_{\alpha, \pi}$. A solution with boundary value $\phi \in L^2_{k-1/2}$ exists if and only if $\phi$ is in the closure of the span of eigenvectors with negative eigenvalues. Because $H_1^{\pi, \epsilon} = H_1^{\epsilon, 0}$ by assumption on $\epsilon$, we see that the operator $Q_{P^* \alpha, \pi}^{-\epsilon}$ is invertible, as desired. However, for the other cylindrical end, we have $H_2^{\pi, \epsilon} = H_2^{\epsilon, 0}$, we see that we have cokernel equal to $H_2^0 = \ker(L_{\alpha, \pi}) = g_\alpha$, by the assumption that $\alpha$ is nondegenerate.

We use these facts about the index to construct a comparison map between certain determinant line bundles.

**Definition 4.17.** Let $A_1$ and $A_2$ be connection on cobordisms $W_1$ and $W_2$ as above, going between $\alpha$ and $\beta$ or $\beta$ and $\gamma$, respectively. If $A_{12}$ is a connection on the composite $W_{12}^T$ which is uniformly close to $A_1$ on $W_1^T$ and $A_2$ on $W_2^{2-T}$, the there is a canonical isomorphism
\[ \rho_{an} : \det(Q_{A_1, \pi}) \otimes \det(g_\beta) \otimes \det(Q_{A_2, \pi}) \cong \det(\tilde{Q}_{A_{12}, \pi}), \]
given by decomposing
\[ \det(Q_{\mathbf{A}_1, \pi}) \cong \det \left( Q_{C_{1, \pi}} \right) \otimes \det(g_\beta)^* \]
and
\[ \det \left( Q_{\mathbf{A}_2, \pi}^{-\varepsilon} \right) \cong \det \left( Q_{C_{2, \pi}}^{-\varepsilon} \right) \otimes \det(g_\gamma)^* , \]
as above, and applying the canonical isomorphism \( \det(g_\beta)^* \otimes \det(g_\gamma)^* \cong \mathbb{R} \), as well as the isomorphism
\[ \det \left( Q_{C_{1, \pi}}^{-\varepsilon} \right) \otimes \det \left( Q_{C_{2, \pi}}^{-\varepsilon} \right) \cong \det \left( Q_{C_{12, \pi}}^{-\varepsilon} \right) \]
given by Lemma 4.50, by first applying a small homotopy taking \( C_1 \) to the restriction of \( C_2 \).

We call \( \rho_{an} \) the analytic gluing map. There is a corresponding version of this map for the operators \( \tilde{Q} \) which have a factor of \( g \) to account for the SO(3)-action; this is written
\[ \tilde{\rho}_{an} : \det \left( \tilde{Q}_{\mathbf{A}_1, \pi} \right) \otimes \det(g_\beta)^* \otimes \det \left( \tilde{Q}_{\mathbf{A}_2, \pi} \right) \cong \det \left( \tilde{Q}_{\mathbf{A}_{12}, \pi} \right) ; \]
given by decomposing
\[ \det \left( \tilde{Q}_{\mathbf{A}_1, \pi} \right) \cong \det \left( Q_{\mathbf{A}_1, \pi} \right) \otimes \det(g) \]
and using the canonical isomorphism \( \det(g) \cong \det(g_\beta) \otimes \det(g_\gamma) \), where here we implicitly demand that \( g = g_\beta \otimes g_\gamma \) is an oriented splitting if we choose an orientation on any two of these.

In particular, this gives an isomorphism
\[ \Lambda^{W_1}_{\mathbb{Z}/2}(\alpha, \beta) \times_{\mathbb{Z}/2} \Lambda(\beta, \gamma) \cong \Lambda^{W_1}_{\mathbb{Z}/2}(\alpha, \gamma), \]
where \( \Lambda(\beta, \gamma) \) is the set of orientations of \( g_\beta \).

Given any compact family of \( \mathbf{A}_i \) as above, the analytic gluing map \( \rho_{an} \) gives an isomorphism of line bundles over the parameter space, and identically with \( \tilde{\rho}_{an} \).

Note that this is usually relevant to us in the case where one of \( W_i \) is a cylinder. If neither are cylinders, the orientations that actually appear in practice are of the parameterized ASD operator, with parameter given by some family of broken metrics. From now on, we restrict to the case where one of the \( W_i \) is a cylinder for convenience of discussion; there is essentially no change in the case of composite cobordisms (and families of metrics and perturbations).

We also have a similar isomorphism coming from a different source: the inverse function theorem. Recall from Section 4.7 that, for certain connected SO(3)-invariant open subsets \( U_{\alpha, \beta} \subset \widetilde{M}_{E_1, z_1}(\alpha, \beta) \) and \( U_{\beta, \gamma} \subset \widetilde{M}_{E_2, z_2}(\beta, \gamma) \), Proposition 4.43 gives a diffeomorphism \( \text{gl} : U_{\alpha, \beta} \times_{\beta} U_{\beta, \gamma} \rightarrow U_{\alpha, \gamma} \). (In the compactification by broken trajectories, the closure of \( \text{gl}(U_{\alpha, \beta} \times_{\beta} U_{\beta, \gamma}) \) includes a chosen component of \( \widetilde{M}_{\alpha, \beta} \times_{\beta} \widetilde{M}_{\beta, \gamma} \).) We remind the reader here that when considering \( \widetilde{M}_{E,k, \delta} \) in the case where \( W \) is a cylinder, this is the moduli space of parameterized trajectories: we do not reduce by the \( \mathbb{R} \)-action.

In particular, suppose we are given a pair of framed instantons \( ([\mathbf{A}_1, p_1], [\mathbf{A}_2, p_2]) \) which project to the same framing at \( \beta \). Taking the derivative of \( \text{gl} \), we obtain an isomorphism
\[ \tilde{\rho}_{gm} : \det \left( \tilde{Q}_{\mathbf{A}_1, \pi} \right) \otimes \det(g_\beta)^* \otimes \det \left( \tilde{Q}_{\mathbf{A}_2, \pi} \right) \cong \det \left( \tilde{Q}_{\mathbf{A}_{12}, \pi} \right) ; \]
note that the $SO(3)$-action on a point gives an isomorphism $T_\beta \cong g^\perp_\beta$, which explains this factor. Because the gluing map is equivariant, this descends to an isomorphism

$$\rho_{gm} : \det (Q_{A_1,\pi}) \otimes \det(g^1_\beta) \otimes \det (Q_{A_2,\pi}) \cong \det (Q_{A_{12},\pi}).$$

We call $\rho_{gm}$ the geometric gluing map. If the open sets $U_{\alpha\beta}$ and $U_{\beta\gamma}$ are chosen so that the restriction of $A \in U_{\alpha\beta}$ to each end is close enough to the constant trajectory, and similarly for the restriction of $A \in U_{\beta\gamma}$ to each end, that we may apply Lemma 4.50. Then we obtain two isomorphisms

$$\rho : \det (Q_{A_1,\pi}) \otimes \det(g_\beta) \otimes \det (Q_{A_2,\pi}) \cong \det (Q_{A_{12},\pi}).$$

The next part of this section is dedicated to showing that these two isomorphisms are positive scalar multiples of one another, and hence both induce the same orientation on $\det (Q_{A_{12},\pi})$.

The map $\rho_{gm}$ (or rather, $\tilde{\rho}_{gm}$) is not difficult to understand at the level of kernels and cokernels; it is less clear how to understand $\tilde{\rho}_{an}$ at this level, so this is our next immediate goal.

**Lemma 4.52.** Let $V, W$, and $W_b$ be Hilbert spaces; suppose we have a continuous family of Fredholm maps $A_t = (A, r_t) : V \to W \oplus W_b$, with $A : V \to W$ surjective. Suppose that $A_0$ is surjective, and that the map

$$r_t - r_0 : \ker(A) \to W_b$$

has sufficiently small norm, uniformly in $t$. Let $J \subset W_b$ be a finite-dimensional subspace so that $\bar{A}_t : V \oplus J \to W \oplus W_b$, given by

$$\begin{pmatrix} A & 0 \\ r_t & 1 \end{pmatrix},$$

is surjective for all $t$. Then the composite

$$\ker(\bar{A}_0) \to V \oplus J \to \ker(\bar{A}_t)$$

is an isomorphism, where the first map is inclusion and the second map is orthogonal projection. Similarly, the map $\ker(A_0) \to \ker(A_1)$ is injective, and the above gives us an isomorphism $\ker(A_0) \oplus \ker(A_1) \cong \ker(A_1)$. This gives an isomorphism between orientation sets $\Lambda(A_0) \cong \Lambda(A_1)$, the same as that induced by the homotopy through Fredholm operators $A_t$.

**Proof.** First, we argue in slightly more generality. Suppose we are given two Fredholm maps $T_i : X \to Y$ of Hilbert spaces, and that $\|T_2\|_{\ker(T_1)} < \epsilon$. Write $\pi_2 : X \to \ker(T_2)$ for the orthogonal projection, and $C_2 = \ker(\pi_2)$, the orthogonal complement to $\ker(T_2)$. Then the map $T_2 : C_2 \to Y$ is an isomorphism onto its (closed) image, and in particular enjoys an inequality of the form $\|T_2v\|_Y \geq c\|v\|_{C_2}$ for some $c > 0$. For any $v \in \ker(T_1)$, we have

$$\|(1 - \pi_2)v\| \leq \frac{1}{c}\|T_2(1 - \pi_2)v\| = \frac{1}{c}\|T_2v\| \leq \frac{c}{c}\|v\|;$$

the equality follows because $T_2\pi_2v = 0$, by definition of $\pi_2$. As soon as $\epsilon < c$, we see that $\pi_2v$ must be nonzero. In particular, so long as $\|T_2\|_{\ker(T_1)}$ is sufficiently small, the composite $\ker(T_1) \to X \to \ker(T_2)$ is injective.

Given an element $(v, j)$ of $\ker(\bar{A}_0)$, we have

$$\bar{A}_t(v, j) = (A v, r_t v + j) = (0, (r_t - r_0)v);$$
because \( v \in \ker(A) \), we see that
\[
\| \tilde{A}_t \|_{\ker(A)} \leq \| (r_t - r_0) \|_{\ker(A)}.
\]
So if \( \| (r_t - r_0) \|_{\ker(A)} \) is sufficiently small, we see that \( \ker(\tilde{A}_0) \to \ker(\tilde{A}_t) \) as defined above is injective, for all \( t \); because the operators \( \tilde{A}_t \) are all homotopic through Fredholm operators and assumed to be surjective, these kernels all have the same dimension, and hence this injection is an isomorphism.

By the assumption that \( A_0 \) is surjective, we may identify \( \ker(\tilde{A}_0) \) as \( \ker(A_0) \oplus J \).

If we write \( \ker(\tilde{A}_t) \oplus J'_t \cong J \), then choosing a section of the projection \( \ker(\tilde{A}_t) \to J'_t \), we may similarly identify \( \ker(\tilde{A}_t) \cong \ker(A_t) \oplus J'_t \). The isomorphism
\[
\ker(A_0) \oplus J \cong \ker(A_t) \oplus J'_t
\]
is isometric to the direct sum of an isomorphism \( \ker(A_0) \oplus \ker(A_t) \to \ker(A_t) \) with the identity map on \( J'_t \). Here, the map \( \ker(A_t) \to \ker(A_t) \) is given (up to a small homotopy) by first choosing a section \( s \) of the surjective map \( \ker(A) \xrightarrow{r_t} W_b \to \ker(A_t) \), which supplies us with a map \( s' : \ker(A_t) \to \ker(A_0) \oplus J \) by \( s'(j) = (s(j), r_0 s(j)) \); because the space of sections is contractible this map only depends on the choice of section up to an isotopy. Then as usual we apply orthogonal projection to \( s(j) \) to obtain a map \( \ker(A_0) \oplus J \to \ker(A_t) \).

This, then, is identified as an isomorphism of virtual vector spaces \( (\ker(A_0), 0) \to (\ker(A_t), \ker(A_t)) \), the map on kernels just being the composite \( \ker(A_0) \to V \to \ker(A_t) \) as above.

Given that we have a family of such isomorphisms, starting with the identity, identifies this map with the isomorphism \( \Lambda(A_0) \to \Lambda(A_1) \) induced by the homotopy through Fredholm operators.

Let \( W_1 \) and \( W_2 \) be cobordisms, with the positive end of \( W_1 \) and the negative end of \( W_2 \) modeled on \( (Y, E) \). Suppose we have regular framed instantaneous \( (A_1, p_1), (A_2, p_2) \) on the respective cobordisms (possibly cylinders) which limit to the critical orbit \( \beta \); the gluing map provides a framed instanton \( (A_1^T, p) \) on the glued-up cobordism \( W_1^T \cup_Y W_2^{\geq -T} \), obtained by truncating the two cobordisms at some large \( \{ \pm T \} \times Y \) and pasting them together. The defining property of \( (A_1^T, p) \), arising from the inverse function theorem, is as follows. First, for large enough \( T \) the restriction to \( W_1^{< T/2} \) and \( W_2^{\geq -T/2} \) is uniformly close to \( (A_1, p_1) \) and \( (A_2, p_2) \), respectively (here we use parallel transport to move the framing between various basepoints). Second, the restriction to the region in between these two pieces, isometric to \( [-T/2, T/2] \times Y \), is uniformly small. We write \( [-T/2, T/2] \times Y =: Z^{T/2} \). If \( \pi : E_b \to E_b/T \beta \) is the projection and \( q \) the framing at some basepoint \( b \) on \( Z^{T/2} \), then \( \pi q \) is uniformly close to the common limiting value \( q \) of \( p_1 \) and \( p_2 \).

In this context, we consider
\[
V = \Lambda_{k, \delta}^1(W_1^{< T/2}) \oplus \Omega_{k, \delta}^1(Z^{T/2}) \oplus \Omega_{k, \delta}^1(W_2^{\geq -T/2}),
\]
while
\[
W = (\Omega^{k, +} \oplus \Omega^0)_{k-1, \delta}(W_1^{< T/2}) \oplus (\Omega^{k, +} \oplus \Omega^0)_{k-1, \delta}(Z^{T/2}) \oplus (\Omega^{k, +} \oplus \Omega^0)_{k-1, \delta}(W_2^{\geq -T/2}),
\]
and
\[
W_b = (\Omega^0 \oplus \Omega^1)_{k-1/2}([-T/2, T/2] \times Y).
\]
The operator $A : V \to W$ is given by $(d^*, d^+)$, while there are two restriction operators $V \to W_0$ of interest to us: one is $r_0 = (\mathrm{res}, -\mathrm{res})$, taking the restriction on each positive boundary component and negative the restriction on each negative boundary component. The other is $r_1 = (\Pi^{> - \epsilon}, \Pi^{< - \epsilon})$, the spectral projection onto the $> - \epsilon$ eigenspaces on positive ends, and the $< \epsilon$ eigenspaces on negative ends. The operator $A_1$ has the property that it splits as a direct sum of operators corresponding to the pieces $W_1^{< t/2}, Z^{t/2}, W_2^{> - t/2}$, while the operator $A_0$ has kernel and cokernel naturally isomorphic to $Q_{A_{12}, \pi}$ itself. Here the appropriate choice of $L_{k, \pi}$ norm on the direct sum in fact has a weight on the $Z^{t/2}$ component, as in Section 4.7, essentially weighted by a symmetric function equal to $e^{\epsilon(t+T/2)}$ on $[-T/2, -t)$ for some small $t > 0$.

We may interpolate through these projections via $r_t = (\Pi^{> - \epsilon} + t\Pi^{< - \epsilon}, -t\Pi^{> - \epsilon} - \Pi^{< - \epsilon})$; as discussed in [KM07, Section 20.3], this gives a homotopy $A_t = (A, r_t)$ through Fredholm operators. Furthermore, if $T$ is chosen large enough, then $\| (r_t - r_0)_{\ker(A)} \|$ is arbitrarily small, uniformly in $t$. Observe here that $r_t - r_0 = t(\Pi^{< - \epsilon}, -\Pi^{> - \epsilon})$, so we may focus attention on the case $t = 1$. For convenience, we only pay attention to the factor $\Pi^{< - \epsilon}$ as we approach $\{T/2\} \times Y$ from the left.

Write $\omega_{T/4} = \omega|_{\{T/4\} \times Y}$ and $\omega_{T/2} = \omega|_{\{T/2\} \times Y}$. For $T$ very large, we have that $A_{12}|_{W_1^{[T/4, T/2]}}$ is uniformly close to the constant trajectory at the $\pi$-flat connection $\beta$, as follows because $A_1$ decays exponentially. For some constant $C$, independent of $T$, we have $\| \omega_{T/4} \|_{k-1/2} \leq C \| \omega \|_{k, \delta}$ for all $\omega \in L_{k, \delta}^2$; this is just the claim that the trace along a hypersurface is a continuous map (and the fact that a neighborhood of $\{T/4\} \times Y$ is isometric to $(-t, t) \times Y$, independent of $T$). Writing the ASD operator for the constant trajectory at $\beta$ as an ODE in the eigenvalues of $L_{\beta, \pi} : (\Omega^0 \oplus \Omega^1)_{k-1/2}(Y) \to (\Omega^0 \oplus \Omega^1)_{k-3/2}(Y)$, and writing $\omega_{-T/2} = \omega_{-T/2}^+ \oplus \omega_{-T/2}^-$ for the spectral decomposition, we see that

$$\| \omega_{-T/2}^- \| \leq \frac{e^{-\delta T/4}}{2} \| \omega_{-T/4}^- \| \leq \frac{e^{-\delta T/4}}{C} \| \omega \|$$

for any $\omega \in \ker(A)$, where $\delta$ is the absolute value of the least eigenvalue of $L_{\beta, \pi}$, and the factor of 2 is simply a fudge factor to account for the fact that $A_1$ is not literally the constant trajectory at $\beta$ in the relevant portion of the cobordism. Here we use that $-\epsilon < 0$ to conclude that this component of $r_t - r_0$ is uniformly small.

For the negative boundary components, a similar argument applies, but now one must exploit the exponential weights in the definition of the $L_{k, \delta}^2$ norm on the compact cylinder and the fact that $\epsilon$ is chosen less than $\delta$ to get the desired bound (now instead there is a factor of $e^{(\epsilon - \delta)T/4}$).

This is almost sufficient to apply the above lemma, except for the condition that $A_0$ is surjective; rather, we have a canonical identification $\mathrm{coker}(A_0) \cong \mathfrak{g}_{A_{12}}$, the tangent space to the stabilizer $\Gamma_{A_{12}}$. This is easy enough to dispatch; recall the definitions of the extended operators $\tilde{Q}_{A, \pi}$ from earlier in this section, depending on a certain choice of map $p : \mathfrak{g} \to \Omega^0(W)$. These were chosen precisely so that $\tilde{Q}_{A, \pi}$ is surjective for a regular instanton $\mathbf{A}$. In this context, take $\tilde{V} = V \oplus \mathfrak{g} \oplus \mathfrak{g}$ (thinking of each $\mathfrak{g}$ as being a choice of framing on one of the components $W_1^{< t/2}$ or $W_2^{> - t/2}$, based at say $\gamma(-T)$ and $\gamma(T)$ for the base curve $\gamma$), while $\tilde{W} = W \oplus \mathfrak{g}$ and $\tilde{W}_b = W_b$. The map $\mathfrak{g} \oplus \mathfrak{g} \to \tilde{W}$ is the expected maps $p_\gamma$ on the $\Omega^0(W_1)$ factors, while the map $\mathfrak{g} \oplus \mathfrak{g} \to \tilde{W}$ is the identity in the first factor, and $-\mathrm{Hol}_{A_{12}}^{\gamma(T) \to \gamma(-T)}$ in
the second factor; if the points $\gamma(\pm T)$ are chosen to lie on the boundary of the two pieces, then the composite of this map with the projection $g \to g_\beta$ is very close to the identity, because $A_{12}$ is sufficiently close to the constant trajectory at $\beta$.

Then the map $\tilde{A} : \tilde{V} \to \tilde{W}$ is surjective, as is $\tilde{A}_0 : \tilde{V} \to \tilde{W} \oplus \tilde{W}_b$, and the maps $p_i$ contribute minimally to the boundary-evaluation, so these satisfy the assumptions of the lemma.

**Corollary 4.53.** If $T$ is sufficiently large, the index-theoretic isomorphism $\tilde{\Lambda}(A_{12}) \cong \tilde{\Lambda}(A_1^{< T/2}) \Lambda(g)^* \tilde{\Lambda}(A_2^{> -T/2})$ is given, at the level of kernels and cokernels, by an injection

$$\ker(\tilde{Q}_{A_{12}}) \to \ker(\tilde{Q}_{A_1^{< T/2}}) \oplus_g \ker(\tilde{Q}_{A_2^{> -T/2}}),$$

obtained first by restriction to $\Omega^1(W_1^{< T/2} \sqcup W_2^{> -T/2})$ and second by projection to the kernel. (Here $\oplus_g$ means we take the kernel of the natural projection of the direct sum to $g$.)

Similarly, for sufficiently large cutoffs $T$, the index-theoretic isomorphism

$$\tilde{\Lambda}(A_1) \Lambda(g^\perp) \Lambda(\tilde{A}_2) \to \tilde{\Lambda}(A_1^{< T/2}) \Lambda(g)^* \tilde{\Lambda}(A_2^{> -T/2})$$

may be described at the level of kernels and cokernels as a map

$$\ker(\tilde{Q}_{A_1}) \oplus_{g^\perp} \ker(\tilde{Q}_{A_2}) \to \ker(\tilde{Q}_{A_1^{< T/2}}) \oplus \ker(\tilde{Q}_{A_2^{> -T/2}}),$$

which may be described explicitly again as restriction to the corresponding manifolds with boundary, and then projection to the latter kernel.

**Proof.** The only point not outlined above is a discussion of the middle piece, isometric to $[-T/2, T/2] \times Y$. Given that $A_{12}$ is uniformly close to the constant trajectory at $\beta$ for large enough $T$, and that for large enough $T$ the ASD map on the cylinder (with boundary conditions) is an isomorphism for the constant trajectory at $\beta$, we see that its determinant line is canonically trivial, and does not enter into the discussion.

Now consider the diagram

$$\tilde{\Lambda}(A_1) \Lambda(g^\perp) \Lambda(\tilde{A}_2) \longrightarrow \tilde{\Lambda}(A_1^{< T/2}) \Lambda(g)^* \tilde{\Lambda}(A_2^{> -T/2})$$

The vertical map may be taken to be either $\Lambda(\tilde{\rho}_gm)$ or $\Lambda(\tilde{\rho}_an)$. The horizontal map and upper-right map are as discussed above: they may be understood either as the inverses of index-theoretic gluing isomorphisms, or via projection maps between various kernels.

If the vertical map is taken to be $\tilde{\rho}_an$, then this diagram commutes (in fact, $\rho_{an}$ is defined so that this diagram commutes). If we take the vertical map to be $\tilde{\rho}_gm$, this diagram still commutes: we are claiming that the composite map $(A_1, p_1, T, A_2, p_2) \to (A_1^{< T/2}, p_1, A_2^{> -T/2}, p_2')$ is uniformly close to restriction, and hence this map commutes (up to a small homotopy) at the level of kernels. Because the diagram commutes with either choice, $\Lambda(\tilde{\rho}_gm) = \Lambda(\tilde{\rho}_an)$: the two gluing maps are the same.

If we have fixed an orientation of $g_\beta$, then via the recipe given by the analytic gluing map, an orientation of $A_2^{W_2}(\alpha, \beta)$ and an orientation of $A_2^{W_2}(\beta, \gamma)$ induces
an orientation of $\Lambda_w^W(\alpha, \gamma)$; it also naturally an induces an orientation of the fiber product $\tilde{\mathcal{M}}_{E_1,z,k,\delta}(\alpha, \beta) \times_{\beta} \tilde{\mathcal{M}}_{E_2,z,w,\delta}(\beta, \gamma)$, and what we learned above is that the geometric gluing map, which gives rise to a diffeomorphism between an open subset of this fiber product and $\tilde{\mathcal{M}}_{E_1,z,w,k,\delta}(\alpha, \gamma)$ is orientation-preserving, having oriented the latter via the recipe given by the analytic gluing map.

Suppose that $W_1$ is a cylinder. Then if $\tilde{\mathcal{M}}_{E_1,z}(\alpha, \beta)$ is the ‘compactification’ of $\tilde{\mathcal{M}}_{E_1,z}(\alpha, \beta)$ by broken trajectories, quotienting by $\mathbb{R}$ gives the usual $\tilde{\mathcal{M}}_{E_1,z}(\alpha, \beta)$ of unparameterized broken trajectories. We always orient this so that the diffeomorphism

$$\mathbb{R} \times \tilde{\mathcal{M}}_{E_1,z,k,\delta}(\alpha, \beta) \cong \tilde{\mathcal{M}}'_{E_1,z,k,\delta}(\alpha, \beta)$$

is orientation-preserving.

The notation in the following is slightly different than in Proposition 4.43, to allow for a more uniform discussion.

**Proposition 4.54.** If $(Y, E)$ is a 3-manifold equipped with regular perturbation, then if $\tilde{\mathcal{M}}_{E,k,\delta}(\alpha, \beta)$ is the compactified moduli space of unparameterized flowlines, then Proposition 4.43 gives a decomposition of the boundary

$$\partial \tilde{\mathcal{M}}_{E,z,\pi}(\alpha, \beta) = \bigcup_{\gamma \in E_{\pi}} \tilde{\mathcal{M}}_{E,z_1,\pi}(\alpha, \gamma) \times_{\gamma} \tilde{\mathcal{M}}_{E,z_2,\pi}(\gamma, \beta).$$

Suppose we orient $\Lambda_{z}(\alpha, \beta)$ using the analytic gluing map and fixed orientations of the three of $\Lambda_{z_1}(\alpha, \gamma), \Lambda_{z_2}(\gamma, \beta)$, and $g_{\gamma}$. If $d$ is the dimension of $\tilde{\mathcal{M}}_{E, z_1, \pi}(\alpha, \gamma)$, then the boundary orientation on

$$\tilde{\mathcal{M}}_{E,z_1,\pi}(\alpha, \gamma) \times_{\gamma} \tilde{\mathcal{M}}_{E,z_2,\pi}(\gamma, \beta)$$

differs from the fiber product orientation by a factor of $(-1)^d$.

Now let $(W, E)$ be a cobordism equipped with regular perturbation. By Proposition 4.43, each compactified moduli space $\tilde{\mathcal{M}}_{E,z,\pi}$ has boundary given as the union

$$\partial \tilde{\mathcal{M}}_{E,z,\pi}(\alpha, \beta) = \bigcup_{\gamma \in E_{\pi}} \tilde{\mathcal{M}}_{E_1,z_1,\pi}(\alpha, \gamma) \times_{\gamma} \tilde{\mathcal{M}}_{E_2,z_2,\pi}(\gamma, \beta)$$

$$\bigcup_{\zeta \in E_{\pi}} \tilde{\mathcal{M}}_{E_1,z_1,\pi}(\alpha, \zeta) \times_{\zeta} \tilde{\mathcal{M}}_{E_2,z_2,\pi}(\zeta, \beta).$$

This decomposes the boundary into two types of components: whether breaking occurs at the negative end or at the positive end.

Then if we orient $\Lambda_{z}(\alpha, \beta)$ using the analytic gluing map and fixed orientations of the three of $\Lambda_{z_1}(\alpha, \gamma), \Lambda_{z_2}(\gamma, \beta)$, and $g_{\gamma}$, the orientation of

$$\tilde{\mathcal{M}}_{E_1,z_1,\pi}(\alpha, \gamma) \times_{\gamma} \tilde{\mathcal{M}}_{E_2,z_2,\pi}(\gamma, \beta)$$

as a boundary stratum agrees with the orientation induced by the fiber product. If the dimension of $\tilde{\mathcal{M}}_{E_1,z_1,\pi}$ is $d$, then the orientation of

$$\tilde{\mathcal{M}}_{E_1,z_1,\pi}(\alpha, \zeta) \times_{\zeta} \tilde{\mathcal{M}}_{E_2,z_2,\pi}(\zeta, \beta)$$

as a boundary stratum is $(-1)^d$ that of the orientation given as a fiber product.

\[12\] To make sense of this, the broken trajectories are not parameterized on each component, but rather the stratum corresponding to a $k$-broken trajectory is quotiented by the action of $\mathbb{R}^{k-1} \subset \mathbb{R}^k$, sitting inside as the subset with zero sum.
Proof: The argument is no different from [KM07, Proposition 20.5.2] and [KM07, Proposition 25.1.1], respectively. We write $\mathcal{M}_{E,z,\pi}^0(\alpha, \beta)$ for the quotient of the space of parameterized flowlines by the $\mathbb{R}$ action, not compactified. Then in the first case, the sign arises from the orientation-preserving local homeomorphism

$$\mathcal{M}_{E,z,\pi}^0(\alpha, \gamma) \times [0, \infty) \times \gamma \times \mathcal{M}_{E,z,\pi}^0(\gamma, \beta) \to \mathcal{M}_{E,z,\pi}(\alpha, \beta);$$

to compare to the boundary orientation, we should commute the $[0, \infty)$ factor across $\mathcal{M}_{E,z,\pi}^0(\alpha, \gamma)$, which has dimension $d - 1$, as above; the final sign $(-1)^d$ comes because we restrict to the negative boundary component $\{0\}$.

The argument in the case of a cobordism is identical, and the difference in signs arises because in the case of breakings along the negative end, we do not need to commute anything.

What remains is to find a recipe so that, given some choices for each $\alpha$ and $\beta$, and a choice depending only on the underlying cobordism (in the case that it is not just a cylinder), we are given natural orientations of each $\Lambda^W_{z_0}(\alpha, \beta)$. These must compose appropriately under gluing. If we change one of the choices for $\alpha$ then the sign of the orientation on each $\Lambda_{z_0}(\alpha, \beta)$ should change, and similarly with swapping the choice for $\beta$. If one changes the choice for $W$, then every orientation should change uniformly.

To do this, we will ultimately require that the cobordism (or possibly some larger cobordism) admits a reducible connection. Already this places the demand that $\beta w_2(E) = 0$, but this is not enough to pin down orientations.

Supposing that $A$ is an $SO(2)$-reducible, we have a splitting $Q_{A,\pi} \cong Q_{\eta,\pi} \oplus Q_{\theta,\pi}$. If we chose a complex structure on $\eta$, then the term $Q_{\eta,\pi}$ is complex linear with respect to this complex structure, which would give an isomorphism $\det(Q_{A,\pi}) \cong \det(Q_{\theta,\pi})$, thus reducing discussion to the case of the trivial connection (possibly perturbed), but a different choice of complex structure will give a different isomorphism. We need to ensure that a choice of such a complex structure is part of our data.

For the first time in this text we need to choose (for each 3-manifold $Y$ and each cobordism $W$) not an $SO(3)$-bundle over the manifold, but rather a $U(2)$-bundle. An $SO(3)$-bundle has a lift to a $U(2)$-bundle if and only if $\beta w_2(E) = 0 \in H^3(\gamma; \mathbb{Z})$. If $\tilde{E}$ is a rank 2 complex vector bundle with fixed connection on its determinant line, then a reducible connection $A$ induces a splitting $\tilde{E} \cong \eta \oplus \zeta$, where $\eta \otimes \zeta \cong \det(\tilde{E})$ and $\eta, \zeta$ are complex line bundles. Thus for the trace-0 adjoint bundle we obtain an isomorphism $\varrho_{\tilde{E}} \cong \mathbb{R} \oplus (\eta \otimes \zeta^{-1})$, respecting the splitting induced by $A$; thus we have fixed a complex structure on the second component of the above splitting.

We begin this with some preliminaries.

**Lemma 4.55.** Let $(W, \tilde{E})$ be a complete Riemannian manifold equipped with a $U(2)$-bundle with no boundary components and some number of cylindrical ends, modelled on either $(-\infty, 0] \times Y_i$ or $[0, \infty) \times Y_i$; in the former case we say $Y_i$ is a negative end, in the latter a positive end. Suppose $W$ is equipped with a perturbation which is regular on each end, and a choice of $\alpha_i \in C_+(Y_i)$ for each 3-manifold $Y_i$ the ends are modelled on. Write $\tilde{E}_{E,k,\delta}^+(\alpha)$ for the configuration space of $E^2_{k,\delta}$ connections, asymptotic to the $\alpha_i$ on the corresponding ends.

Then $\pi_0 \tilde{E}_{E,k,\delta}^+(\alpha) \cong \mathbb{Z}$, this isomorphism affine over $\pi_1 \tilde{E}_{E,\delta} \cong \mathbb{Z}$ for any end $(Y_i, E_i)$.
If all of the $\alpha_i$ are trivial, then there is a unique component $z \in \pi_0 B^c_{E,k,\delta}(\alpha)$ which supports a reducible connection.

Proof. The statement about components in the space of connections is little more than the classification of $U(2)$-bundles over a compact 4-manifold with boundary with fixed isomorphism class on the boundary in terms of their first Chern class $c_1 \in H^2(W; \mathbb{Z})$ and Pontryagin class $p_1 \in \Lambda \subset H^4(W, \partial W; \mathbb{R})$. Here $\Lambda$ is a subset affine over $8\pi^2 \mathbb{Z}$, the latter defined by a curvature integral with respect to some connection with fixed boundary components. Gluing in the nontrivial positive (determined by the same curvature integral) generator of $\pi_1 B^c_E$, increases this by $8\pi^2$, as expected. We denote this operation as $z \mapsto z + 1$.

The fact that one and only one component supports a reducible connection follows from the enumeration of reducible components in Proposition 1.7 (or rather, a version allowing more ends).

Lemma 4.56. In the situation above, there is a canonical isomorphism $\Lambda^W_+(\alpha) \cong \Lambda^W_{\gamma+1}(\alpha)$, compatible with the gluing maps.

Suppose all of the $\alpha_i$ are trivial, and $z$ is the corresponding component supporting a reducible connection, and the perturbation is zero on the ends of $W$. Then if we write

$$Q^W_\theta: \Omega^+_{k,\delta}(W; \mathbb{R}) \rightarrow \Omega^+_{k,\delta}(W; \mathbb{R}) \oplus \Omega^0_{k,\delta}(W; \mathbb{R}),$$

for $(d^+, d^a)$, we have an isomorphism $\Lambda^W_+(\alpha) \cong \Lambda^W(\theta)$, where the latter is the set of orientations of $Q^W_\theta$. This is true for any choice of Sobolev indices on the various ends, so long as they never coincide with eigenvalues of $\text{Hess}_{\alpha_i}$ or $\text{Hess}_{\beta}$.

Proof. The first isomorphism follows by gluing charge-1 instantons on $S^4$ into the cobordism. By the assumption that the perturbation is compactly supported in $W$, we may modify $Q^W_{A,p}$ by a homotopy through Fredholm operators to $Q^W_A$.

We discussed the second claim above for $SO(2)$-reducible connections on $SO(3)$-bundles. When we have a lift of the $SO(3)$-bundle to a $U(2)$-bundle, if the reducible corresponds to some splitting $E \cong \mathbb{R} \oplus \xi$, then we obtain a canonical complex orientation on the complement of the $\mathbb{R}$-factor, giving the desired claim. For $SO(3)$-reducible connections, instead we have $Q^W_A = Q^W_g \otimes g$, and so an orientation of one canonically induces an orientation of the other (supposing an orientation of $g$ is given).

Thus we drop $z$ from the notation $\Lambda^W(\alpha, \beta)$.

If one of $\alpha$ or $\beta$ is irreducible, we cannot appeal to reducible connections on the cobordism as above. So we need to 'cap off' the two ends, as appropriate. Being precise about this is a little intricate, as described in the following definition.

Definition 4.18. Let $(W, \hat{E})$ be a cobordism $(Y_1, E_1) \rightarrow (Y_2, E_2)$ equipped with $U(2)$-bundle, a choice of perturbation on the ends (possibly zero, allowing for it to fail to be regular), and a choice of critical orbits $\alpha, \beta$ on the negative and positive end.

The symbol $\Lambda^W(\alpha, \beta)$ always means the usual two-element set of orientations of $\text{det}(Q_{A,\pi})$, where $A$ is a connection asymptotic to $\alpha$ and $\beta$ on the appropriate ends, and $Q_{A,\pi}$ has domain $\Omega^1_{\delta}$ and codomain $\Omega^2_{\delta} \oplus \Omega^0_{\delta}$. (If one of these terms $\alpha, \beta$ is trivial, we sometimes do not write the corresponding $\theta$.) When unadorned, the symbol $\Lambda^W(\theta)$ denotes the same for the trivial connection on the trivial bundle.
If \((W, \tilde{E})\) is a cobordism from some \((Y_-, \text{triv})\) to \((Y_1, E_1)\), for which the perturbation is zero on the incoming end, we say \(W\) is an incoming cap; if \((Y_2, E_2)\) is the incoming end and \((Y_-, \text{triv})\) is the outgoing end, with zero perturbation on the outgoing end, we call \((W, \tilde{E})\) an outgoing cap, and otherwise call \(W\) an intermediate cobordism.

If \(W\) is an incoming, intermediate, or outgoing cobordism respectively, we write \(\Lambda_W^W(\theta), \Lambda_W^W(\theta), \) and \(\Lambda_W^W(\theta)\) for the two-element sets of orientations of the following three operators:

\[
\begin{align*}
Q^-_\theta : \Omega^1_{-\delta} & \to (\Omega^{2,+} \oplus \Omega^0)_{-\delta} \\
Q^+_\theta : \Omega^1_{+\delta} & \to (\Omega^{2,+} \oplus \Omega^0)_{+\delta} \\
Q^0_\theta : \Omega^1_{-\delta,\delta} & \to (\Omega^{2,+} \oplus \Omega^0)_{-\delta,\delta}
\end{align*}
\]

Here the subscripts indicate the Sobolev weights on the ends; if only one subscript appears, it is the Sobolev weight on both ends, and if two appear, they are the Sobolev weights on the negative and positive ends, in that order.

Whenever we have two-element sets \(\Lambda\) and \(\Lambda'\), we write \(\Lambda\Lambda'\) to denote \(\Lambda \times_{Z/2} \Lambda'\), the \(Z/2\) action the canonical free involution on both sets.

The point of the choices of Sobolev indices is that these operators (for the unperturbed trivial connection) enjoy an additivity property, immediate from spectral flow arguments.

**Lemma 4.57.** Let \(W_0, W, W_1\) denote incoming, intermediate, and outgoing cobordisms equipped with \(U(2)\)-bundles, as above. If \(\hat{W}\) denotes the composite of these, then we have canonical isomorphisms

\[
\Lambda^{W_0(\alpha)} \Lambda(\varrho_\alpha) \Lambda^{W}(\alpha, \beta) \Lambda(\varrho_\beta) \Lambda^{W_1}(\beta) \cong \Lambda^{\hat{W}}(\theta)
\]

and

\[
\Lambda^{W_0(\theta)\Lambda^{W_0}(\theta)} \Lambda^{W_1}(\theta) \cong \Lambda^{\hat{W}}(\theta).
\]

Given a 3-manifold with \(U(2)\)-bundle \((Y, \tilde{E})\), and \((W_0, \tilde{E}), (W_0, \tilde{E}'), \) the sets \(\Lambda(\varrho_\alpha)\Lambda^{W_0(\alpha)}\Lambda^{W_0}(\alpha)\Lambda^{W_0}(\theta)\) and \(\Lambda(\varrho_\alpha)\Lambda^{W_0(\alpha)}\Lambda^{W_0}(\alpha)\Lambda^{W_0}(\theta)\) are canonically isomorphic. The appropriate modification is true for the outgoing caps, as well.

**Proof.** These follow from the usual gluing of operators on manifolds with cylindrical ends (for which the Sobolev weights on the ends match up appropriately, or there is a corresponding term to account for the spectral flow). The composite of the terms in the first displayed equation are naturally isomorphic, rather, to the orientation set of a nontrivial connection over \(W\), but as above we may reduce this to the case of a reducible connection and then to the trivial connection, as above.

Once we have this, using twice the orientation-reversal \(\hat{W}_0\) as a positive cap, using these canonical isomorphisms we further get canonical isomorphisms

\[
\Lambda(\varrho_\alpha)\Lambda^{W_0(\alpha)}\Lambda^{W_0}(\theta) \cong \Lambda^{\hat{W}_0(\alpha)}\Lambda^{\hat{W}_0}(\theta) \cong \Lambda(\varrho_\alpha)\Lambda^{W_0(\alpha)}\Lambda^{W_0}(\theta),
\]

as desired. \(\blacksquare\)
The collection of isomorphisms above between the different possible 2-element sets $\Lambda(g_\alpha)\Lambda^W(\alpha)\Lambda^{-W}(\theta)$ determine an equivalence relation on the set

$$
\bigsqcup_{(W,E)} \Lambda(g_\alpha)\Lambda^W(\alpha)\Lambda^{-W}(\theta),
$$

$W$ incoming cap

whose quotient is a 2-element set we write $\Lambda_-(\alpha)$. Similarly the 2-element set $\Lambda_+(\alpha)$ is a quotient of various copies of $\Lambda^W(\alpha)\Lambda^{-W}(\alpha)\Lambda(g_\alpha)$ as $W$ varies over positive caps.

We thus have from the first part of the above lemma a natural isomorphism

$$
\Lambda_-(\alpha)\Lambda^W(\theta)\Lambda_+(\beta) \cong \Lambda^W(\alpha,\beta).
$$

This was essentially our goal, though we are not quite done.

**Lemma 4.58.** There is a canonical isomorphism $\Lambda_+(\beta) \cong \Lambda(g_\beta)\Lambda_-(\beta)$.

**Proof.** Writing this out explicitly, we have chosen incoming and outgoing caps $W_0, W_1$, we are trying to construct a trivialization of

$$
\Lambda^W_{+1}(\theta)\Lambda^{W_1}(\beta)\Lambda(g_\beta)\Lambda^{W_0}(\beta)\Lambda^{W_0}(\theta),
$$

here exploiting the isomorphism between orientations on a vector space and their dual to reverse the order of the first few terms. By gluing the middle three terms, we obtain an isomorphism to $\Lambda^W_{+1}(\theta)\Lambda^{W_1}(\theta)\Lambda^{W_0}(\theta)$; splitting up the middle term, this is isomorphic to $\Lambda^W_{+1}(\theta)\Lambda^{W_0}(\theta)\Lambda^{W_1}(\theta)\Lambda^{W_0}(\theta)$. As this takes the form $\Lambda^2$ for some 2-element set $\Lambda$, it is canonically trivial. $\blacksquare$

Therefore, if we simply write $\Lambda(\alpha) := \Lambda_+(\alpha)$, we have above found a canonical isomorphism $\Lambda(g_\alpha)\Lambda(\alpha)\Lambda^W(\theta)\Lambda(\beta) \cong \Lambda^W(\alpha,\beta)$; then we obtain the gluing isomorphism $\Lambda^W(\alpha,\beta)\Lambda(g_\beta)\Lambda^{W'}(\beta,\gamma) \cong \Lambda^{W\cup W'}(\alpha,\gamma)$ simply by paing off adjacent like terms.

The last thing to be clear about is precisely what the middle orientation set $\Lambda^W(\theta)$ is.

**Definition 4.19.** A homology orientation of $W$, a cobordism with cylindrical ends and incoming end $(-\infty,0] \times Y_1$, is an orientation of the real vector space $H^1(W) \oplus H^{2,+}(W) \oplus H^1(Y_1)$.

Because $W$ and $Y$ are connected, note that we have a canonical isomorphism $H^0(W) \oplus H^0(Y_1) \cong \mathbb{R}^2$, and in particular carries a canonical orientation. Because we have $\ker(Q^1_\theta) \cong H^1(W)$ and $\coker(Q^1_\theta) \cong H^{2,+}(W) \oplus H^1(Y_1) \oplus H^0(W) \oplus H^0(Y_1)$, a homology orientation canonically induces an orientation of $\Lambda^W(\theta)$.

**Remark 4.3.** Homology orientations have a natural associative composition law, given by the index-gluing described above. However, the composition law may be described explicitly, without passing to a discussion of Fredholm operators. An explicit formula for this law was found in [Sca15], where an explicit understanding was crucial to discuss the differentials in a spectral sequence to Khovanov homology. We do not need this here, and so will not discuss Scaduto’s results in any more detail.

We assemble the content of this section into a proposition.
Proposition 4.59. For any 3-manifold \( Y \) equipped with \( U(2) \)-bundle \( \tilde{E} \) and regular perturbation \( \pi \), there are canonical 2-element sets \( \Lambda(\alpha) \) for each critical orbit \( \alpha \in \mathcal{C}_\pi \), and we have the canonical isomorphism \( \Lambda(\alpha, \beta) \cong \Lambda(g_\alpha)\Lambda(\alpha)\Lambda(\beta) \). Therefore, the moduli space \( \tilde{\mathcal{M}}_2(\alpha, \beta) \) may be given an orientation if we choose an element of \( \Lambda(\alpha) \) and \( \Lambda(\beta) \), and an orientation of the orbit \( \alpha \). Choosing the other element of either set will negate the orientation of \( \tilde{\mathcal{M}}_2(\alpha, \beta) \).

If one fixes a choice of element of each of \( \Lambda(\alpha) \), \( \Lambda(\beta) \), \( \Lambda(\gamma) \), and an orientation of \( \alpha \), then the orientation these induce on \( \tilde{\mathcal{M}}_2(\alpha, \beta) \) differs from the orientation induced as the fiber of a component of \( \tilde{\mathcal{M}}_{E,\pi}(\alpha, \gamma) \) by a sign of \( (-1)^d \), where \( d \) is the dimension of \( \tilde{\mathcal{M}}_{E,\pi}(\alpha, \gamma) \).

Let \( (W, \tilde{E}) \) be a cobordism from \( (Y_1, E_1) \) to \( (Y_2, E_2) \), equipped with a regular perturbation \( \pi \) that restricts to \( \pi_i \) on the ends. Then a choice of element of each \( \Lambda(\alpha) \) and \( \Lambda(\beta) \), as well as a homology orientation on \( W \) and orientation of \( \alpha \), give rise to an orientation of \( \tilde{\mathcal{M}}_{E,\pi}(\alpha, \beta) \); swapping any one of these elements will negate this orientation.

The boundary components of \( \tilde{\mathcal{M}}_{E,\pi}(\alpha, \beta) \) arise in two pieces: those of the form \( \tilde{\mathcal{M}}_{E_1,z_1,\pi_1}(\alpha, \gamma) \times_\gamma \tilde{\mathcal{M}}_{E_2,z_2,\pi}(\gamma, \beta) \), where \( \gamma \in \mathcal{C}_{\pi_1} \), and those of the form \( \tilde{\mathcal{M}}_{E_1,z_1,\pi_1}(\alpha, \zeta) \times_\zeta \tilde{\mathcal{M}}_{E_2,z_2,\pi_2}(\zeta, \beta) \), where \( \zeta \in \mathcal{C}_{\pi_2} \).

Given a choice of element of each \( \Lambda(\alpha), \Lambda(\beta), \Lambda(\gamma) \), and \( \Lambda(\zeta) \) as above, as well as a choice of homology orientation on \( W \) and orientation of \( \alpha \), the orientations on \( \partial \tilde{\mathcal{M}} \) and on the fiber products agree in the first case; they differ by a sign of \( (-1)^{d_W} \) in the second case, where \( d_W \) is the dimension of \( \tilde{\mathcal{M}}_{E_1,z_1,\pi}(\alpha, \zeta) \).

Similarly, if \( S \) is a 1-parameter family of metrics on \( W \) abutting to a broken metric, the decomposition of the boundary of \( \tilde{\mathcal{M}}_{E,z,S}(\alpha, \beta) \) of Proposition 4.46 gives the boundary components orientations which differ by the orientation induced by analytic gluing by a factor of \( 1, (-1)^{d_1}, (-1)^{d_{12}}, \) and \( -1 \), respectively (the last term coming from the boundary orientation of \( \{0\} \subset [0,1] \)), where \( d_1 \) is the dimension of \( \tilde{\mathcal{M}}_{E_1,z_1}(\alpha, \gamma) \), while \( d_{12} \) is the dimension of \( \tilde{\mathcal{M}}_{E_1,z_1,S}(\alpha, \eta) \).
5. Floer homology

5.1. Geometric homology. It will be useful in what follows to have a chain complex computing the singular homology of a smooth manifold \(X\), whose generators are smooth maps from oriented manifolds with corners (as opposed to continuous maps from simplices). Fix a principal ideal domain \(R\).

The definitions below are modeled on [Lin18, Chapter 4.1], which in turn follow the simpler [Lip14]. For technical reasons, we must introduce the notion of strong \(\delta\)-chain. The reader will not be led astray in what follows by pretending every occurrence of “strong \(\delta\)-chain” means “compact smooth manifold with corners”; if we could achieve that level of smoothness on the instanton moduli spaces, this sequence of definitions would not be necessary to set up our homology theory.

**Definition 5.1.** A \(d\)-dimensional strong \(\delta\)-chain is a compact topological space \(P\) with a stratification

\[
P^d \supset P^{d-1} \supset \cdots \supset P^0 \supset \emptyset
\]

by closed subsets, so that \(P^e \setminus P^{e-1}\) is decomposed as a finite disjoint union of smooth manifolds of dimension \(e\), written \(\sqcup_{i=1}^{n_e} M^e_i\), the top stratum \(P^d\) has only one open face \(M_1^d = P^d \setminus P^{d-1}\) in its decomposition. (Note that \(M^e_i\) need not be connected!) We denote the closure of any one of these manifolds an \(e\)-dimensional face, and write it as \(\Delta\). We write \(\Delta^e\) for the interior of a face (one of the manifolds \(M^e_i\) in the disjoint union \(P^e \setminus P^{e-1}\)) and call it an ‘open face’.

We demand that whenever a codimension \(e\) face \(\Delta_0\) is contained in a codimension \((e - 2)\) face \(\Delta_2\), there are exactly two codimension \((e - 1)\) faces \(\Delta'\) with \(\Delta_0 \subset \Delta' \subset \Delta_2\).

Whenever \(\Delta_1 \subset \Delta_2\), we assign a set \(n(\Delta_1, \Delta_2)\) (which we will write as \(n\), or \(n_{12}\) when the faces are not implicit) with cardinality \(\dim \Delta_1 - \dim \Delta_2\), an open neighborhood \(\Delta_1^e \subset W(\Delta_1, \Delta_2) \subset \Delta_2\), and a map \(r : W \to [0, \varepsilon)^n\) so that \(\Delta_1^e = r^{-1}(0)\).

We also assign a space \(W \leftarrow EW(\Delta_1, \Delta_2)\), where \(EW\) is a topological manifold with corners and a smooth structure on each stratum of dimension \(d + m(\Delta_1, \Delta_2)\), for which the map from \(W\) is smooth on each stratum, and equipped with a map \(\tilde{r} : EW(\Delta_1, \Delta_2) \to [0, \varepsilon)^n\) extending \(r : W \to [0, \varepsilon)^n\). We demand the following.

- There is a vector bundle \(V(\Delta_1, \Delta_2) \to EW\) of rank \(m(\Delta_1, \Delta_2)\), and a section \(\sigma\) of \(V\), smooth and transverse to the zero section on each stratum, so that \(\sigma^{-1}(0) = W\).

- Write \(EW^k\) for the inverse image of \(R_k \subset [0, \varepsilon)^n\), the set of points for which exactly \(k\) coordinates are zero; we demand that the map \(EW^k \to R_k\) is a smooth submersion.

- \(\tilde{r}\) is a fiber bundle projection. The restriction of \(\sigma\) to \(\tilde{r}^{-1}(0)\) is transverse to the zero section, whose fiber above zero is \(\Delta_1^e\). In particular, \(EW\) is diffeomorphic to \([0, \varepsilon)^{n_{12}} \times \tilde{r}_{12}^{-1}(0)\), and \(\tilde{r}_{12}^{-1}(0)\) is diffeomorphic to a neighborhood of the zero section in the restriction of \(V\) to \(\Delta_1^e\).

These are compatible in the following sense. Associated to a sequence of inclusions \(\Delta_1 \subset \Delta_2 \subset \Delta_3\) we have an inclusion of sets \(n(\Delta_1, \Delta_2) \hookrightarrow n(\Delta_1, \Delta_3)\), an embedding \(EW(\Delta_1, \Delta_2) \hookrightarrow EW(\Delta_1, \Delta_3)\), as well as an embedding of vector bundles \(V(\Delta_1, \Delta_2) \hookrightarrow V(\Delta_1, \Delta_3)\) covering this; the section \(\sigma_{12}\) is the restriction of \(\sigma_{13}\). The map \(n_{12} \to n_{13}\) of sets induces a stratum-preserving embedding \([0, \varepsilon)^{n_{12}} \hookrightarrow [0, \varepsilon)^{n_{13}}\) so that, with respect to this inclusion, \(\tilde{r}_{12}\) is the restriction of \(\tilde{r}_{13}\). Finally, we
demand that with respect to these embeddings,
\[ EW(\Delta_1, \Delta_2) = \tilde{\tau}^{-1}_{13} ([0, \varepsilon]^{n_{12}}). \]

This complicated definition is in fact more or less forced on us by a few simple requirements. First, our chains should include compact topological manifolds with corners with a smooth structure on each stratum. Second, they should be closed under transverse intersections (in particular, the inverse image of a regular value should be a chain). This is the property we need to ensure that excision holds in our coming homology theory based on maps from strong $\delta$-chains.

Already we run into trouble: because the smooth structure on the strata do not interact, all we see is that the transverse intersection of two topological manifolds with smooth structures on each stratum is a space \textit{stratified} by smooth manifolds. For instance, it is not hard to construct a $\theta$-shaped graph as the zero set of a continuous function $f$ on the $[0, 1] \times \mathbb{R}$ which is smooth on each stratum and for which 0 is a regular value on each stratum. This is what forces us to think of spaces equipped with “local thickenings” which actually \textit{are} manifolds. In this case, the three open arcs of the graph form the unique top-dimensional open face, whose local thickening is a small open neighborhood of these in the strip; the vector bundle is the trivial line bundle, and $\sigma$ is the function $f$. The stratum $P^0$ consists of two points, for which the neighborhood $W(\Delta_0, \Delta_1)$ is a small neighborhood of one of these points (which looks like a chicken foot). The local thickenings are small 2-dimensional neighborhoods $[0, \varepsilon] \times (-t, t)$ and the map $\tilde{r}$ is projection to $[0, \varepsilon)$. Again, the vector bundle is the trivial line bundle and $\sigma = f$.

The third requirement is that faces of a chain should also be chains, and thus we are forced to define local thickenings not just for the whole space, but also for inclusions $\Delta_1 \subset \Delta_2$. Finally, it is crucial that the combinatorial boundary operator satisfies $\partial^2 = 0$; this is why we demand the combinatorial condition on faces. At the level of manifolds with corners, it says that a bigon is a manifold with corners, but a teardrop (unigon) is not.

Essentially, the definition of strong $\delta$-chain is one possible approach to formalize a space with local charts modelling a neighborhood of each point as the zero set of a smooth function (smooth section of a vector bundle) on a compact topological manifold with smooth structures on each stratum, and compatibility relations between the charts.

The notion of strong $\delta$-chain here is a slight modification of the notion of $\delta$-chain given in [KM07, Definition 24.7.1] and [Lin18, Definition 4.1.1].

There are two differences. First, for a strong $\delta$-chain the set $W \subset EW$ is cut out as the zero set of a transverse (on each stratum) section of a vector bundle over $EW$; for a $\delta$-chain, instead the map is simply to some vector space $\mathbb{R}^k$. The definition of $\delta$-chain is attempting to formalize the kind of spaces that appear as the monopole moduli spaces, while the definition of strong $\delta$-chain is meant to capture all transverse intersections of compact topological manifolds with corners and smooth structures on each stratum. These transverse intersections may be written as the inverse image of a diagonal $\Delta \subset Y \times Y$; the vector bundle in question is $N\Delta$, and if $Y$ has interesting topology this bundle may be nontrivial, so we need to include zero sets of sections of vector bundles. Locally near any point, these are the same notion; but they may not be the same in a neighborhood of an entire stratum.
Second, the map $EW \to \mathbb{R}^k$ in the definition of $\delta$-chain is required to satisfy certain positivity properties, depending on the values of $\tilde{r}(EW) \subset [0, \varepsilon]^n$; we do not make any such demand. This requirement originates from the boundary-obstructedness phenomenon in monopole moduli spaces, which we do not encounter.

**Definition 5.2.** Suppose we are given the following data.

- For each pair of faces $\Delta_1 \subset \Delta_2$, open subsets $U_{12} \subset W_{12}$ and $U'_{12} \subset W'_{12}$, as well as open subsets $EU_{12}, EU'_{12}$ of the extensions,
- homeomorphisms $\varphi_{12} : U_{12} \to U'_{12}$, $\tilde{\varphi}_{12} : EU_{12} \to EU'_{12}$, which are diffeomorphisms over each stratum,
- an isomorphism of vector bundles $V_{12} \cong V'_{12}$ over these open subsets $EU$ and $EU'$, covering the diffeomorphism $\tilde{\varphi}_{12}$.

all of which respect all the associated structure and are compatible for triples $\Delta_0 \subset \Delta_1 \subset \Delta_2$. Then we say the two strong $\delta$-chains are germ equivalent.

Suppose now that we are given two strong $\delta$-chains with the same $P, W$, and maps $r$, but different thickenings $EW, E'W$. Suppose we are given the following.

- A vector bundle $F_{12}$ over $EW_{12}$ for each inclusion of faces,
- an embedding $F_{12} \to F_{13}$ covering the embedding $EW_{12} \to EW_{13}$ corresponding to any triple of faces,
- a diffeomorphism $\varphi_{12} : E'W_{12} \cong F_{12}$ (the total space of the vector bundle),
- an isomorphism $\psi_{12} : \pi^*(V_{12} \oplus F_{12}) \cong V'_{12}$ covering $\varphi_{12}$, taking $\sigma \oplus 0$ to $\sigma' \oplus 0$.

all of which are compatible for triples of faces, then we say that the strong $\delta$-chains are stably equivalent.

This definition fits with the above intuition for strong $\delta$-chains as spaces equipped with a special kind of chart: if we pass to smaller open subsets, or stabilize, this should still present a perfectly good, equivalent chart at a point.

Next, we will show the important property that strong $\delta$-chains are, in a sense, closed under fiber products. Stable equivalence was partly introduced as a means to this end. To start, we should define what a map from a strong $\delta$-chain is.

**Definition 5.3.** Suppose $X$ is a smooth manifold, and $P$ is a strong $\delta$-chain. We say a map $f : P \to X$ is a continuous map $f : P \to X$ on the underlying topological space which is smooth on each stratum, and for each inclusion of faces $\Delta_1 \subset \Delta_2$, an extension $Ef_{12} : EW_{12} \to X$ which is smooth on each stratum. Given a sequence of faces $\Delta_0 \subset \Delta_1 \subset \Delta_2$, recall that $EW_{02} \to EW_{12}$; we demand that $Ef_{02}$ is the restriction of $Ef_{12}$.

If $f_1 : P_1 \to X$ and $f_2 : P_2 \to X$ are two maps from strong $\delta$-chains to $X$, we say that they are transverse if they are transverse on each stratum, and for any pair of faces in $P_1$ and any pair of faces in $P_2$, the extensions $Ef_1$ and $Ef_2$ are transverse on each stratum in a neighborhood of $W_1 \times W_2$ in $EW_1 \times EW_2$.

We say that a map $f$ is submersive if $Ef_{12}$ is a submersion on each stratum.

Note that the definition of submersive says nothing about the underlying map $f$. For a map to be submersive $EW$ has to be quite large, but we may do this at the cost of a stabilization; this is why we introduced stable equivalence above.

**Lemma 5.1.** Let $f : P \to X$ be a map from a strong $\delta$-chain to a smooth manifold $X$ without boundary. Then $f$ extends to a submersive map from the stabilization of
Proof. Equip $X$ is equipped with a complete Riemannian metric. Then $E^1f_{12}(p, v)$ is $\exp(E_{f12}(p), v)$, the exponential of the tangent vector $v \in T_{E(f)}(p)$ using the metric on $X$. This map is submersive because the derivative of the exponential map is the identity.

**Proposition 5.2.** Suppose $f_1 : P_1 \to X$ are transverse maps from germ equivalence classes of strong $\delta$-chains $P_1, P_2$ to a complete Riemannian manifold $X$, and suppose that $f_2$ is submersive. Then $P_1 \times_X P_2$ has the natural structure of a germ equivalence class of strong $\delta$-chain.

**Proof.** The compact space $P_1 \times_X P_2$ is still stratified by smooth manifolds (the fiber products of the original open faces). For a pair of inclusions $\Delta_i^1 \subset \Delta_i^2$ of faces of $P_i$, the open set $W$ corresponding to $\Delta_i^1 \times_X \Delta_i^2 \subset \Delta_i^1 \times_X \Delta_i^2$ is the fiber product of the corresponding open sets. $EW$ is the fiber product $EW_1 \times_X EW_2$; this is still a topological manifold with corners and a smooth structure on each stratum by the assumption that $Ef_2$ is a submersion on each stratum. All of the additional structures (the vector bundle $V$, the section $\sigma$, the map $r$, and so on) is given as the fiber product of the structures on the $EW_i$; for instance the vector bundle $V$ is $V_1 \times_X V_2$, whose fibers above $EW_1 \times_X EW_2$ are isomorphic to the direct sum of fibers $V_1 \oplus V_2$.

To use coefficients other than $\mathbb{Z}/2$, we must introduce a notion of orientation.

**Definition 5.4.** Let $P$ be a strong $\delta$-chain. Suppose we have an orientation on the top stratum of $P$, and for a codimension 1 face $\Delta_1 \subset P$, suppose we have oriented the open manifold $\Delta_1^\circ$. We may find an isomorphism $\det(TEW_{12}) \cong \det V_{12}$ above $W_{12}^\circ$ using the fiber bundle isomorphism $EW \cong [0, \varepsilon) \times \tilde{r}_{12}^{-1}(0)$, using that $\tilde{r}_{12}^{-1}(0)$ is isomorphic to a neighborhood of the zero section of the restriction of $V_{12}$ to $\Delta_1^\circ$, and that we have oriented $\Delta_1^\circ$. Then $W_{12}^\circ := r_{12}^\circ(0, \varepsilon)^n = \tilde{r}_{12}^{-1}(0, \varepsilon)^n \cap \sigma^{-1}(0)$ may be oriented as the zero set of the section $EW_{12}^\circ \to V_{12}$ using the above isomorphism of determinant bundles. This gives an orientation on an open subset of $P^d \setminus P^{d-1}$. If the same as the orientation already given, we say that $\Delta_1$ has the boundary orientation; we express that this is possible for some orientation on $\Delta_1$ by saying that $\Delta_1$ is consistently orientable with respect to $P$.

For any sequence of codimension 1 faces $\Delta_k \subset \cdots \subset \Delta_1 \subset P$, we may thus inductively define the notion of any sequence being consistently orientable with respect to $P$.

We say that $P$ is an oriented $\delta$-chain if $P$ is equipped with an orientation on its top stratum $P^d \setminus P^{d-1}$, satisfying the following conditions:

- Any sequence of codimension 1 faces of $P$ is consistently orientable with respect to $P$.
- Given a codimension 2 inclusion $\Delta_0 \subset \Delta_2$, there are two intermediate faces $\Delta_1$ and $\Delta_1'$; we demand that if we fix an orientation on $\Delta_2$ (induced from $P$ by some sequence of codimension 1 faces as above), then the boundary
orientation that arises on \( \Delta_0 \) from the sequence \( \Delta_0 \subset \Delta_1 \subset \Delta_2 \) is negative that from the sequence \( \Delta_0 \subset \Delta'_1 \subset \Delta_2 \).

The definition of boundary orientation above is relevant for the following calculation for strong \( \delta \)-chains. This is proved for \( \delta \)-chains as [KM07, Theorem 21.3.2]; instead of subdividing our strong \( \delta \)-chains into pieces that may be given the structure of \( \delta \)-chains in the sense of Kronheimer-Mrowka, instead we spell out their proof in simpler language as an exercise in understanding the definition of strong \( \delta \)-chains. The proof follows similar lines as a discussion in [SS10, Section 2c].

**Lemma 5.3.** If \( P \) is a 1-dimensional oriented strong \( \delta \)-chain, then \( P_0 \) consists of a finite number of oriented points whose count, signed by the boundary orientation, is equal to zero.

**Proof.** If \( x \in P_0 \), there is an open subset \( W_x \subset P \) with a map to \( [0, \varepsilon] \), as well as a larger space \( EW_x \cong [0, \varepsilon] \times V_x \) so that \( W_x \hookrightarrow EW_x \), with \( x \) mapping to \( (0, 0) \), and \( W_x^{\sigma} \) mapping into \( (0, \varepsilon) \times V_x \); the neighborhood \( W_x \subset P \) is the zero set of a map \( \sigma : [0, \varepsilon] \times V_x \to V_x \) which is smooth and transverse to zero on each stratum. Write \( \sigma_1(x) = \sigma(t, x) \). That \( \sigma \) is transverse to zero on the boundary implies that \( \sigma^{-1}(0) \cap (\{0\} \times V_x) \) is discrete (and in particular discrete inside \( W \)). In particular, \( P_0 \) is a discrete set in the compact space \( P \), so it is finite. For convenience, after passing to a possibly smaller neighborhood and reparameterizing, we assume \( \sigma_0 = \text{Id} \) in our coordinates.

We also see that \( \sigma \) does not vanish on a sufficiently small sphere \( \{0\} \times S_\delta(V_x) \) in the boundary of the local thickening near \( x \); the inclusion map from this sphere to \( V_x \setminus \{0\} \) is degree 1 (regardless of how we orient \( V_x \), as long as we orient these compatibly). Extending this to a disc \( D_x \) in \( [0, \varepsilon] \times V_x \) which only intersects \( \{0\} \times V_x \) in \( S_\delta(V_x) \) and is transverse to \( \sigma^{-1}(0) \), we see that \( D_x \cap \sigma^{-1}(0) \) is a finite set of points whose oriented sum is 1, orienting them as \( \det(\sigma_1)(y) \) at an intersection point \((t, y)\).

\( D_x \) bounds a ball one dimension larger; deleting the part of \( W_x \) contained in the interior of this ball for each \( x \), what is left of \( P^1 \) is a compact oriented 1-manifold \( L \) with boundary \( \bigcup_{x \in P^1} (D_x \cap \sigma^{-1}(0)) \). The boundary orientation convention is that a point \((t, y)\) in this intersection is oriented positively as the boundary of \( L \) if the sign of \( \det(d\sigma_t)(y) \) agrees with the sign of \( x \). In particular, the signed sum of points in \( \partial L \) is

\[
\sum_{x \in P^0} \text{sgn}(x) \# (D_x \cap \sigma^{-1}(0)) = \sum_{x \in P^0} \text{sgn}(x),
\]

where the signed count is as in the previous paragraph. The signed count of points in the boundary of an oriented compact 1-manifold is zero, and thus the desired count is zero. \( \blacksquare \)

Now that these objects have been introduced, we can introduce the geometric chain complex computing singular homology. First we will write down the generating set.

**Definition 5.5.** Let \( X \) be a smooth manifold. Fix once and for all a Hilbert space \( H \) equipped with a Hilbert bundle \( V_H \).

A basic chain of degree \( d \) in \( X \) is a stable germ equivalence class of maps \( \sigma : P \to X \), where \( P \subset H \) is a compact connected subspace equipped with the structure of
an oriented strong $\delta$-chain of dimension $d$, whose thickenings $EW_{12}$ are compatibly embedded subspaces of $H_\ast$, and whose vector bundles $V_{12}$ are subbundles of $V_H$.

Two basic chains $\sigma_1 : P_1 \to X$ are isomorphic if there is an orientation-preserving homeomorphism $f : P_1 \to P_2$ which is a diffeomorphism on each stratum, with $\sigma_2 f = \sigma_1$, and an extension for each $\Delta_1 \subseteq \Delta_2$ to $E f_{12} : EW_{12} \to EW_{22}$ which are diffeomorphisms on each stratum compatible with restriction. A basic chain $\sigma : P \to X$ is isomorphic to itself with the opposite orientation, and chiral if it is not achiral. A basic chain $\sigma : P \to X$ has small image if there is map $g : Q \to X$ from a $\delta$-chain $Q$ of strictly smaller dimension.

Write $\tilde{C}_\ast(X; R)$ for the graded vector space freely generated in degree $d$ by the basic chains of degree $d$. One step closer to the desired chain complex, there is a quotient $\overline{C}_\ast(X; R)$ obtained from the relations

$$-[\sigma : P \to X] = [\sigma : \partial P \to X],$$

and if $2 \neq 0 \in R$ we further set any achiral basic chains $\sigma$ equal to zero.

This is still a free graded $R$-module, freely generated by isomorphism classes of chiral basic chains, with exactly one choice of orientation for each nontrivial basic chain appearing as a generator. (The achiral chains are precisely the basis elements of $\tilde{C}_\ast(X; R)$ which are sent to zero in $\overline{C}_\ast(X; R)$; if $2 = 0 \in R$ then no basis elements are set to zero.)

There is a geometric boundary operator $\partial : \tilde{C}_\ast(X; R) \to \tilde{C}_{\ast-1}(X; R)$, sending each (isomorphism class of) basic chain $\sigma : P \to X$ to the sum of the faces of $P_{d-1}$ equipped with their boundary orientation. Because this is compatible with orientation-reversal, it descends to an operator $\overline{\partial} : \overline{C}_\ast(X; R) \to \overline{C}_{\ast-1}(X; R)$.

**Lemma 5.4.** The operator $\overline{\partial}^2 = 0$.

**Proof.** This immediately follows from the combinatorial conditions on the definition of $\delta$-chain and orientation thereof.

Given any face $\Delta_2 \subseteq P$ of codimension 2, there are two intermediate faces of codimension 1. The boundary of $\partial P$ then contains two copies of the face $\Delta_2$: once from each of the intermediate faces. The assumption on orientations were that these two copies were oriented negatively to one another, so they sum to zero. Therefore, $\overline{\partial}^2 \sigma = 0 \in \overline{C}_\ast(X; R)$.

As is familiar already from cubical definitions of singular homology, the homology of $\overline{C}_\ast(pt; R)$ is not obviously concentrated in degree zero unless we impose some further degeneracy requirements. (These degeneracy requirements will later turn out to be essential in our definition of the instanton Floer complex.) This is furnished by Lipyanskiy’s notion of small image. However, the basic chains of small image do not span a subcomplex of $\overline{C}_\ast(X; R)$: for instance, a basic chain of dimension larger than $X$ is automatically of small image, but if the chain is of dimension $\dim X + 1$, there is no such guarantee that its boundary has small image. This inspires us to make the following definition.

**Definition 5.6.** A basic chain $\sigma : P \to X$ is degenerate if both $\sigma$ has small image and $\overline{\partial} \sigma$ is a disjoint union of basic chains of small image and an achiral chain. The span of the degenerate basic chains forms a subcomplex $D_\ast(X; R) \subseteq \overline{C}_\ast(X; R)$; this is precisely the submodule spanned by basic chains with small image and whose
boundary in $\mathcal{C}_\ast(X; R)$ is a sum of elements of small image. We define the geometric chain complex of $X$ (with coefficients in $R$) as

$$C^\text{gm}_\ast(X; R) := \mathcal{C}_\ast(X; R)/D_\ast(X; R).$$

Again, $C^\text{gm}_\ast(X; R)$ is degreewise $R$-free (as opposed to strong $\delta$-chains). It is functorial under smooth maps, and is supported in degrees $[0, \dim X + 1]$; the map $C_{\dim X + 1}(X; R) \to B_{\dim X}(X; R)$. It is also well-behaved with respect to transverse intersections.

**Definition 5.7.** Suppose $f_i : P_i \to X$ is a countable family $F$ of maps from $\delta$-chains to $X$. The subcomplex of $C^\text{gm}_\ast(X; R)$ spanned by nondegenerate chains transverse to all of the $f_i$ is written $C^\text{gm,F}_\ast(X; R)$.

The following lemma can be proved as an inductive application of transversality theorems to each stratum, using that $P^c \setminus P^{c-1}$ is a manifold and $P^c$ is compact. A proof in the only slightly different setting of $\delta$-chains (not strong) is given in [Lin18, Lemma 4.1.14].

**Lemma 5.5.** The inclusion of $C^\text{gm,F}_\ast(X) \hookrightarrow C^\text{gm}_\ast(X)$ is a quasi-isomorphism.

Most importantly for us, these chain complexes have chain-level fiber product maps, as long as we’re potentially willing to pass to a quasi-isomorphic subcomplex. First we need an additional definition.

**Definition 5.8.** Let $f : Z \to X$ be a map from a strong $\delta$-chain to a smooth manifold $X$ which is a submersion on each stratum. We say that a fiber orientation of $f$ is an orientation of the $\delta$-chain $f^{-1}(x)$ for any $x \in X$.

**Lemma 5.6.** Suppose $X$ is a connected smooth manifold. Let $Z$ be a strong $\delta$-chain, and suppose we have maps $e_- : Z \to X, e_+ : Z \to Y$, for which $e_-$ is a submersion on each stratum of $Z$. Further, suppose $e_-$ is equipped with a fiber orientation. Then there is an induced map $C^\text{gm}_\ast(X; R) \to C^\text{gm}_\ast(Y; R)$ of degree $(\dim Z - \dim X)$ given by sending a basic chain $\sigma : P \to X$ to $e_+ : P \times e_- Z \to Y$, where

$$P \times e_- Z = \{(p, z) \in P \times Z \mid \sigma(p) = e_-(z)\}$$

and $e_+(p, z) = e_+(z)$. We denote this chain $\sigma \times e_- Z$. Here we orient the fiber product so that

$$T_p P \oplus T^\text{fib}_Z Z \cong T_{p,z} P \times_X Z$$

is an oriented splitting.

If $\dim Z > \dim X + \dim Y + 1$, this chain map is identically zero.

These maps satisfy the formula

$$\partial(\sigma \times e_- Z) = \partial \sigma \times e_- Z + (-1)^{\dim P + \dim X} \sigma \times e_- \partial Z.$$

The second statement follows because $C^\text{gm}_\ast(Y; R)$ vanishes in degrees larger than $\dim Y + 1$, and the final statement because taking the boundary in the final case involves commuting the desired outward-pointing normal across the first two factors (of $P$ and the base $X$).

If $M$ and $N$ are smooth manifolds, Cartesian product of basic chains defines a product $C^\text{gm}_\ast(M; R) \otimes C^\text{gm}_\ast(N; R) \to C^\text{gm}_\ast(M \times N; R)$. Combining this with the group multiplication $m : G \times G \to G$ of a Lie group, the geometric chain complex of a Lie group inherits the structure of a dg-algebra, and if $G$ acts on a smooth manifold $X$, the same product endows $C^\text{gm}_\ast(X; R)$ with the structure of a dg-module.
over $C^\text{gm}_*(G; R)$. In the situation of the above lemma, if $X, Y, Z$ are $G$-manifolds and the maps $e_{\pm}$ are $G$-equivariant, the fiber product map $- \times e_{\pm} Z$ is a $C^\text{gm}_*(G; R)$-module homomorphism. The most important case of Lemma 5.6 for us will be when $X$ and $Y$ are orbits of a compact Lie group $G$, $Z$ is a compact oriented $G$-manifold, and the $e_{\pm}$ are equivariant; that the endpoint maps are submersions then follows from equivariance and that $G$ acts transitively on $X$ and $Y$.

Now we should justify the claim that this is just a fancy way to write down singular homology with desirable chain-level properties. To make sense of the Eilenberg-Steenrod axioms for smooth manifolds, we work with the notion of admissible pairs $(X, A)$, where $X$ is a smooth manifold without boundary, and $A$ is a closed (in the sense of point-set topology) submanifold of $X$ of codimension zero. These were introduced in [Sch93] to prove the equivalence between Morse homology and singular homology, and used in [Lin18, Theorem 4.1.13] to prove that a very similar homology theory to ours (using strong $\delta$-chains, instead of strong $\delta$-chains) agrees with singular homology; our proof follows similar lines as his.

**Theorem 5.7.** The functor $H^\text{sm}_*(X; R)$ from smooth manifolds and maps to graded $R$-modules satisfies the Eilenberg-Steenrod axioms for a homology theory:

1. The induced map $H^\text{sm}_*(X; R) \to H^\text{sm}_*(Y; R)$ is a homotopy invariant,
2. There is a natural relative long exact sequence relating $H^\text{sm}_*(X, Y) := H(\text{C}^\text{gm}_*(X) / \text{C}^\text{gm}_*(Y))$ to $H^\text{sm}_*(X)$ and $H^\text{sm}_*(Y)$,
3. $H^\text{sm}_*(\ast; R)$ is a copy of $R$ concentrated in degree zero, and
4. If $A$ is codimension zero, and $Y = \partial A$ is a codimension 1 submanifold of $X$, bounding $A$ on one side and $B$ on the other, then the inclusion induces an isomorphism $H^\text{sm}_*(B; Y) \cong H^\text{sm}_*(X, A)$.

As a result, there is an isomorphism $H^\text{sing}_*(X; R) \cong H^\text{sm}_*(X; R)$ for all $X$, natural under smooth maps.

**Proof.** We prove each property separately, and then explain why this restricted class of Eilenberg-Steenrod axioms is enough.

Homotopy invariance follows the expected strategy: write down a chain homotopy of the induced map, given by sending a basic chain $P$ to $P \times I$, with map given by composing the map $\sigma : P \to X$ with the homotopy $f_t$.

The relative long exact sequence is a matter of homological algebra (it is induced by a short exact sequence of chain complexes).

To check the first nontrivial axiom, observe that $C^\text{gm}_*(\ast; R)$ is concentrated in degrees 0 and 1. We saw in Lemma 5.3 that the boundary map $C^\text{gm}_1(\ast; R) \to C^\text{gm}_0(\ast; R) = R$ is identically zero; but a chain of small image with zero boundary has boundary of small image, and is in particular degenerate, so $C^\text{gm}_1(\ast; R) = 0$. Therefore $H^\text{gm}_1(\ast) = C^\text{gm}_1(\ast) = R(0)$.

Excision is the hardest property to verify. If we write $C^\text{gm}_*(A \cup B)$ for the image of $C^\text{gm}_*(A) \oplus C^\text{gm}_*(B)$ in $C^\text{gm}_*(X)$, then there is a diagram of exact sequences

\[
\begin{array}{cccccc}
0 & \longrightarrow & C^\text{gm}_*(A) & \longrightarrow & C^\text{gm}_*(A \cup B) & \longrightarrow & C^\text{gm}_*(B, Y) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & C^\text{gm}_*(A) & \longrightarrow & C^\text{gm}_*(X) & \longrightarrow & C^\text{gm}_*(X, A) & \longrightarrow & 0.
\end{array}
\]

By the five lemma and the induced map on homology long exact sequences, if we can show the middle vertical arrow is a quasi-isomorphism, so too will be
the last arrow, as desired. Now, if we write $C^\text{gm,Y}_*(X)$ for the quasi-isomorphic subcomplex of strong $\delta$-chains transverse to the submanifold $Y$, there is a map $\rho : C^\text{gm,Y}_*(X) \to C^\text{gm}_*(A \cup B)$ by sending each basic chain $\sigma : P \to X$ to the sum of $P \cap A$ and $P \cap B$; these are again $\delta$-chains because of the transversality hypothesis, and their boundary is a subdivision of the original boundary into $(\partial P) \cap A$ and $(\partial P) \cap B$, respectively, and so this is a chain map. If $P$ is a cycle, then $\rho(P)$ is homologous to $P$ itself, the bounding chain given by $P \times I$, with the usual $\delta$-structure on one end, and the ‘broken’ $\delta$-structure $P = (P \cap A) \cup (P \cap B)$ on the other. In particular, we see that the inclusion $C^\text{gm}_*(A \cup B) \to C^\text{gm}_*(X)$ is surjective on homology. Injectivity is similar: given a chain whose boundary is transverse to $P$, we can represent it instead by a chain transverse to each admissible pair via Morse theory, we have a natural inclusion from the homotopy category of pairs of finite CW complexes. Putting a relative CW structure on one end, and the ‘broken’ $\delta$-structure on the other. In particular, we see that the inclusion $C^\text{gm}_*(A \cup B) \to C^\text{gm}_*(X)$ is surjective on homology. Injectivity is similar: given a chain whose boundary is transverse to $P$, we can represent it instead by a chain transverse to $P$ with homologous boundary still transverse to $P$; cutting it into two pieces, we get a chain in $C^\text{gm}_*(A \cup B)$ whose boundary is homologous to the original.

Now we should explain why the homotopy category of admissible pairs of smooth manifolds (with the homotopy type of a finite CW complex) is equivalent to the homotopy category of pairs of finite CW complexes. Putting a relative CW structure on each admissible pair via Morse theory, we have a natural inclusion from the category of admissible pairs to the category of finite CW complexes; by smooth and (relative) smooth approximation, this is a fully faithful functor, and it suffices to show that it’s surjective on objects. Given a pair $(X, A)$, find an embedding of this in some large Euclidean space; then a small open neighborhood $U_X$ of the image of $X$ (called a regular neighborhood) is homotopy equivalent to $X$ itself and so that $\partial U_X$ is a manifold, and we may choose an open neighborhood $U_A$ of $A$ satisfying the same property, with $\overline{U_A} \subset U_X$. Then the pair $(U_X, \overline{U_A})$ is the desired admissible pair.  

If $R$ has characteristic 2, we can simplify definitions by removing all reference to orientations; this is even desirable, as it allows us to refer to the identity map of a non-oriented compact manifold as a chain.

5.2. Equivariant instanton homology. We are now ready to define the framed instanton chain complex $\tilde{CI}(Y, E, \pi; R)$, using the instanton moduli spaces. The material of sections 1-4 amounts to the following package of theorems about these instanton moduli spaces. Recall from Definition 3.2 that for a $U(2)$-bundle $E$ over a 3-manifold $Y$ to be weakly admissible, either $\beta_1(Y) = 0$ or $c_1(E)$ is not twice some element of $H^2(Y; \mathbb{Z})$.

In the following, the regular perturbation $\pi$ is chosen from an open subset $\mathcal{P}_{E, \delta}$ of the Banach space $\mathcal{P}_E$ of perturbations, defined in Definition 4.3.

**Theorem 5.8.** Let $Y$ be a 3-manifold equipped with a weakly admissible $U(2)$-bundle $E$, a basepoint $b$, and a choice of metric and regular perturbation $\pi$ (which always exists). Then:

1. The collection of critical orbits of the perturbed Chern-Simons function $cs + \pi$ on the configuration space $\tilde{S}_E$ is a finite set of $SO(3)$-orbits. We write this set of orbits as $\mathcal{E}_{\pi}$.

2. Write $c_1(E) = \lambda \in H^2(Y; \mathbb{Z})$. Define $\text{Pair}(H^2)$ to be the set of pairs of integral cohomology classes on $Y$; that is, the set $H^2(Y; \mathbb{Z}) \times H^2(Y; \mathbb{Z})$ modulo the relation $(z, w) \sim (w, z)$. The set of reducible critical orbits may
be identified with the set
\[ \mathcal{R}_\lambda(Y,E) \subset \text{Pair}(H^2) \]
given by those \((z,w)\) with \(z + w = \lambda\). The fully reducible critical orbits are those for which \(z = w\), and there are hence \(H^1(Y;\mathbb{Z}/2)\) of them if \(Y\) is a rational homology sphere with \(c_1(E) = 0\), and none otherwise.

(3) There is a number \(\text{gr}_z(\alpha,\beta) \in \mathbb{Z}\), assigned to each pair of critical orbits \(\alpha,\beta\) and homotopy class of path \(z\) between them in \(\mathcal{B}_E\). If \(w\) is a path from \(\beta\) to \(\gamma\) and \(z \ast w\) is the concatenation of paths, we have
\[ \text{gr}_{z \ast w}(\alpha,\gamma) = \text{gr}_z(\alpha,\beta) + \text{gr}_w(\beta,\gamma). \]
For different paths \(z,w\), we have \(\text{gr}_z(\alpha,\beta) - \text{gr}_w(\alpha,\beta) \in 8\mathbb{Z}\), so that \(\text{gr}(\alpha,\beta) \in \mathbb{Z}/8\) is well-defined.

(4) If \(\alpha\) and \(\beta\) are reducible, then \(\text{gr}(\alpha,\beta)\) is even. If \(\alpha\) and \(\beta\) are fully reducible, then \(\text{gr}(\alpha,\beta)\) is divisible by 4.

(5) Associated to each pair \((\alpha,\beta)\) of critical orbits and homotopy class \(z\) is a smooth \(\text{SO}(3)\)-manifold (possibly empty) \(\bar{M}_{E,z,\pi}^0(\alpha,\beta)\) of dimension \(\text{gr}_z(\alpha,\beta) + \dim \alpha - 1\). It comes equipped with equivariant smooth maps
\[ \alpha \xleftarrow{e_-} \bar{M}_{E,z,\pi}^0(\alpha,\beta) \xrightarrow{e_+} \beta. \]

(6) For each critical orbit \(\alpha\), there is an associated 2-element set \(\Lambda(\alpha)\). A choice of element of each of \(\Lambda(\alpha)\) and \(\Lambda(\beta)\) induces an orientation on the fiber of
\[ e_- : \bar{M}_{E,z,\pi}^0(\alpha,\beta) \]
for all \(z\); in the language of Definition 5.8, this is a fiber orientation of \(e_-\). Negating either one of these choices negates the resulting fiber orientation.

(7) If \(\text{gr}_z(\alpha,\beta) \leq 10 - \dim \alpha\), there is a natural compactification
\[ \bar{M}_{E,z,\pi}^0(\alpha,\beta) \subset \mathcal{M}_{E,z,\pi}(\alpha,\beta) \]
into a compact topological \(\text{SO}(3)\)-manifold with corners and a smooth structure on each stratum. The endpoint maps extend to equivariant maps from \(\mathcal{M}^0\) which are smooth on each stratum; we will use the same notation \(e_{\pm}\) for these extended maps. In the trivial case \(\alpha = \beta\) with homotopy class \(z = 0\), we consider \(\mathcal{M}_{E,0,\pi}(\alpha,\alpha)\) to be empty.

(8) The action of \(\text{SO}(3)\) on \(\mathcal{M}_{E,z,\pi}(\alpha,\beta)\) is free.

(9) Given any choice of element of each of \(\Lambda(\alpha),\Lambda(\beta),\) and \(\Lambda(\gamma)\), so that all of the relevant moduli spaces are fiber oriented, there is a decomposition respecting fiber orientations
\[ \partial \mathcal{M}_{E,z,\pi}(\alpha,\beta) \cong \bigsqcup_{\gamma,w_1,w_2} (-1)^{\text{gr}_{w_1}(\alpha,\gamma)} \mathcal{M}_{E,w_1,\pi}(\alpha,\gamma) \times_{\gamma} \mathcal{M}_{E,w_2,\pi}(\gamma,\beta). \]
Observe that the relation \(\text{gr}_z(\alpha,\beta) = \dim \mathcal{M}_{E,z,\pi}(\alpha,\beta) - \dim \alpha + 1\) is compatible with the dimensions of the manifolds involved in the gluing formula.
Proof. For a fixed metric $g$, the existence of an admissible perturbation $\pi$ is the combination of Theorem 3.6 and Theorem 4.34; these guarantee that the critical orbits are isolated, and that the moduli spaces of trajectories between them are smooth manifolds of the appropriate dimension, respectively. That there are only finitely many critical orbits follows from Lemma 3.3 (that the derivative of our perturbed Chern-Simons functional is a proper map). The enumeration of reducible critical orbits is a combination of Corollary 1.3 and Proposition 3.5; as mentioned after Definition 4.3, one of the reasons we demand $\pi \in P_{E,\delta}$ is so that this enumeration remains true. The integer $gr_z(\alpha, \beta) = \overline{gr}_z(\alpha, \beta) - \dim \alpha$ is defined in Definition 4.5, where we also explain why $\overline{gr}_z$ counts the dimension of the fiber of $\hat{M} \cong \mathbb{R} \times \hat{M}^0$ above a point in $\alpha$. That the grading is well-defined modulo 8 is Corollary 4.27. The calculation of relative gradings between full reducibles is Proposition 4.32 and arbitrary reducibles is Proposition 4.33. The existence of a compactification only in terms of fiber products of lower-dimensional moduli spaces is Corollary 4.13; that the resulting object is a topological manifold with corners with a smooth structure on each stratum is Proposition 4.43. That the $SO(p,q)$-action is free means that there are no reducible solutions; this is guaranteed by Proposition 4.14. The moduli spaces carry compatible fiber orientations by Proposition 4.59, writing the sign in terms of the grading function $gr_z$. The signs here are different from those in Proposition 4.59 by a factor of $p - 1$. This arises because in the above formula, we take the boundary *before* passing to the fiber above $\alpha$. \[\square\]

We will also need a similar package in the case of cobordisms $W$. We include this here as well. Recall the definition of weakly admissible bundle over a cobordism from Definition 4.14, as well as Definition 4.19 of homology orientations of a cobordism $W$.

In the following, we choose the 4-manifold perturbation $\pi$ from a space $P_{E,L,\delta}^{(4)}$ of perturbations which restrict to elements of $P_{E,\delta}$ on the ends. These perturbations are allowed to be nontrivial in the complement of the ends; the symbol $L$ indicates that we include a finite-dimensional vector space $V_L$ consisting of holonomy perturbations on the interior of $W$, either with identical or non-overlapping domain. See Definition 4.2 for more details. Splitting this space as $P_{E,\delta}^{(4)} \oplus V_L$, we usually demand that the $V_L$ factor of a perturbation is taken very small.

**Theorem 5.9.** Suppose $W$ is an oriented 4-manifold with two cylindrical ends, with incoming end modeled on $Y_1$ and outgoing end modeled on $Y_2$. Furthermore suppose that $E$ is a weakly admissible $U(2)$-bundle over $W$. Then we have the following.

1. For every pair of critical orbits $\alpha \subset \mathcal{C}_{\pi_1}(Y_1)$ and $\beta \subset \mathcal{C}_{\pi_2}(Y_2)$, there is a set of configurations of framed connections on $W$ from $\alpha$ to $\beta$, denoted $\mathcal{B}_E(\alpha, \beta)$. The set of components is written $\pi_0 \mathcal{B}_E(\alpha, \beta)$, and carries a free and transitive action of both $\pi_1 \mathcal{B}_E(\alpha)$ and similarly with $E_2$; in particular, it carries an affine identification

$$\pi_0 \mathcal{B}_E(\alpha, \beta) \cong \mathbb{Z}.$$ 

For each $z$ in this set, there is an integer $gr_z^W(\alpha, \beta)$, satisfying the same additivity formula as in the previous theorem, which is independent of $z$ after reducing modulo 8.
(2) Let $\pi \in \mathcal{P}^{(1)}_{E,L,\delta}$ be a regular perturbation on $W$, restricting to regular perturbations $\pi_i$ on the cylindrical ends corresponding to $Y_i$. For each pair of critical orbits $\alpha, \beta$, with $z$ as above, we associate a smooth $SO(3)$-manifold $\tilde{M}_{E,z,\pi}(\alpha, \beta)$ of dimension $\text{gr}_z^W(\alpha, \beta) + \dim \alpha$, so long as $\tilde{M}(\alpha, \beta)$ contains an irreducible connection. The term $\text{gr}_z^W(\alpha, \beta)$ is independent of $z$ modulo 8. When $W$ is equipped with a smoothly embedded path $\gamma : \mathbb{R} \to W$ with $\gamma(-t) = (-t, b_1) \in (-\infty, 0] \times Y_1$ and $\gamma(1 + t) = (t, b_2) \in [0, \infty) \times Y_2$, then $\tilde{M}$ is imbued with smooth equivariant endpoint maps to $\alpha$ and $\beta$.

(3) If $\text{gr}_z^W(\alpha, \beta) \leq 10 - \dim \alpha$, then there is a natural compactification of this $SO(3)$-manifold to a compact topological $SO(3)$-manifold with corners and a smooth structure on each stratum, $\bar{M}_{E,z,\pi}(\alpha, \beta)$. The endpoint maps extend to $\bar{M}^W$.

(4) If $W$ is given a homology orientation as in Definition 4.19, and we choose generators of the 2-element sets $\Lambda(\alpha)$ and $\Lambda(\beta)$ from the previous proposition, these induce a fiber orientation on every $\bar{M}_{E,z,\pi}(\alpha, \beta) \to \alpha$. Negating any one of these choices negates the corresponding orientation.

(5) Suppose $b_1(W) = b^+(W) = 0$. So long as the perturbation $\pi$ is taken sufficiently small, the set of reducible orbits in $\bar{M}_{E,z,\pi}(\alpha_1, \alpha_2)$ may be identified as the following set, written $\text{Red}(W, E)$. In all other cases no reducible configurations arise. Write $c_1(E) = \lambda \in H^2(W; \mathbb{Z})$, and write $\lambda_i = \lambda|_{Y_i}$. There is an induced map $\text{Pair}(H^2W) \to \text{Pair}(H^2Y_i)$ given by restriction; the orbit $\alpha_i$ corresponds to some $r_i \in \mathcal{R}_{\lambda_i}(Y_i, E_i) \subset \text{Pair}(H^2Y_i)$. Then $\text{Red}(W, E)$ is the subset of $\text{Pair}(H^2W)$ consisting of pairs $\{z, w\}$ restricting to $r_i$ on the corresponding ends, and with $z + w = \lambda$. The set of full reducibles is taken to the subset with $z = w$.

(6) Assuming we have chosen a homology orientation of $W$ and an element of all relevant orientation sets, there is an oriented decomposition

\[
\partial \bar{M}^W_{E,z,\pi}(\alpha, \beta) \cong \bigcup_{\gamma \in \mathcal{C}_{\varepsilon_1, \varepsilon_2}, \varepsilon_1, \varepsilon_2, \varepsilon_1' \varepsilon_2'} (-1)^{\text{gr}_{\varepsilon_1'}(\alpha, \gamma)} \bar{M}^W_{\varepsilon_1,\varepsilon_2,\pi_1}(\alpha, \gamma) \times \gamma \bar{M}^W_{\varepsilon_2,\varepsilon_1,\pi_2}(\gamma, \beta)
\]

(7) There is a constant $\epsilon(W)$ depending only on $W$ and its metric so that if the perturbations on $Y_i$ are $\epsilon(W)$-small, there is a regular perturbation connecting them, and in fact a dense subset of $\epsilon(W)$-small perturbations are regular.

Proof. That such a regular perturbation exists is Theorem 4.36. The grading function was defined in Definition 4.5 and seen to be well-defined mod 8 in Corollary 4.27, the compactification with no bubbling is given in Corollary 4.13, and the manifold-with corners structure (and decomposition of the boundary) given by
Proposition 4.43. The orientations are provided by Proposition 4.59. The identification of reducible orbits was given by Proposition 1.7 and Proposition 4.15.

The path $\gamma$ is necessary to choose a basepoint in $W$ (as $\gamma(1/2)$, say) and then to parallel transport the chosen framing to the basepoints $\gamma(0), \gamma(1)$ on the boundary components.

The fiber orientations and oriented decomposition were provided by Proposition 4.59; as in the previous theorem, there is a sign difference of $(-1)^{\dim \alpha}$ between what is written here and in that proposition.

Lastly, to prove invariance of these induced maps, we will want a version of this package for families of perturbations $\pi$. Recall both the definition of broken metric and regular family of metrics and perturbations from Section 4.8. We only use this notation for the case of families indexed by the interval $[0, 1]$.

Also recall that Definition 4.14 of weakly admissible cobordisms partitions these into two classes: the admissible cobordisms for which one end is an admissible bundle (so supports no reducible flat connections), and the weakly admissible cobordisms with rational homology sphere ends and dim (1/2) (so supports no reducible flat connections), and the weakly admissible cobordisms (with rational homology sphere ends and $b^+(W) > 0$, but which either fail to satisfy the ‘strong monotonicity inequality’ $\rho_{\pi_1}(t) \leq \rho_{\pi_2}(s) - 2b^+(W)$ or support bad reducibles (meaning that $H_1W \rightarrow H_1Y_1 \oplus H_1Y_2$ is surjective).

**Theorem 5.10.** Suppose $I = [0, 1]$ parameterizes a family of metrics on $W$ (with fixed cylindrical ends); the metrics $g_0$ and $g_1$ may be cut along a separating 3-dimensional submanifold of $W$, as in Section 4.8. Then the following is true.

1. Suppose $(W, E)$ is a weakly admissible cobordism that supports no reducible connections and $\pi_0, \pi_1 \in \mathcal{P}_{L, \delta}$ are a pair of regular perturbations, equal sufficiently far on the ends. Then expanding $L$ to a larger set $L'$ if necessary, there is a path $\pi(t) : [0, 1] \rightarrow \mathcal{P}_{L', \delta}$ connecting them, constant on the ends, which forms a regular family of metrics and perturbations. Write $\pi_\pm$ for the restriction of this family to the $3$-manifolds $(Y_1, E_1)$ and $(Y_2, E_2)$, respectively.

2. If $(W, E)$ is a weakly admissible cobordism with $b_1(W) = b^+(W) = 0$ and with $E$ trivializable, if $\pi_0, \pi_1$ are a pair as above, then for any path $\pi(t)$ for which the fully reducible connections are regular for all $\pi(t)$, we may perturb this path on the interior to make it a regular family of metrics and perturbations. In particular, if $\pi_0$ and $\pi_1$ are sufficiently small, we may always find such a path.

3. If $b^+(W) = 1$, then for $\pi_0, \pi_1$ as above and any path $\pi_1$ that supports no reducible instantons, we may perturb $\pi_1$ on the interior to form a regular family of metrics and perturbations.

4. If $\pi(t)$ is a regular family of perturbations on $[0, 1] = I$, there are smooth $SO(3)$-manifolds $\widetilde{M}_{E, \pi(t)}(\alpha, \beta)$ of dimension $\dim \mathfrak{g}(\alpha, \beta) + \dim \alpha + 1$.

5. A homology orientation of $W$ and a choice of element of each of $\Lambda(\alpha)$ and $\Lambda(\beta)$ induces a fiber orientation on $e_- : \widetilde{M}_{E, \pi(t)}(\alpha, \beta) \rightarrow \alpha$, as before. These orientations negate under orientation-reversal of any one of these choices.
(6) As long as \( \text{gr}^W_\alpha(\alpha, \beta) \leq 9 - \dim \alpha \), the moduli space \( \overline{M}_{E,z,\pi}^{1}(\alpha, \beta) \) has a natural compactification \( \overline{M}_{E,z,\pi}^{(1)}(\alpha, \beta) \) satisfying the same properties as before.

(7) If an element is chosen from each relevant orientation set as above, then there is an oriented decomposition

\[
\partial \overline{M}_{E,z,\pi}^{1}(\alpha, \beta) = \bigcup_{\gamma \in E_{\pi,\alpha} \cup \{\gamma\} \cup 2} \overline{M}_{E_{\alpha},\gamma,\pi}^{1}(\alpha, \gamma) \times_{\gamma} \overline{M}_{E,z,\pi}^{1}(\gamma, \beta)
\]

\[
\bigcup_{\zeta \in E_{\pi,\beta} \cup \{\zeta\} \cup 2} (-1)^{\text{gr}^W_\alpha(\alpha, \zeta)} \overline{M}_{E,z,\pi}^{1}(\alpha, \zeta) \times_{\zeta} \overline{M}_{E,z,\pi}^{1}(\zeta, \beta)
\]

\[
\cup \overline{M}_{E,z,\pi}^{W}(\alpha, \beta) \cup -\overline{M}_{E,z,\pi}^{W}(\alpha, \beta).
\]

If the perturbation \( \pi \) is broken along some submanifold \( Y \) with limiting perturbation \( \pi_b \), along \( Y \), and restricting to perturbations \( \pi_i \) on the piece \( (W_1, E_1) \), then we have the decomposition

\[
\overline{M}_{E,z,\pi}^{W}(\alpha, \beta) = \bigcup_{\eta \in E_{\pi,\alpha} \cup \{\eta\} \cup 2} (-1)^{\text{gr}^W_\alpha(\alpha, \eta)} \overline{M}_{E_{\alpha},\pi}^{W}(\alpha, \eta) \times_{\eta} \overline{M}_{E,z,\pi}^{W}(\eta, \beta).
\]

Proof. This precisely the content of Section 4.8: in particular, the transversality and gluing results in the absence of broken metrics are given by Proposition 4.44 and Proposition 4.46, respectively; the extension to the case of broken metrics, including the third item above, is given by Proposition 4.47.

We will use the fiber product maps of Lemma 5.6 associated to the \( SO(3) \)-equivariant endpoint maps

\[
e_-, e_+: \overline{M}_{E,z,\pi}^{1}(\alpha, \beta) \to \alpha, \quad e_+: \overline{M}_{E,z,\pi}^{1}(\alpha, \beta) \to \beta
\]

to define the framed instanton differential, as opposed to the usual counting of points in 0-dimensional moduli spaces. (Note that the degree of the fiber product map is precisely \( \text{gr}^W_\alpha(\alpha, \beta) - 1 \) for moduli spaces on the cylinder, and precisely \( \text{gr}^W_\alpha(\alpha, \beta) \) on a cobordism.) Lemma 5.6 guarantees that when \( \dim \overline{M}(\alpha, \beta) \geq \dim \alpha + \dim \beta + 2 \), the map \( \sigma \mapsto e_- \overline{M}(\alpha, \beta) \) is identically zero on the chain level. Because all orbits have dimension at most 3, and \( \dim \overline{M}(\alpha, \beta) = \text{gr}^W_\alpha(\alpha, \beta) + \dim \alpha + 1 \), we see that these fiber product maps vanish when \( \text{gr}^W_\alpha(\alpha, \beta) \geq 6 \). As a result, we need not panic about the Uhlenbeck bubbling arising in large-dimensional moduli spaces. This same observation arose in the original definition of equivariant instanton homology given in [AB96]. In their case, the differential was defined via pullback and integration-over-the-fiber of differential forms, and whenever the differential form is in degree lower than the dimension of the fiber, the integral is zero.

**Definition 5.9.** A relative \( \mathbb{Z}/8 \) grading on a set \( S \) is defined by a function \( r: S \times S \to \mathbb{Z}/8 \), with \( r(x, y) + r(y, z) = r(x, z) \). Such an additive function is equivalent to a function \( i: S \to \mathbb{Z}/8 \) considered up to translation; that is, \( i \sim i' \) if \( i(x) = i'(x) + c \), for some fixed \( c \). Given a function \( i \), one obtains an additive function from its differences:

\[
r_i(x, y) = i(y) - i(x).
\]
This gives the same function for any equivalent \( i, i' \). Conversely, evaluation defines functions \( i_s(x) = r(x, s) \), with \( i_s(x) = i_t(x) + i(t, s) \), so that \( i_s \sim i_t \).

A relatively graded chain complex splits as a direct sum \( C = \bigoplus_{i \in \mathbb{Z}/8} C_i \), where the differential is the direct sum over \( d_i : C_i \to C_{i-1} \).

The relevant relative grading to us is \( \text{gr} : E \times E \to \mathbb{Z}/8 \). Picking a critical orbit \( \rho \) arbitrarily, we write \( i(\alpha) = \text{gr}(\rho, \alpha) \). If \( E \) is trivial, we set \( i(\alpha) = i(\theta, \alpha) \), where \( \theta \) is the trivial connection. This gives us an absolute grading when \( E \) is trivial (though changing the trivialization may change \( i \) by a multiple of 4).

Just as there is a relative grading on critical orbits, the framed instanton chain complex \( \mathcal{CI} \) has a relative grading. As a relatively graded \( R \)-module, \( \mathcal{CI} \) is defined by

\[
\mathcal{CI}_q(Y,E,\pi;R) := \bigoplus_{\alpha \in \mathcal{E}_x} C^\text{gm}_q(\alpha;R)[i(\alpha)] \otimes_{R[\mathbb{Z}/2]} R[\Lambda(\alpha)].
\]

Here \( \Lambda(\alpha) \) is the 2-element orientation set discussed in Theorem 5.8 (6). The group \( \mathbb{Z}/2 \) acts on \( \Lambda(\alpha) \) by swapping the two elements, and on \( C^\text{gm}_q(\alpha;R) \) by negation.

If \( \sigma_1 : P \to \alpha \) and \( \sigma_2 : Q \to \beta \) are basic chains, the relative grading \( |\sigma_1| - |\sigma_2| \in \mathbb{Z}/8 \) is given as

\[
|\sigma_1| - |\sigma_2| = (\dim P + i(\alpha)) - (\dim Q - i(\beta)) \mod 8.
\]

Because \( i(\alpha) - i(\beta) = \text{gr}(\rho, \alpha) - \text{gr}(\rho, \beta) \), this simplifies using the additivity formula of Theorem 5.8 (2) to

\[
|\sigma_1| - |\sigma_2| = \dim P - \dim Q + \text{gr}(\alpha, \beta).
\]

This further implies \((|\sigma_1| - |\sigma_2|) + (|\sigma_2| - |\sigma_3|) = |\sigma_1| - |\sigma_3|\), as expected.

The differential is given (termwise on basic chains \( \sigma : P \to \alpha \)) by

\[
\partial_{\mathcal{CI}} \sigma = \partial \sigma + \sum_{\beta \in \mathcal{E}_x} (-1)^{\dim \sigma + \dim \alpha} \sigma \times_{e_-} \mathcal{M}_{E,z,\pi}(\alpha, \beta).
\]

Here \( \partial \sigma \) denotes the differential inside \( C^\text{gm}_\ast(\alpha) \) and the chain \( \sigma \times_{e_-} \mathcal{M} \) is defined as in Lemma 5.6; we orient the moduli spaces using choices of elements of \( \Lambda(\alpha) \) and \( \Lambda(\beta) \). Because swapping elements of these orientation sets negates the orientation on the fiber product, the map \( C^\text{gm}_\ast \otimes R[\Lambda(\alpha)] \to C^\text{gm}_\ast \otimes R[\Lambda(\beta)] \) descends to the quotient under the two \( \mathbb{Z}/2 \) actions. We drop the orientation sets from notation as much as is reasonably possible.

The index demand on the sum ensures that all of the moduli spaces \( \mathcal{M} \) appearing in the sum are compact oriented topological manifolds with corners and a smooth structure on each stratum (as a consequence of Theorem 5.8 (7)). The reason we can do this without concern is that the fiber product with any larger-dimensional moduli spaces is identically zero — we can consider our sum as being a formal truncation of what “should be” the instanton differential \( \partial_{\mathcal{CI}} \), where we throw out moduli spaces of dimension too large to contribute. (This happens precisely when the degree of the fiber product map is larger than \( \dim \beta + 1 \); because \( SO(3) \)-orbits have dimension at most 3, this is true when \( \text{gr}_z(\alpha, \beta) - 1 > 4 \).

First observe that the differential decreases the relative grading by one: if \( \sigma : P \to \alpha \) is a basic chain, taking the fiber product gives a basic chain \( \sigma' : P \times_{e_-} \mathcal{M}_z \to \beta \). The relative grading between these is

\[
|\sigma| - |\sigma'| = \dim P - \dim (P \times_{e_-} \mathcal{M}_z) + \text{gr}_z(\alpha, \beta).
\]
Lemma 5.6 tells us that the dimension of the fiber product is \( \dim P + \dim \overline{\mathcal{M}}_z - \dim \alpha \). Combining these with \( \dim \overline{\mathcal{M}}_z = \text{gr}_z(\alpha, \beta) + \dim \alpha - 1 \), we see that the relative grading

\[
|\sigma| - |\sigma'| = \dim P - (\dim P + \text{gr}_z(\alpha, \beta) - 1) + \text{gr}_z(\alpha, \beta) = 1.
\]

**Lemma 5.11.** \( \widetilde{C}^I(Y, E, \pi; R) \) is a chain complex. That is, \( \partial^2 \overline{\mathcal{M}} = 0 \). Furthermore, the right action of \( C^g_\text{sm}(SO(3); R) \) on \( \widetilde{C}^I \), acting on each \( \otimes C^g_\text{sm}(\alpha; R) \) on the right (induced by the right action of \( SO(3) \) on \( \alpha \)).

**Proof.** It is clear that, for a basic chain \( \sigma : P \to \alpha \),

\[
\partial^2 \overline{\mathcal{M}} = \partial^2 \sigma + \sum_{\beta \in \mathcal{E}_+} (-1)^{\dim \sigma + \dim \alpha - 1} (\partial \sigma) \times_{e-} \overline{\mathcal{M}}_{E,z,\pi}(\alpha, \beta)
\]

\[
+ \sum_{\beta \in \mathcal{E}_+} (-1)^{\dim \sigma + \dim \alpha} \partial (\sigma \times_{e-} \overline{\mathcal{M}}_{E,z,\pi}(\alpha, \beta))
\]

\[
+ \sum_{\gamma, \beta \in \mathcal{E}_+} (-1)^{\dim \sigma + \dim \alpha + \text{gr}_z(\alpha, \gamma) - 1} \sigma \times_{e-} (\overline{\mathcal{M}}_{E,z,\pi}(\alpha, \gamma) \times_\gamma \overline{\mathcal{M}}_{E,z,\pi}(\gamma, \beta))
\]

First, \( \partial^2 \sigma = 0 \) because \( C^g_\text{sm}(\alpha; R) \) is a chain complex. Using the decomposition

\[
\partial (\sigma \times_{e-} \overline{\mathcal{M}}) = \partial \sigma \times_{e-} \overline{\mathcal{M}} + (-1)^{\dim \sigma + \dim \alpha} \sigma \times_{e-} \overline{\mathcal{M}}
\]

of Lemma 5.6, and cancelling the \( (\partial \sigma) \times_{e-} \overline{\mathcal{M}} \) terms, this reduces to

\[
\partial^2 \overline{\mathcal{M}} = \sum_{\beta \in \mathcal{E}_+} \sigma \times_{e-} (\partial \overline{\mathcal{M}}_{E,z,\pi}(\alpha, \beta))
\]

\[
+ \sum_{\gamma, \beta \in \mathcal{E}_+} (-1)^{\text{gr}_z(\alpha, \gamma) - 1} \sigma \times_{e-} (\overline{\mathcal{M}}_{E,z,\pi}(\alpha, \gamma) \times_\gamma \overline{\mathcal{M}}_{E,z,\pi}(\gamma, \beta))
\]

The terms in the second sum can only be nonzero when \( \text{gr}_z(\alpha, \gamma) + \text{gr}_w(\gamma, \beta) \leq 5 \), which is to say that \( \text{gr}_z(\alpha, \beta) \leq 5 \). After eliminating terms in the sum which vanish for dimension reasons, we’re left with

\[
\partial^2 \overline{\mathcal{M}} = \sum_{\beta \in \mathcal{E}_+} \sigma \times_{e-} (\partial \overline{\mathcal{M}}_{E,z,\pi}(\alpha, \beta))
\]

\[
+ \sum_{\gamma, \beta \in \mathcal{E}_+} (-1)^{\text{gr}_z(\alpha, \gamma) - 1} \sigma \times_{e-} (\overline{\mathcal{M}}_{E,z,\pi}(\alpha, \gamma) \times_\gamma \overline{\mathcal{M}}_{E,z,\pi}(\gamma, \beta))
\]

This is zero by the decomposition of the boundary given in Theorem 5.8 (9); here we need to use those moduli spaces with \( \text{gr}_z(\alpha, \beta) = 5 \).

That the action of \( C^g_\text{sm}(SO(3); R) \) makes \( \widetilde{C}^I \) into a \( C^g_\text{sm}(SO(3); R) \)-module is clear from the fact that each summand \( C^g_\text{sm}(\alpha; R) \) is, and that the fiber product map of Lemma 5.6 is a \( C^g_\text{sm}(SO(3); R) \)-module homomorphism. ■
Suppose we have a weakly admissible bundle $E$ over a cobordism $W$ from $(Y_1, E_1)$ to $(Y_2, E_2)$. To fix the orientations of these moduli spaces, we need to choose a homology orientation on $W$; we will suppress this from notation and just refer to “a cobordism $W$”.

**Lemma 5.12.** Suppose 
\[(W, E, \pi) : (Y_1, E_1, \pi_1) \to (Y_2, E_2, \pi_2)\]

is a cobordism where $E$ is a weakly admissible bundle and $\pi$ is a choice of metric and regular perturbation restricting to the $\pi_i$ on the ends. Furthermore suppose $W$ is equipped with an embedded path $\gamma$ between the basepoints $b_1$ and $b_2$. Then there is an induced $C^\infty_\text{reg}(\text{SO}(3); R)$-equivariant chain map

\[\tilde{\mathcal{F}}_{W, E, \pi, \gamma} : \tilde{\mathcal{I}}(Y_1, E_1, \pi_1; R) \to \tilde{\mathcal{I}}(Y_2, E_2, \pi_2; R).\]

**Proof.** The map is defined analogously to the differential itself; we only need to define its value on a basic chain $\sigma : P \to \alpha$, where $\alpha \subset \mathcal{C}_{\pi_1}$ is a critical orbit on $Y_1$. Here the value is

\[\tilde{\mathcal{F}}_{W, E, \pi, \gamma} (\sigma) = \sum_{\beta \in \mathcal{C}_{\pi_2}, z, \text{gr}_z (\alpha, \beta) \leq 4} (-1)^{\dim \sigma + \dim \alpha} \sigma \times_{e_+} \mathcal{M}_{E, z, \pi},\]

where the map to $\beta$ is defined using the positive endpoint map $e_+ : \mathcal{M}_{E, z, \pi} \to \beta$. Defining this endpoint map is the essential place the path $\gamma : R \to W$ is used. Here the sum is up to $\text{gr}_z (\alpha, \beta) = 4$ instead of 5, because for cobordisms $\dim \mathcal{M} = \text{gr} + \dim \alpha$, whereas there is an extra factor of $+1$ on the cylinder, account for the translation action.

That this is an equivariant chain map follows as in the previous lemma from the decomposition of the moduli space $\mathcal{M}_{E, z, \pi}$ of Theorem 5.9 (6).

These maps are essentially independent of the perturbation chosen on the cobordism, except in the case that $W$ is weakly admissible and $b^+(W) = 1$. (In the statement of the following, recall the definition of weakly admissible from Definition 4.14.)

**Lemma 5.13.** Given two regular perturbations $\pi_0, \pi_1$ on a weakly admissible cobordism $(W, E)$, there is a regular family of perturbations $\pi(t)$ parameterized by $[0, 1]$ interpolating between them, restricting to the same fixed perturbations on the ends for all $t$, or possibly some chain of them, so long as either $b^+(W) > 1$, or $E$ supports no reducible connections, or $b^+(W) = 0$ and the $\pi_i$ are chosen sufficiently small. This induces a $C^\infty_\text{reg}(\text{SO}(3); R)$-equivariant chain homotopy between

\[\tilde{\mathcal{F}}_{W, E, \pi_0, 1} \sim \tilde{\mathcal{F}}_{W, E, \pi_1, 1}.\]

If $(W, E)$ is weakly admissible $b^+(W) = 1$, then consider the set of regular perturbations on $W$, equipped with the equivalence relation $\pi_0 \sim \pi_1$ if there is a path $\pi_t$ between them so that no $\pi_t$ supports a reducible $\pi_t$-ASD connection. If $\pi_0 \sim \pi_1$, then we have a chain homotopy as above.

**Proof.** When $W$ supports no reducible connections, or has $b^+(W) > 1$, then the existence of such a regular family of perturbations is Theorem 5.10 (1), and when $b^+(W) = 0$ or $b^+(W) = 1$ it is points (2) or (3), respectively. That the moduli spaces of relevant dimension have compactifications to manifolds with corners and
a smooth structure on each stratum follows from Theorem 5.10 (5). The chain map is given on a basic chain $\sigma : P \to \alpha$ by sending

$$
\sigma \mapsto \sum_{\beta \in C_{Y_2}} (-1)^{\dim \sigma + \dim \alpha} \sigma \times_{e_{\eta}} \overline{M}_{E_i, \gamma, \pi(t)},
$$

That this is a chain homotopy from $\tilde{F}_{W, t_{\pi(t)}, \gamma}$ to $\tilde{F}_{W, t_{\pi(t)}, \gamma}$ follows from the decomposition of the moduli space given by Theorem 5.10 (6).

Some care should be taken about perturbations chosen from different $P_{E, L_i, \delta}$, where $L_i$ are two different (but not necessarily disjoint) finite sets of interior holonomy perturbations; Theorem 5.10 only gives us paths when the $\pi_i$ are chosen from the same $P_{E, L_i, \delta}$. However, Lemma 4.37 guarantees that if some

$$
\pi_i \in P_{E, L_i, \delta}
$$

are cut out transversely, one may find an $L_3$ so that $L_1 \cup L_3$ and $L_2 \cup L_3$ are both families of thickened loops with disjoint domain, and so that one may achieve transversality as well in $P_{E, L_i, \delta}$. In particular, thinking of elements of $L_i$ as elements of either $L_1 \cup L_3$ or $L_2 \cup L_3$, we first are able to find a regular path $\pi_i$ connecting $\pi_0$ to a perturbation whose interior part is supported in $V_{L_i}$; and then a path from this perturbation to $\pi_1$. (This is why we use the phrase “or possibly a chain” in the statement of the theorem.)

The main interesting point to make here is that these moduli spaces may be nonempty even though $\text{gr}_{L_i}(\alpha, \beta)$ is negative. This corresponds to the existence of $\pi(t)$-instantons for $t \in (0, 1)$ that are regularly cut out in the family — but because of the index they cannot possibly be cut out regularly (considered as an instanton for the single perturbation $\pi(t)$).

We need to relate maps arising from the geometric composition of cobordisms and the composition of chain maps.

**Lemma 5.14.** Suppose we have two weakly admissible cobordisms $(W_i, E_i, \pi_i)$ from $Y_i$ to $Y_{i+1}$ with regular perturbations $\pi_i \in P_{E, L_i, \delta}^{(4)}$. Further suppose each $W_i$ is equipped with a path $\gamma_i$, so that the positive end of $W_1$ agrees with the negative end of $W_2$, and the paths concatenate to form a smooth path $\gamma$ in the composite cobordism. Denote the composite $(W, E)$. Then there is a regular family of metrics and perturbations $\pi(t)$ so that $\pi(0)$ is a regular perturbation and metric on $W$, while $\pi(1)$ is the broken metric (and perturbation) corresponding to the obvious way to glue the two cobordisms. This induces a $C^m_\ast(SO(3); R)$-equivariant chain homotopy

$$
\tilde{F}_{W_2, E_2, t_{\pi_2}, \gamma_2} \circ \tilde{F}_{W_1, E_1, t_{\pi_1}, \gamma_1} \simeq \tilde{F}_{W, E, t_{\pi(0)}, \gamma}.
$$

**Proof.** Recall that the moduli spaces for the broken metric are by definition given by

$$
\overline{M}_{E, \tau, t_{\pi(1)}}(\alpha, \gamma) = \bigcup_{\beta \in C_{Y_2}} \overline{M}_{E_1, \tau_1, t_{\pi_1}} \times_{\beta} \overline{M}_{E_2, \tau_2, \gamma_2}.
$$

Thus the chain map given by a broken metric is precisely the composite of its component cobordisms.

Lemma 4.29 guarantees that the composite of weakly admissible cobordisms remains weakly admissible. In both cases, we may use Proposition 4.47 to construct
a metric and perturbation close to the broken metric and perturbation, and a path
to the broken metric, so that we achieve transversality. Observe that here, the
broken metric is assumed regular at all fully reducible points and to support \(SO(2)\)-
reducible instantons if and only if \(b^+(W) = 0\). In particular, if \(\pi_t\) is the path from
a broken metric to a non-broken metric, the same will hold for sufficiently small \(t\);
these were the constraints in Theorem 5.10 (2)-(3).

Then as before, we may define the induced map as

\[
\sigma \mapsto \sum_{\beta} (-1)^{\dim \sigma + \dim \alpha} \sigma \times_{e_-} \mathcal{M}_{E,\pi(t)}(\alpha, \beta).
\]

There are no additional difficulties in verifying this is a chain homotopy between
the map induced by the perturbation \(\pi(0)\) and the map induced by the broken
metric/perturbation \(\pi(1)\).

The chain maps \(\tilde{F}\) are invariant under diffeomorphisms of the cobordisms.

**Lemma 5.15.** Suppose \(W_1\) and \(W_2\) are cobordisms equipped with weakly admissible
bundles \(E_i\), regular perturbations \(\pi_i\), and paths \(\gamma_i\) between the basepoints of the
ends. Furthermore suppose that the ends of the two cobordisms \((W_i, E_i, \pi_i, \gamma_i)\) agree
(so that we think of them as cobordisms between the same manifolds). Suppose
\(\varphi : W_1 \to W_2\) is a diffeomorphism, constant on the ends and with \(\varphi(\gamma_1) = \gamma_2\),
and \(\Psi : \varphi^* E_2 \cong E_1\) an isomorphism of \(U(2)\)-bundles so that \(\Psi\) takes \(\varphi^* \pi_2\) to \(\pi_1\).
Furthermore suppose \(\varphi\) preserves the homology orientations on the \(W_i\). Then the
chain maps \(\tilde{F}_{W_i, E_i, \pi_i, \gamma_i}\) are identical.

**Proof.** This data induces a diffeomorphism

\[
\Psi : \mathcal{M}_{E_1, \pi_1}(\alpha, \beta) \to \mathcal{M}_{E_2, \pi_2}(\alpha, \beta).
\]

That this diffeomorphism preserves the endpoint maps follows because \(\varphi(\gamma_1) = \gamma_2\);
\(\Psi\) preserves orientation because \(\varphi\) preserves the homology orientation of the
cobordisms. Given any basic chain \(\sigma : P \to \alpha\), the map

\[
P \times_{e_-} \mathcal{M}_{E_1} \to P \times_{e_-} \mathcal{M}_{E_2}
\]

induced on the fiber product is a diffeomorphism preserving \(e_+ : P \times_{e_-} \mathcal{M}_{E_1} \to \beta\).
By definition, this means these two basic chains are isomorphic, hence equal in
\(C_{gm}^\infty(\beta; R)\). \(\blacksquare\)

**Remark 5.1.** In dimension 4, any two embedded paths which agree near the ends
and are homotopic relative to their boundary are in fact isotopic relative to their
boundary. The isotopy extension theorem then provides a diffeomorphism of \(W\)
fixing the boundary and taking \(\varphi(\gamma_2) = \gamma_1\). Hence from the previous lemma
\(\tilde{F}_{W, E, \pi, \gamma_1} = \tilde{F}_{W, E, \varphi^* \pi, \gamma_2}\). So homotopic paths induce the same map in homology.

Combining all of these, we see that we have a functor from a sort of cobordism
category to a homotopy category of chain complexes. We make this precise in the
following definition. Recall Definition 3.3 of the finite set \(\sigma(Y, E)\) of signature data
on \((Y, E)\).

There are two categories of cobordisms relevant to us. The first includes
the data of the perturbations; the second removes it as much as possible.
Definition 5.10. Let \( \text{Cob}^{U(2),w,\pi}_{3,b} \) denote the following category.\(^\text{13}\) The objects are closed oriented Riemannian 3-manifolds \( Y \), equipped with a choice of basepoint \( b \in Y \), a weakly admissible \( U(2) \)-bundle \( E \to Y \), and a regular perturbation \( \pi \in \mathcal{P}_{E,\delta} \) for some \( \delta > 0 \).

The morphisms in \( \text{Cob}^{U(2),w,\pi}_{3,b} \) will be formal concatenations of morphisms of the following type (called ‘unbroken cobordisms’).

An unbroken cobordism \( (Y_1, E_1, \pi, b_1) \to (Y_2, E_2, \pi_2, b_2) \) is given by the data of \((W, \tilde{E}, \pi, \gamma, \varphi, \psi)\). Here \( W \) is an oriented, homology oriented, Riemannian 4-manifold with two cylindrical ends, with chosen oriented isometries of these ends to \((-\infty, 0] \times Y_1 \) and \([0, \infty) \times Y_2 \), respectively, while \( \tilde{E} \) is a weakly admissible \( U(2) \)-bundle with specified isomorphisms to the pullback of \( E_i \) on the cylindrical ends. The data \( \pi \) is a regular perturbation in\(^\text{14} \) \( \mathcal{P}_{E,\delta} \) for some \( \delta > 0 \), restricting to the perturbations \( \pi_i \) on the ends. Lastly, \( \gamma : \mathbb{R} \to W \) is an embedded path which agrees with \((t, b_i)\) on the ends, following the specified isometries above.

A morphism in \( \text{Cob}^{U(2),w,\pi}_{3,b} \) is simply a finite sequence of morphisms \( W_i \) (for \( 1 \leq i \leq n \)) so that the target of \( W_i \) is the source of \( W_{i+1} \) when \( 1 \leq i < n \). These morphisms are thought of as Riemannian manifolds with ‘broken metric and perturbation’.

Let \( \text{Cob}^{U(2),w}_{3,b} \) denote the following category. Its objects are closed oriented 3-manifolds \( Y \) equipped with a basepoint \( b \in Y \), a weakly admissible \( U(2) \)-bundle \( E \to Y \), and a signature datum \( \sigma \in \sigma(Y,E) \). A morphism \( (Y_1, E_1, \sigma_1, b_1) \to (Y_2, E_2, \sigma_2, b_2) \) is given by the data of \((W, \tilde{E}, [\pi], \gamma, \varphi, \psi)\). Here \( W \) is a compact oriented, homology oriented, 4-manifold; \( \tilde{E} \) is a weakly admissible \( U(2) \)-bundle; \([\pi]\) is an equivalence class of small regular perturbation as in Lemma 5.13\(^\text{15} \) restricting to perturbations \( \pi_i \) on the ends with associated signature data \( \sigma_i = \sigma_1 \); and \( \gamma : [0, 1] \to W \) is an embedded path which is cylindrical in a collar of the boundary; \( \varphi : \partial W \to Y_1 \sqcup Y_2 \) is a diffeomorphism sending \( \gamma(i) \) to \( b_{i+1} \); \( \psi \) is an isomorphism \( E_1 \sqcup E_2 \to \varphi^*E \).

Two such cobordisms \( (W_i, \tilde{E_i}, [\pi_i], \gamma_i, \varphi_i) \) are to be considered the same morphism in \( \text{Cob}^{U(2),w}_{3,b} \) if there is a diffeomorphism \( \phi : W_1 \to W_2 \) with \( \varphi_2 \phi = \varphi_1 \), a bundle isomorphism \( \Psi : \tilde{E}_1 \to \phi^* \tilde{E}_2 \) taking \( \Psi^*[\pi_2] = [\pi_1] \) and a homotopy relative to the endpoints between \( \gamma_1 \) and \( \phi^{-1}(\gamma_2) \).

The target category is the following.

Definition 5.11. Let \( \text{Kom}_{\text{C}^\ast_{\text{SO}(3);R}}^{r,\mathbb{Z}/8} \) denote the category with objects \( \text{dg-modules over } \text{C}^\ast_{\text{SO}(3);R} \) equipped with a relative \( \mathbb{Z}/8 \) grading, and whose morphisms are relatively graded \( \text{C}^\ast_{\text{SO}(3);R} \)-equivariant chain maps. There is an equivalence relation on the morphisms in this category — \( \text{C}^\ast_{\text{SO}(3);R} \)-equivariant chain

\(^{13}\)The notation \( b \) is meant to stand for based, and \( w \) for weakly admissible; the \( \pi \) signifies that the perturbation data are included in an essential way.
\(^{14}\)Recall that this includes a smallness condition depends on \( E \), and in particular not every ‘small regular perturbation’ on the bounding 3-manifolds extends to a small regular perturbation on \( E \).
\(^{15}\)By definition, unless \( b^+(W) = 1 \), all perturbations are equivalent, and so this additional data is vacuous!
The following is essentially immediate from definitions, and we record it as a lemma.

**Lemma 5.16.** Sending \((Y, E, b, \pi)\) to the framed instanton complex \(\tilde{\mathcal{I}}(Y, E, \pi; R)\) defines a functor

\[
\text{Cob}^{U(2), w, \pi}_{g_{3,b}} \to \text{Kom}^{r, \mathbb{Z}/8}_{C_+SO(3); R'}
\]

**Proof.** Because morphisms \(\text{Cob}^{U(2), w, \pi}_{g_{3,b}}\) are given by formal concatenations of a given generating set of morphisms, this merely asserts that we have defined induced maps for that generating set. This was done in Lemma 5.12.

Finally, we see that framed instanton homology is functorial on the category of cobordisms without perturbation data.

**Theorem 5.17.** There is a functor

\[
\tilde{I} : \text{Cob}^{U(2), w}_{g_{3,b}} \to \text{Mod}^{r, \mathbb{Z}/8}_{H_+SO(3); R'}
\]

so that for any regular perturbation \(\pi\) on \((Y, E)\), the group \(\tilde{I}(Y, E, \pi; R)\) is canonically isomorphic to the homology groups \(\tilde{I}(Y, E, \pi; R)\) of the chain complexes defined above. Following this canonical isomorphism, the induced map of a cobordism \((W, E, [\pi])\) is given by the induced map on homology of the map of chain complexes defined above.

**Proof.** The previous lemmas showed that \(\tilde{\mathcal{I}}\) is a complex of the appropriate type, and showed that cobordisms \(W\) in this category — when equipped with a perturbation \(\pi\) — induce chain maps, as long as \(\pi\) is sufficiently small. An isomorphism between \((W, E, \gamma_1, \varphi, \psi)\) (here, taking the path \(\gamma_1\) diffeomorphically to \(\gamma_2\)) takes the moduli spaces (and their endpoint maps) of the first to those of the second, and hence the chain maps are the same on the nose. In particular, given regular perturbations \(\pi_1, \pi_2\) on \((Y, E)\) with signature data \(\sigma\), any path \(\pi\) between them gives a sufficiently small perturbation on \(\mathbb{R} \times Y\) (possibly with time-varying metric), and hence gives an induced map \(\tilde{F}_{\mathbb{R} \times Y, \pi \ast E, \pi_1, b}\); by Lemma 5.13, any two such chain maps are homotopic.

We will use this to pin down the group \(I(Y, E, \sigma)\) as something independent of the choice of metric and perturbation.

Write \(\rho_{12} : \tilde{I}(Y, E, \pi_1) \to \tilde{I}(Y, E, \pi_2)\) for the unique isomorphism between the groups associated to the perturbations \(\pi_i\) (abbreviated for the moment \(\tilde{I}(\pi_i)\)) defined by choosing a perturbation along \(\mathbb{R} \times Y\) as above. Set

\[
\tilde{I}(Y, E, \pi; R) := \left( \bigoplus_{\pi; \sigma_1 = \sigma} \tilde{I}(Y, E, \pi; R) \right) / \left( (x \in \tilde{I}(\pi_1) \sim \rho_{12} x \in \tilde{I}(\pi_2) \right).
\]
The map $\tilde{I}(Y,E,\pi;R) \to \tilde{I}(Y,E;R)$ induced by inclusion into the direct sum is an isomorphism for any regular perturbation $\pi$.

A cobordism with cylindrical ends $(W,E,\pi)$ restriced to $\pi_i$ on the appropriate ends induces a map on each summand $\tilde{F}_{W,E,\gamma} : \tilde{I}(Y_1,E_1,\pi_1) \to \tilde{I}(Y_2,E_2,\pi_2)$. The invariance result of Lemma 5.13 implies that this map is the same, independent of the choice of $\pi$ extending the $\pi_i$ on the ends; furthermore, Lemma 5.14 implies that these compose as expected with the canonical isomorphisms $\rho_{12}$ above. Therefore, so long as there exist $\pi_i$ on the ends that extend to a small regular perturbation over $(W,E)$, this defines an induced map $\tilde{F}_{W,E,\gamma} : \tilde{I}(Y_1,E_1,\sigma_1) \to \tilde{I}(Y_2,E_2,\sigma_2)$ by attaching cylindrical ends on each boundary component and choosing some sufficiently small regular perturbation.

Because the union of the spaces $\bigcup_{\delta \to 0} P_{E_i,\delta}$ of regular perturbations is dense near 0, we see that as long as $\delta$ is taken sufficiently small, there is some sufficiently small regular perturbation on the ends.

We saw that the chain maps were independent of the perturbation $\pi$ and homotopy class $\gamma$ up to chain homotopy in Lemma 5.13 (unless $b^+(W) = 1$, in which case there is an equivalence relation on perturbations), and that they compose as a functor should up to chain homotopy in Lemma 5.14. ■

Remark 5.2. There is a natural partial ordering on $\sigma(Y,E)$: an element $\sigma$ defines a function $f_\sigma : \text{Red}(Y,E) \to \mathbb{Z}_{\geq 0}$, and we say that $\sigma \leq \sigma'$ if $f_{\sigma'}(\alpha) \leq f_\sigma(\alpha)$ for all $\alpha$. The problem with attempting to show that the instanton homology groups are invariant of the signature datum is that the cylinder is only a weakly admissible cobordism $(Y,E,\sigma) \to (Y,E,\sigma')$ if $\sigma \leq \sigma'$. Any attempt to resolve this needs somehow to cope with reducible connections on the cylinder which cannot be made to be cut out transversely; perhaps the obstruction bundle technique, as described briefly at the end of [Don02] and used to great effect in [Tau84], is one such tool.

Remark 5.3. Suppose we work instead in the category $\text{Cob}^{SO(3),a}_{3,b}$ of admissible cobordisms. Then in fact, a much stronger invariance property of the maps $\tilde{F}_{W,E}$ is true: the homotopies between induced maps are homotopic, and so on.

A precise formulation of this statement uses the language of quasicategories (sometimes called $(\infty,1)$-categories); a good introduction to the language is [HLS16]. There is a quasicategory $\pi\text{Cob}^{U(2),a}_{3,b}$ of 3-manifolds equipped with perturbations, whose morphism spaces are simplicial sets whose $n$-simplices are $\Delta^n$-indexed admissible families of (possibly broken) perturbations on cobordisms. The framed instanton chain complex lifts to a functor from this quasicategory to the quasicategory of chain complexes, and furthermore the forgetful functor $\pi\text{Cob}^{U(2),a}_{3,b} \to \text{Cob}^{U(2),a}_{3,b}$ to the homotopy category is an equivalence of quasicategories. This is essentially the statement that the space of admissible perturbations on a given cobordism is contractible, and a version of this would be provided by a natural extension of Theorem 5.10 for simplices; the only obstructions to such an extension lies at reducible connections, which the admissibility assumption is used to avoid. Using this, it’s possible to find a chain-level version of $\tilde{C}\tilde{I}$ that is functorial under cobordisms on the nose, as opposed to up to homotopy, and well-defined up to essentially unique natural equivalence. In fact, this is done using the quasicategorical Kan extension; the definition of $I(Y,E,\sigma)$ above is a very special case of a Kan extension. We don’t see any need for this structure, and so leave the details to the interested reader.
Any extension of such a result to all weakly admissible \((W, \tilde{E})\) would require a substantially different notion of “regular family”, as there are obstructions to achieving transversality in the standard sense at reducible connections for families.

In this framework, the space of perturbations on a fixed 3-manifold form a quasi-category (actually, a Kan complex): 0-simplices are pairs \((Y, \tilde{E})\) of a weakly admissible 3-manifold and regular perturbation \(\pi\), and an \(n\)-simplex starting at \((Y, \tilde{E}, \pi_1)\) and ending at \((Y, \tilde{E}, \pi_2)\) is a regular family of perturbations on \(\mathbb{R} \times Y\) parameterized by the \((n - 1)\)-dimensional simplex, so that for all perturbations in this family, the restriction of \(\pi\) to the the corresponding end is the fixed perturbation \(\pi_i\). Then 5.10 (1) guarantees that the space of perturbations with fixed signature perturbation is contractible, and that this \(\infty\)-category is equivalent to the poset whose elements are signature data on \(Y\), and there is a morphism \(\sigma \to \sigma' \iff \sigma \leq \sigma'\).

5.3. The index filtration. Recall that the relatively \(\mathbb{Z}/8\)-graded complex

\[
\hat{CI}(Y, E, \pi; R)
\]

is defined as a graded \(C^\text{gm}_g(SO(3); R)\)-module to be

\[
\bigoplus_{\alpha \in \mathbb{E}_\pi} C^\text{gm}_g(\alpha; R)[i(\alpha)],
\]

where \(i(\alpha) = \text{gr}_z(\rho, \alpha) \in \mathbb{Z}/8\); this grading is well-defined up to a translation (arising from choosing a different base orbit \(\rho\)), and thus defines a relative grading on \(\hat{CI}\).

Here we drop the orientation sets \(\Lambda(\alpha)\) from notation; if so desired, there is no harm in making a choice of orientation for each critical orbit.

Now if \(z\) is a homotopy class of path from \(\rho\) to \(\alpha\), write

\[
i(\alpha, z) = \text{gr}_z(\rho, \alpha) \in \mathbb{Z}/8.
\]

There is a \(\mathbb{Z}\)-graded complex \(\hat{CI}_{\text{unr}}(Y, E, \pi; R)\) given as

\[
\bigoplus_{\alpha \in \mathbb{E}_\pi} C^\text{gm}_g(\alpha; R)[i(\alpha, z)].
\]

If \((\alpha_i, z_i)\) are two such labelled critical orbits, there is a unique homotopy class of path \(w\) from \(\alpha_1\) to \(\alpha_2\) so that \(z_2 \simeq w * z_1\). Then using the additivity of \(\text{gr}\), we see that the degree difference between \(\sigma_1 : P \to \alpha_i\) is given as

\[
|\sigma_1| - |\sigma_2| = (\dim P_1 - i(\alpha_1, z_1)) - (\dim P_2 - i(\alpha_2, z_2))
\]

\[
= \dim P_1 - \dim P_2 + (\text{gr}_{z_2}(\rho, \alpha_2) - \text{gr}_{z_1}(\rho, \alpha_1))
\]

\[
= \dim P_1 - \dim P_2 + \text{gr}_w(\alpha_1, \alpha_2).
\]

We thus see that the grading induced by choosing \(\rho\) as a basepoint induces the expected relative grading.

We write a basic chain \(\sigma : P \to \alpha\) corresponding to a critical orbit \(\alpha\) labelled by \(z\) as \((\sigma, z)\). The differential of \(\sigma : P \to (\alpha, z)\) is given by

\[
\partial \sigma(z) = (\partial \sigma, z) + \sum_{\beta \in \mathbb{E}_\pi, \begin{array}{c}w \in \pi_1(\tilde{B}_E, \alpha, \beta) \\ \text{gr}_z(\alpha, \beta) \leq 5 \end{array}} (\sigma \times_{\text{ex.}} \mathcal{M}_{\tilde{E}, w, \pi}(\alpha, \beta), z * w).
\]

Each critical orbit \(\alpha\) has a unique homotopy class of path \(1 \in \pi_1(\tilde{B}_E, \alpha)\) with \(\text{gr}_1(\alpha, \alpha) = 8\); the element 1 is a cyclic generator of the fundamental group, which
is isomorphic to $\mathbb{Z}$. The complex $\widehat{CI}_{\text{uni}}(Y, E, \pi; R)$ has a periodicity isomorphism, sending the component $C^g_{\ast}(\alpha; R)$ labelled by $z$ identically to the component labelled by $z+1$. That this commutes with the differential above is only the statement $w+1 = 1+w$ for any path $w$.

The $8\mathbb{Z}$-periodic, $\mathbb{Z}$-graded complex $\widehat{CI}_{\text{uni}}(Y, E, \pi; R)$ is called the unrolled complex of $\widehat{CI}(Y, E, \pi; R)$ in Section A.8; the quotient by the above periodicity isomorphism just forgets about the labelling by $z$.

$\widehat{CI}_{\text{uni}}(Y, E, \pi; R)$ carries an honest filtration by index. We write

$$F_z\widehat{CI}_{\text{uni}}(Y, E, \pi; R) = \bigoplus_{\substack{\alpha \in \mathcal{E}_x \cr z \in \pi_1(B_E, \rho, \alpha) \cr i(\alpha, z) \leq s}} C^g_{\ast}(\alpha; R)[i(\alpha, z)].$$

Because a nonempty moduli space $\overline{M}_{E,w,\pi}(\alpha, \beta)$ can only exist if

$$\text{gr}_z(\rho, \alpha) > \text{gr}_{z+w}(\rho, \beta),$$

the differential decreases index and hence preserves the filtration. Furthermore, because the filtration is defined by taking a direct sum of $C^g_{\ast}(\alpha; R)$ for $(\alpha, z)$ satisfying the index bound, this is a filtration by $C^g_{\ast}(SO(3); R)$-modules.

Because $\text{gr}_{z+1}(\rho, \alpha) = \text{gr}_z(\rho, \alpha) + 8$, this is a periodic filtration in the sense of Definition A.9. The associated graded module is

$$\text{gr}_p\widehat{CI}_{\text{uni}}(Y, E, \pi; R) \cong \bigoplus_{\substack{\alpha \in \mathcal{E}_x \cr i(\alpha) = p \mod 8}} C^g_{\ast}(\alpha; R)[p].$$

In particular, because $C^g_{\ast}(X; R)$ is supported in degrees at most $\dim X + 1$, and all of our components $\alpha$ are $SO(3)$-orbits, each associated graded piece is bounded, supported in degrees $[0, 4]$.

Now note that the instanton differential $\partial_{CI}: \widehat{CI}_{\text{uni}} \to \widehat{CI}_{\text{uni}}$ decomposes as $\partial_{CI} = \partial_0 + \partial_1 + \cdots + \partial_5$, where $\partial_0(\sigma, z) = (\partial\sigma, z)$ and for $k > 0$,

$$\partial_k(\sigma, z) = \sum_{\substack{\beta \in \mathcal{E}_x \cr w \in \pi_1(B_E, \rho, \beta) \cr \text{gr}_w(\alpha, \beta) = k}} (\sigma \times_{\mathcal{E}_x} \overline{M}_{E,w,\pi}(\alpha, \beta), z * w).$$

This is the component of the differential that decreases filtration by $k$, and is given by those fiber product maps with moduli spaces which increase dimension by $k-1$. The decomposition of $\partial_{CI}$ into the $\partial_k$ ends at $\partial_3$ because fiber products with moduli spaces with $\text{gr}_w(\alpha, \beta) > 5$ are identically zero.

Furthermore, observe that $\dim \overline{M}_z(\alpha, \beta) = \dim \alpha + \text{gr}_z(\alpha, \beta) - 1$, while for any nonempty moduli space we have $\dim \overline{M}_z(\alpha, \beta) \geq 3$ because by Theorem 5.8 (8), the $SO(3)$-action is free. Therefore, if $\overline{M}_z(\alpha, \beta)$ is nonempty and $\alpha$ is an $SO(2)$-reducible, necessarily $\text{gr}_z(\alpha, \beta) > 1$; if $\alpha$ is fully reducible, then necessarily $\text{gr}_z(\alpha, \beta) > 3$. In particular, $\partial_1$ is identically zero on reducible orbits; in addition both $\partial_2$ and $\partial_4$ vanish on fully reducible orbits.

Therefore, we have the following, which a special case of Theorem A.29, item (3).

**Theorem 5.18.** Let $(Y, E)$ be a weakly admissible bundle and $\pi$ a regular perturbation. There is a $(\mathbb{Z}/8, \mathbb{Z})$-bigraded spectral sequence of $H_w(SO(3); R)$-modules so
that the \( \mathbb{Z}/8 \)-grading is relative, whose \( E^1 \) page is
\[
E^1_{p,q} = \bigoplus_{\alpha \in \mathcal{E}} H_q(\alpha; R)
\]
this spectral sequence converges to \( E^\infty_{p,q} \cong \text{gr}_p \hat{I}_{p+q}(Y,E,\pi; R) \), the \( p \)th component of the associated graded vector space of \( \hat{I}(Y,E,\pi; R)_{p+q} \).

On the unrolled \((\mathbb{Z}, \mathbb{Z})\)-graded spectral sequence, we have
\[
E^\infty_{p,q} \cong \text{gr}_p \hat{I}^\text{unr}_{p+q}.
\]
For a class \([x] \in E^r\) which has a representative \( x \in \hat{C}I \) with \( d_i x = 0 \) for \( i < r \), the spectral sequence differential \( d_r[x] \) may be identified with \([\hat{\partial} x]\). In particular, the differential \( d_1 \) on \( E^1 \) is only nonzero on the irreducible components, where it is given by counting points in 0-dimensional moduli spaces \( \overline{\mathcal{M}}_{E,z,\pi}(\alpha, \beta)/SO(3) \) (note that \( \beta \) need not be irreducible).

The index spectral sequence for \( \hat{I} \) degenerates on the \( E^5 \) page for dimension reasons.

Proof. The only conditions we need to check to apply Theorem A.29 is that the algebra \( A = C^\text{gm}_*(SO(3); R) \) and each associated graded module \( \text{gr}_p \hat{C}I_{\text{unr}} \) (for each fixed \( p \)) are bounded in degree. These follow by definition: \( C^\text{gm}_*(X; R) \) is supported in degrees \([0, \dim X + 1]\), and \( \text{gr}_p \hat{C}I_{\text{unr}} \) is a direct sum of \( SO(3) \)-orbits with \( \iota(\alpha) = p \), so is supported in degrees \([p, p + 4]\).

That the differentials on elements \([x]\) with lifts that have \( d_i x = 0 \) for \( i < r \) are induced by \( \hat{\partial} \) is an elementary diagram chase. In fact, that this is true holds more generally: Wall’s notion of a multicomplex (see Definition A.4) is a complex so that the differential splits nicely into components \( d_r \) that decrease the filtration by \( r \), and \( C \) is equipped with the induced differential
\[
d = d_0 + d_1 + d_2 + \cdots
\]
Observe that our choice of \( \hat{\partial}_k \) is precisely the component of the differential that decreases the filtration level of \( \hat{C}I \) by \( k \).

The invariants \( RE^\infty \) and \( W \) of Proposition A.3 are both zero for this spectral sequence, because it degenerates at the 5th page.

By the second part of Proposition A.3, and Definition A.3 of strong convergence, we see that \( E^\infty(\hat{C}I) = \text{gr} \hat{I} \).

As an immediate corollary, we may define and calculate the Euler characteristic \( \chi(\hat{I}(Y,E,\sigma)) \), where \( \sigma \) is a signature datum.

**Corollary 5.19.** Let \((Y,E,\sigma)\) be a 3-manifold equipped with a weakly admissible bundle and signature datum \( \sigma \). Then \( \hat{I}(Y,E,\sigma) \) is finitely generated, and
\[
\chi(\hat{I}(Y,E,\sigma)) = |H^2(Y; \mathbb{Z})| = |H_1(Y; \mathbb{Z})|. \]
If \( b_1 Y > 0 \), we interpret the right-hand-side to be zero.

This follows because
\[
\chi(\hat{I}(Y,E,\sigma)) = \chi(\text{gr} \hat{I}(Y,E,\sigma)) \]
because \( \chi(SO(3)) = 0 \), the only contribution to this Euler characteristic is from the reducibles. Then the result follows from the enumeration of reducibles given by Proposition 1.4, which have even index by the calculation of Proposition 4.33.
(and so all contribute positively to the Euler characteristic sum). The isomorphism $H^2(Y; \mathbb{Z}) = H_1(Y; \mathbb{Z})$ is Poincaré duality.

**Remark 5.4.** In general it’s rare for a multicomplex that a class $[x] \in E^r$ has a representative with all $d_r x = 0$ for $i < r$, but is relatively common in the $I$ spectral sequence. The discussion simplifies substantially if $\frac{1}{2} \in R$, so we make this assumption; then we know that $H_*(SO(3); R) = R \oplus R[3]$ and $H_*(SO(3)/SO(2); R) = H_*(S^2; R) = R \oplus R[2]$.

Choosing an arbitrary class $\rho \in \mathcal{C}_x$, we write the absolute grading $i(\alpha) = \text{gr}_\rho \alpha$.

We define

$$C_p^{\text{irr}} = \bigoplus_{\substack{\alpha \in \mathcal{C}_x \\ i(\alpha) = p}} R$$

$$C_p^{SO(2)} = \bigoplus_{\substack{\alpha \in \mathcal{C}_x \\ \gamma_\alpha \cong SO(2) \\ i(\alpha) = p}} R$$

$$C_p^{\theta} = \bigoplus_{\substack{\alpha \in \mathcal{C}_x \\ \gamma_\alpha = SO(3) \\ i(\alpha) = p}} R$$

With this notation, the $E^1$ page is given as

$$E^1_{p,q} = \begin{cases} 
C_p^{\text{irr}} \oplus C_p^{SO(2)} \oplus C_p^{\theta} & q = 0 \\
C_p^{SO(2)} & q = 2 \\
C_p^{\text{irr}} & q = 3 
\end{cases}$$

The $E^1$ differential on $C_p^{\text{irr}}$ counts $SO(3)$-orbits in 3-dimensional moduli spaces. For $\alpha \in E^1_{p,0}$, the first page differential is given by counting points in unframed moduli spaces with $\text{gr}(\alpha, \beta) = 1$;

$$d_1 \alpha = \sum_{\beta, \text{gr}(\alpha, \beta) = 1} (\# \mathcal{M}(\alpha, \beta)/SO(3)) \beta,$$

and $d_1 \alpha$ is zero if $\alpha$ is reducible.

Because the $E^1$ differential vanishes on the $SO(2)$-irreducibles, we see that the differential $d_2$ on $[\alpha] \in E^2_{p,2}$, for $\alpha \in C_p^{SO(2)} \subset E^1_{p,2}$, is given by a similar formula as above: counting, for each $\beta$ with $\text{gr}(\alpha, \beta) = 2$, the (signed) number of $SO(3)$-orbits of trajectories between them: $n(\alpha, \beta) = \# \mathcal{M}(\alpha, \beta)/SO(3)$ for $\beta$ irreducible; so

$$d_2[\alpha] = \sum_{\beta, \text{gr}(\alpha, \beta) = 2} n(\alpha, \beta) [\beta]_{p-2,3}.$$

d_2 vanishes everywhere else for dimension reasons. Finally, $d_4$ on $[\theta] \in E^4_{0,0}$ (when $Y$ is a rational homology sphere and so has trivial connections, and $0 \in V$ makes sense) counts $SO(3)$-orbits $\mathcal{M}(\theta, \beta)/SO(3)$ between an irreducible and an irreducible with $\text{gr}(\theta, \beta) = 4$. (This is the map Donaldson calls $D_2$ in [Don02, Section 7.1].)
Once we develop the equivalence to the Donaldson model DCI in Section 6.2, the terms in this spectral sequence become much easier to calculate, and visible at the chain level.

### 5.4. Four flavors of instanton homology.

We will soon apply Theorem A.29 to define three additional flavors of equivariant instanton homology, \( I^+(Y), I^-(Y), \) and \( I^\pm(Y) \). To explain what these different flavors represent, it is useful to analogize to the situation of finite \( G \)-CW complexes. To correctly state certain stabilization phenomena, we must ensure everything in sight has a basepoint. Let \( X_+ \) denote the disjoint union of a \( G \)-space \( X \) with a disjoint fixed basepoint; the smash product of pointed spaces is \( X \wedge Y = (X \times Y)/S^1 \). A pointed \( G \)-CW complex is a \( G \)-space assembled from cells of the form

\[
(G/H)_+ \wedge D^i = (G/H \times D^i)/(G/H \times *),
\]

where the basepoint is in the boundary sphere and attaching maps preserve the basepoint. Given a pointed \( G \)-CW complex \( X \), its most well-known homological invariants are its nonequivariant homology \( H_\ast(X) \) and its Borel equivariant homology

\[
H^G_\ast(X) = H_\ast((X \wedge EG)/G),
\]

both taken relative to a basepoint (equivalently, taking reduced homology). These are both invariants of \( X \) up to equivariant homotopy equivalence. In fact, if \( \Sigma X \) denotes the reduced suspension

\[
X \wedge S^1 = (X \times [0, 1])/(\ast \times [0, 1], X \times \{0\}, X \times \{1\})
\]
of a pointed \( G \)-space, because \( H_{\ast+1}(\Sigma X) \cong H_\ast(X) \) and \( H^G_{\ast+1}(\Sigma X) \cong H^G_\ast(X) \), these are invariants of \( X \) up to stable equivariant homotopy equivalence.

In the stable homotopy category of finite \( G \)-CW complexes, where the suspension operator \( \Sigma \) has an inverse and there is a sphere \( S^n \) for every \( n \in \mathbb{Z} \), there is a contravariantly functorial Spanier-Whitehead duality operator \( D_G \) sending

\[
D_G((G/H)_+ \wedge S^n) \cong (G/H)_+ \wedge S^{-n}.
\]

Explicitly, if \( V \cong \mathbb{R}^N \) is a \( G \)-representation, and \( X \hookrightarrow S^{V+1} \) is an equivariant embedding into the one-point compactification of \( V \oplus \mathbb{R} \), then \( D_G X \) is the desuspension \( \Sigma^{-V}(S^{V+1} \setminus X) \).

In the nonequivariant case, the remarkable Alexander duality theorem identifies \( H^{-\ast}(DX) \cong H_\ast(X) \), by

\[
H^{-\ast}(DX) = H^{-\ast}(\Sigma^{-n}(S^{n+1} \setminus X)) = H^{n-\ast}(S^{n+1} \setminus X) \cong H_\ast(X).
\]

Though the double dual involved in writing \( H^{-\ast}(DX) \) outputs a homology theory (a composition of two contravariant functors is a covariant functor), Alexander duality identifies it as the homology theory we started with, so this operation doesn’t produce anything new. Note that we used negatively graded cohomology, \( H^{-\ast} \), so that the gradings in this formula would work out.

However, in the equivariant case, something altogether new happens: applying the above formula to orbits,

\[
H^{-\ast}_G(D_G((G/H)_+)) = H^{-\ast}_G((G/H)_+) = H^{-\ast}_G(BH),
\]

which is not \( H^G_\ast((G/H)_+) = H_\ast(BH) \), nor some degree shift of it: \( H^{-\ast}(BH) \) is concentrated in nonpositive degrees, and \( H_\ast(BH) \) is concentrated in nonnegative degrees. Thus \( H^{-\ast}_G(D_G X) \) is an altogether different homology theory, which
deserves to be called coBorel homology. [Man16] writes this \( cH^G_\bullet(X) \). Following [Jon87], and because \( cH^G_\bullet(X) \) is usually supported in the negative direction, we prefer to denote it \( H^-_G(X) \). Correspondingly, we write \( H^+_G(X) = H^G_\bullet X \), as this is usually supported in the positive direction.

Using the cap product of cohomology and homology, and pulling back cohomology classes from \( BG \), we find that \( H^+_\pi(X) \) is a module over \( H^{-\pi}(BG) \) (note the negative grading, as cohomology classes contract against homology, decreasing degree). Using the cup product instead, \( H^-_\pi(X) \) is also a module over \( H^{-\pi}(BG) \).

If \( G \) is finite or connected, is a homomorphism relating these two homology theories, the norm map: \( N_G : H^G_\bullet(X)[\dim G] \rightarrow H^-_G(X) \). Note the degree shift by \( \dim G \); we think of the norm map as a sort of averaging operator. For a general compact Lie group there is a twist involving the character \( \pi_0 G \rightarrow \pm 1 \) given by the determinant of the adjoint representation (see Theorem A.22).

There is a final equivariant homology theory, Tate homology, written \( H^\infty_\bullet(X) \) and fitting into a long exact sequence

\[
\cdots \rightarrow H^\ast-\dim G(X) \xrightarrow{N_G} H^-_\pi(X) \rightarrow H^\infty_\pi(X) \rightarrow H^\ast-\dim G-1(X) \rightarrow \cdots
\]

Remarkably, the norm map is an \( H^{-\pi}(BG) \)-module homomorphism, and Tate homology also has the structure of an \( H^{-\pi}(BG) \)-module for which all maps in the above exact sequence are module homomorphisms.

The other essential property of Tate homology is that \( H^\infty_{\ast\pi}(G) = 0 \), so that Tate homology of a finite \( G \)-CW complex can be calculated (essentially) from its subcomplex of points with nontrivial stabilizer. This is extremely useful for calculation, as we will see later in the context of \( I^\pi \).

The following table summarizes the analogies between equivariant homology of \( G \)-spaces and the various Floer homologies of 3-manifolds. The monopole Floer homology is defined to be a sort of \( S^1 \)-equivariant Floer homology, but the \( S^1 \) symmetry is less visible in Heegaard Floer theory. In any case, the monopole Floer and Heegaard Floer homology groups are modules over \( H^{-\ast}(BS^1; \mathbb{Z}) = \mathbb{Z}[U] \), where \(|U| = -2\).

<table>
<thead>
<tr>
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<td>( \hat{HM}_\ast(Y) )</td>
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<tr>
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<tr>
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</table>

In our setting, we have a functorial dg-module with dg-algebra action

\[
\widehat{CI}(Y, \hat{E}, \pi; R) \simeq C^\text{gm}_\ast(SO(3); R),
\]

acting from the right because \( \hat{B}_E \) carries a right \( SO(3) \)-action; these are right orbits. This is a reasonable notion of ‘chain complex with action of \( SO(3) \)’, and we would like to obtain invariants that behave like equivariant homology of a \( G \)-space \( X \). The majority of the appendix develops chain complexes \( C^\ast(A; M) \) whose homology gives us equivariant homology theories for \( \mathbb{Z} \)-graded dg-modules over a dg-algebra.

The definitions for \( \mathbb{Z}/8 \)-graded complexes are more delicate, and carried out in Section A.8. Here we take the point of view that the periodic filtration on the
module $M$ should give rise to a periodic filtration on $C^\bullet_{SO(3)}(M)$, and to do so we require that $\text{gr}_p M$ is bounded. Our complexes $\widehat{CI}$ are equipped with a periodic filtration, and each $\text{gr}_p \widehat{C}_\text{I unr}(Y, \tilde{E}, \pi; R)$ is bounded. In particular, Theorem A.29 is ready made for us to apply to $\widehat{CI}$.

**Theorem 5.20.** Suppose $R$ is a PID. There is a dg-algebra $\mathcal{C}^- \left( C^\text{gm}_{\cdot}(SO(3); R) \right)$ whose homology algebra is graded isomorphic to $H^{-\ast}(BSO(3); R)$. We denote this dg-algebra by $\mathcal{C}^\ast$.

There are functors

$$CI^\ast : \text{Cob}^U(2, w, \pi) \to \text{Ho} \left( \mathcal{C}^- \text{Mod}^\mathbb{Z}/8 \right),$$

where $\bullet \in \{ +, -, \infty \}$, given by applying the constructions

$$C^\ast \left( C^\text{gm}_{\cdot}(SO(3); R), \widehat{C}_\text{I unr}(Y, \tilde{E}, \pi; R) \right)$$

of Theorem A.29 for periodically filtered modules with finite support on each associated graded piece. After taking homology, we then have functors

$$I^\ast : \text{Cob}^U(2, w) \to \text{Mod}^\mathbb{Z}/8(BSO(3); R).$$

These fit into an exact triangle

$$\cdots \to I^+(Y, E; R) \xrightarrow{[3]} I^-(Y, E; R) \to I^X(Y, E; R) \xrightarrow{[-4]} I^+(Y, E; R) \to \cdots$$

where all arrows are $H^{-\ast}(BSO(3); R)$-module maps.

Given any other bounded, non-negatively graded dg-algebra $A$ and right $A$-module $M$, equipped with a periodic filtration each of whose associated graded groups $\text{gr}_p M$ is bounded, there are $A$-homology groups $H^\ast_A(M)$. If there is an algebra map which is a quasi-isomorphism

$$f : C^\text{gm}_{\cdot}(SO(3); R) \to A,$$

and an $f$-equivariant filtered module map $g : \widehat{C}_\text{I}(Y, \tilde{E}, \pi; R) \to M$ whose induced map on associated graded complexes is a quasi-isomorphism, then there is an induced canonical isomorphism $I^\ast \cong H^\ast_A(M)$, equivariant under the actions of $H^\ast_A(R)$ under the induced isomorphism $H^{-\ast}(BSO(3); R) \cong H^\ast_A(R)$.

We also have spectral sequence for the equivariant homology groups. As with the non-equivariant homology, we state this in slightly more detail than in Theorem A.29, point (4).

**Theorem 5.21.** For each of $\bullet \in \{ +, -, \infty \}$, if $(Y, E)$ is a weakly admissible bundle equipped with regular perturbation $\pi$, there is a $(\mathbb{Z}/8, \mathbb{Z})$-bigraded spectral sequence of $H^{-\ast}(BSO(3); R)$-modules from

$$E^1_{p, q} = \bigoplus_{\alpha \in \xi^\ast, \text{gr}(\alpha) = p} H^\bullet_{SO(3)}(\alpha; R)$$

whose target is the associated equivariant instanton homology $I^\ast(Y, E, \pi; R)$.

For $I^-$, we have an isomorphism of the unrolled $(\mathbb{Z}, \mathbb{Z})$-bigraded spectral sequence $E^\infty_{p, q} \cong \text{gr}_p I^\text{ unr}_{p, q}$. For any of the theories, a filtered chain map $CI^\ast(Y, E, \pi) \to CI^\ast(Y', E', \pi')$ which induces an isomorphism on any finite page $E^r$ induces an isomorphism on $I^\ast$. 
Proof. The existence of this spectral sequence, and the fact that we may detect quasi-isomorphisms by isomorphisms on a finite page $E^r$, is precisely Theorem A.29, point (4). As for the specific calculation of the $E^1$ page, our associated graded complex

$$\text{gr}_p \widetilde{CI}_{\text{unif}}(Y, E, \pi; R) = \bigoplus_{\alpha \in \mathbb{C}} C_{\text{gm}}(\alpha; R)$$

is a bounded $C_{\text{gm}}(SO(3); R)$-module. By Proposition A.28, there is a chain of quasi-isomorphisms connecting $C_{\text{gm}}(SO(3); R)$ to the algebra $C_{\text{u}}(SO(3); R)$, and similarly a chain of equivariant quasi-isomorphisms between $C_{\text{gm}}(\alpha; R)$ to $C_{\text{u}}(\alpha; R)$. In particular, we may identify

$$H^\bullet(C_{\text{gm}}(SO(3)), C_{\text{gm}}(\alpha; R)) \cong H^\bullet_{SO(3)}(\alpha; R)$$

using Theorem A.26. ■

As before, these spectral sequences will become significantly more computable after the introduction of the Donaldson models $\overline{\mathcal{DICT}}^\pm$ for $C1^-$ and $C1^+$ in Section 6.2.

Remark 5.5. It is important to note that while the homology theory for $\mathbb{Z}$-graded complexes $H^\bullet_A(M)$ described at first in the appendix sends $A$-equivariant quasi-isomorphisms $M \to M'$ to isomorphisms on $H^\bullet_A$, this is not true for arbitrary $A$-equivariant quasi-isomorphisms between $\mathbb{Z}/8$-graded complexes.

What remains true, and will be used extensively, is that if an equivariant map $M \to M'$ of $\mathbb{Z}/8$ complexes equipped with a periodic filtration induces an isomorphism on the associated graded homology groups (the $E^1$ page of the spectral sequence), then it induces an isomorphism on $H^\bullet_A(M)$.

The reason this happens is that a completion is used to define these objects, which ensures that the associated spectral sequences converge. However, quasi-isomorphisms between the uncompleted complexes that are not filtered do not induce quasi-isomorphisms between the completions.
6. Examples, calculations, and comparisons

For the entirety of Section 6, we choose elements in each orientation set Λ(α) arbitrarily, and suppress these orientation sets from notation.

6.1. Equivariant instanton homology for admissible bundles. When E is a nontrivial admissible U(2)-bundle over an oriented 3-manifold Y, all critical orbits are free. The complex \( \check{C}I(Y, E; R) \) resembles the equivariant Morse complex of a finite-dimensional free \( SO(3) \)-manifold M, so our heuristic is that the equivariant homology groups \( I^*(Y, E; R) \) should behave like the equivariant homology \( H^*_{G}(M) \): \( H^*_G(M; R) \cong H^*_G(M; R) \cong H(M/G; R) \), and \( H^*_G(M; R) = 0 \).

**Theorem 6.1.** Suppose \( E \) is a nontrivial admissible bundle over a 3-manifold Y. Then \( I^\infty(Y, E) = 0 \), and if \( CI_e(Y, E) \) is Floer’s instanton chain complex for admissible bundles, there is a natural quasi-isomorphism \( CI^+(Y, E; R) \to CI_e(Y, E; R) \). In particular, the equivariant Floer homology \( I^+(Y, E) \) of a nontrivial admissible bundle is Floer’s original instanton homology group \( I_e(Y, E) \). Furthermore, there is also a functorial quasi-isomorphism \( CI_e(Y, E; R)[3] \to CI^-(Y, E; R) \).

**Proof.** We first prove the statement for \( CI^\infty \). The \( E^1 \) page of the index spectral sequence is

\[
\bigoplus_{\alpha \in \mathcal{E}, n \in \mathbb{Z}} H^\infty_\ast((\alpha; R)[i(\alpha)]).
\]

Now all orbits are free orbits \( \alpha \cong SO(3) \), and so we can calculate this as

\[
\bigoplus_{\alpha \in \mathcal{E}, n = i(\alpha) \mod 8} H^\infty_\ast((\alpha; R)[i(\alpha)]).
\]

That \( H^\infty_{SO(3)}(SO(3)) \) vanishes is one of the defining features of Tate homology: see Proposition A.12 (4). So the \( E^1 \) page is identically zero, and by the \( I^\infty \) part of Theorem 5.21 we see that the map \( 0 \to CI^\infty(Y, E; R) \) is a quasi-isomorphism, as desired.

Floer’s instanton complex \( CI_e(Y, E) \) is defined as the free \( R \)-module generated by critical orbits \( \alpha \in \mathcal{E} \), the relative grading defined still as \( gr_\ast(\alpha, \beta) \); the differential counts points in zero-dimensional moduli spaces of unframed instantons between \( \alpha \) and \( \beta \). As a graded \( C^\ast \)-module, \( CI^+ \) is given by \( \oplus_{\alpha \in \mathcal{E}} C^+(\alpha; R)[i(\alpha)] \). The differential is a sum of the induced differential on each factor \( C^+(\alpha; R) \) and the maps \( C^+(\alpha; R) \to C^+(\beta; R) \) induced by the \( C^\ast_{gm}(SO(3)) \)-equivariant chain maps

\[
\sigma \mapsto \sigma \times e_{i(\alpha)} \mathcal{M}(\alpha, \beta).
\]

Recall from Lemma A.21 that \( H^+_{SO(3)}(SO(3); R) = R \), concentrated in degree zero. There is a chain map \( \varphi : CI^+(Y, E; R) \to CI_e(Y, E; R) \) given by sending each \( C^+_{\ast-i(\alpha)}(\alpha; R) \) to the generator in degree \( i(\alpha) \) corresponding to \( \alpha \); this map kills everything in degree larger than \( i(\alpha) \) and sends \( C^+_0(\alpha; R) \to H^+_{i(\alpha)} = R \). Said another way, it consists of the augmentation \( C_0(\alpha; R) \to R \) in degree zero and kills everything else. Observe that the chain maps \( (\times e_{i(\alpha)})^+ : C^+(\alpha; R) \to C^+(\beta; R) \) described in the above paragraph are compatible with \( \varphi \): the only component of \( (\times e_{i(\alpha)})^+ \) not automatically sent to zero is that component in degree zero, which is given by counting points in zero-dimensional moduli spaces.
To see that this is a chain map, note that if \( \sigma \in CI(Y,E;R) \) is not sent to zero, it can be written as a sum of points in

\[
\bigoplus_{\alpha \in \mathfrak{P}_*} C_0(\alpha; R).
\]

So it suffices to check that \( \varphi(d\sigma) = d\varphi(\sigma) \) for \( \sigma \) = \( p \), \( p \) a point in \( \alpha \). The only components of \( d\sigma \in CI \) not automatically sent to zero are those in some \( C_0(\beta; R) \subset C^+(\beta; R) \). These arise precisely from components of the differential given by counting points in a fiber of \( \mathcal{M}(\alpha, \beta) \) over \( p \in \alpha \) when these moduli spaces are 3-dimensional \( SO(3) \)-manifolds. Because these are free \( SO(3) \)-manifolds, this is the same as counting the number of points in \( \mathcal{M}(\alpha, \beta)/SO(3) \) — the number of unframed instantons. This is precisely how to write down the differential \( d\varphi(p) \).

Now, on the unrolled complexes, this map is a filtered chain map for the tautological periodic filtration of \( CI_* \) by degree, and Theorem 5.21 says that the above map induces an isomorphism

\[
E^1(CI^+(Y,E;R)) \to E^1(CI_*(Y,E;R)) = CI_*(Y,E;R).
\]

Therefore, by the comparison theorem, the induced map

\[
I^+(Y,E;R) \to I_*(Y,E;R)
\]

is an isomorphism. That these isomorphisms fit into a commutative square with the corresponding cobordism maps is proved the same way.

The statement about \( I^- \) follows because of the exact triangle relating \( I^- \), \( I^+ \), and \( I^- \) and the fact that \( I^0 \) vanishes; one may alternately define a chain map \( CI_3 \to CI^-=CI^- \) explicitly, almost precisely as above except including the fundamental class instead of a point class.

Suppose \( \frac{1}{2} \in R \); then \( H^{-4}(BSO(3);R) = R[U] \), where \( |U| = -4 \). Floer's \( I_* \) also carries an action by this ring (when \( \frac{1}{2} \in R \)), and we should check that the above map preserves the \( U \)-action. Because the \( U \)-action carries things down vertically in the spectral sequence, but \( E^2(CI^+) \) is here concentrated on a single horizontal line, this is not a theorem well-suited to a spectral sequence proof. Instead, we must get our hands dirty at the chain level. This will be delayed until the following section, where we construct a simpler chain-level model of \( \tilde{CI} \).

### 6.2. Comparison with Donaldson's theory.

In this section, we describe a finite-dimensional complex computing framed instanton homology \( \tilde{I} \), and then explain how to use it to calculate the equivariant instanton groups. While it seems likely that this is possible over any PID \( R \), the situation is drastically simpler when \( \frac{1}{2} \in R \).

So for the rest of this section, \( R \) is a PID in which \( 2 \) is invertible; it will be dropped from the notation whenever possible.

If \( 2 \) is invertible in \( R \), then \( H_*(SO(3);R) \) is isomorphic to a single copy of \( R \) in degrees 0 and 3 and zero otherwise. We denote this \( R \)-algebra \( \Lambda(u) \), the exterior algebra on a generator \( u \) in degree 3. There is a dg-algebra homomorphism

\[
i : \Lambda(u) \to C_*^{gm}(SO(3);R)
\]

given by picking out the identity in degree 0 and the fundamental class in degree 3; this map induces the identity on homology. By Proposition A.13, because \( i \) is a
quasi-isomorphism,

$$H^\ast(C^\text{gm}_\ast(SO(3); R), \hat{CI}) \cong H^\ast(\Lambda(u), \hat{CI}).$$

For the purposes of computing the equivariant instanton homology groups $I^+, I^-$, and $I^\infty$, it therefore suffices to consider $\hat{CI}$ as a module over $\Lambda(u)$.

Our goal is to write down a differential on the $\mathbb{Z}/8$-graded $R$-module

$$DCI(Y, E, \pi; R) = \bigoplus_H H_\ast(\alpha; R)[i(\alpha)],$$

which has a periodic filtration by index of the orbit $\alpha$. This differential should decrease the filtration, and there should be a map $\hat{CI}(Y, E, \pi) \to DCI$ which is the identity on the $E^1$ page, or something like it. To produce this, we use what is called the homological perturbation lemma. Recall its statement:

**Lemma 6.2** (Homological perturbation lemma). Suppose $(C, d)$ and $(C', d')$ are chain complexes, equipped with an inclusion $i : C \to C'$ and a projection $p : C' \to C$ so that $pi = 1$, both quasi-isomorphisms, as well as a degree 1 map $h : C' \to C'$ serving as a homotopy witnessing this. That is,

$$ip = 1 + d'h + hd'.$$

This data is called deformation retract data, and is depicted

$$(C, d) \rightarrow_p (C', d') \circ h.$$

Suppose that $C'$ is equipped with a deformation: an additional map $\delta : C \to C$ so that $(d' + \delta)^2 = 0$. Suppose that $h^n = 0$ for sufficiently large $n$. Write $A = \sum_{n=0}^{\infty} (\delta h)^n \delta$.

Then $(C, d + pAi)$ is a chain complex, equipped with deformation retract data

$$(C, d + pAi) \rightarrow_{p + pAh} (C', d' + \delta) \circ h + hAh.$$

If $C$ and $C'$ are dg-modules over a dga, and all of $i, p, h,$ and $\delta$ are dg-module homomorphisms, then the same is true of the perturbed deformation retract data. In particular, $(C, d + pAi)$ is homotopy equivalent to $(C', d' + \delta)$ as a dg-module.

If $C$ and $C'$ are filtered, and $i, p, h,$ and $\delta$ all preserve the filtration (in the sense that $f(F_k) \subseteq F'_k$ for all $k$), then the same is true for the perturbed deformation retract data. In particular, $(C, d + pAi)$ is a filtered complex, which is filtered homotopy equivalent to $(C', d' + \delta)$.

A very explicit reference is [Cra04].

To apply the homological perturbation lemma, first recall that we may write the differential on $\hat{CI}$ as $d + d_M$; the first term is the usual boundary operator on geometric chains, and the second term is the contribution from fiber products with moduli spaces. We want to apply the lemma to $(DCI, 0)$ and $(\hat{CI}, d)$, with $\delta = d_M$.

Choose a basepoint $b_\alpha \in \alpha$ for each critical orbit $\alpha$; if $\alpha$ is an $SO(2)$-reducible, we demand that the stabilizer of $b_\alpha$ is the standard $SO(2) \subset SO(3)$. Later, when considering the $SU(2)$-model, this will translate into the demand that the stabilizer is the standard $U(1) \subset SU(2)$.

Our inclusion map

$$i : DCI \to \hat{CI}$$

is given on $H_0(\alpha; R) \to C^\text{gm}_\ast(\alpha; R)$ by sending the generator to the basepoint $b_\alpha$, and on $H_{dim \alpha}(\alpha; R) \to C^\text{gm}_\ast(\alpha; R)$ by sending a generator to the corresponding
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fundamental class of \( \alpha \). (Note that the choice of fundamental class depends on an orientation of \( \alpha \); we are making this choice by choosing a generator of top homology.)

We will proceed to the appropriate deformation retract data in two steps. First, we will pass from \( CI \) to a slightly different but quasi-isomorphic chain complex, for which we can explicitly define \( p \) and \( h \). Then we will modify the differential on \( CI \) itself to be chain-homotopy equivalent to the complex we started with, but in such a way that the map \( p(d_M h)^n d_M i \) is identically zero for \( n \geq 0 \). This implies that the perturbed differential \( pAi \) on \( DCI \) consists only of the term \( pd_M i \), which has a concrete description.

To begin, note that the map \( H_*(SU(2); R) \to H_*(SO(3); R) \) is an isomorphism. It will be important in what follows to first replace \( CI \) with a quasi-isomorphic complex called the \( SU(2) \) model. Throughout this section, in an attempt to minimize notational overload, we write \( M_{\alpha\beta} \) for the compactified moduli space of framed instantons \( M_{E,z,\pi}(\alpha, \beta) \) for the unique homotopy class \( z \) with \( -2 \leq \text{gr}_z(\alpha, \beta) \leq 5 \), as we ignore all larger moduli spaces. This space can only be nonempty if that grading is positive. Its quotient \( M_{\alpha\beta} \) is denoted simply \( M_{\alpha\beta} \).

The main difficulty is that while we may define the holonomy from one point to another as an element of \( SU(2) \) (using the \( U(2) \)-model for the configuration space and the fact that we are only quotienting by even gauge transformations), it is not true that we can lift the endpoint maps \( M_{\alpha\beta} \to \alpha \times \beta \) to maps to \( SU(2) \) when \( \alpha \) or \( \beta \) is irreducible. For instance, consider the case that \( M_{\alpha\beta} \) is a single point and \( \alpha, \beta \) are irreducible; then \( M_{\alpha\beta} \to \alpha \times \beta \) may be identified with the diagonal \( SO(3) \to SO(3) \times SO(3) \), which is nontrivial on \( \pi_1 \), and therefore cannot be lifted to codomain \( SU(2) \times SU(2) \).

What we would like to do is the following. First, replace every irreducible \( SO(3) \)-orbit \( \alpha \) with its universal cover \( \tilde{\alpha} \); this carries a canonical \( SU(2) \) action lifting the \( SO(3) \) action on \( \alpha \). Second, replace every moduli space \( M_{\alpha\beta} \) with the \( \pi_1(\alpha) \times \pi_1(\beta) \) cover corresponding to the homomorphism

\[
e_\#: \pi_1(M_{\alpha\beta}) \to \pi_1(\alpha \times \beta);
\]
call this cover \( M_{\alpha\beta}^{\text{cov}} \). This cover carries a lift to a map

\[
\tilde{M}_{\alpha\beta}^{\text{cov}} \to \tilde{\alpha} \times \tilde{\beta},
\]
unique up to isomorphism. However, when \( \beta \) is irreducible, the fiber product

\[
\tilde{M}_{\alpha\beta}^{\text{cov}} \times_{\tilde{\beta}} \tilde{M}_{\beta\gamma}^{\text{cov}}
\]
is not isomorphic to a boundary stratum of \( \tilde{M}_{\alpha\gamma}^{\text{cov}} \). For convenience of notation, call this stratum \( \partial_{\beta} \tilde{M}_{\alpha\gamma}^{\text{cov}} \). Instead, there is a canonical covering map

\[
\tilde{M}_{\alpha\beta}^{\text{cov}} \times_{\tilde{\beta}} \tilde{M}_{\beta\gamma}^{\text{cov}} \to \partial_{\beta} \tilde{M}_{\alpha\gamma}^{\text{cov}}
\]
with fiber of cardinality \( |\pi_1(\beta)| \).

Suppose we had some rule allowing us to identify the fundamental class (in some sense) of a compact manifold with half of the fundamental class of its double cover. Formally, set

\[
M_{\alpha\beta} := \frac{1}{|\pi_1(\alpha)|} M_{\alpha\beta}^{\text{cov}}.
\]
Then, given this ‘rule’, we would be able to write
\[
\overline{M}_{\alpha \beta} \times_\beta \overline{M}_{\beta \gamma} = \overline{c}_{\beta} \overline{M}_{\alpha \gamma},
\]
giving us the relation necessary to define the instanton chain complex. We therefore aim to make this idea precise.

Recall the definition of the chain complex \( C_{gm}^\ast (\alpha; R) \) from Section 5.1. Whenever \( 2 \in R \) is invertible, we will construct a further quotient of this suitable to our purposes, written \( C_{gm,2}^\ast (\alpha; R) \). Whenever we have a basic chain given as a composite
\[
\sigma' : P' \xrightarrow{p} P \xrightarrow{f} \alpha,
\]
where \( p \) is a covering map with fiber of order \( 2^n \), we quotient by the additional relation
\[
[\sigma'] = 2^n[\sigma].
\]
Because the composite of two finite covers is again a finite cover, no new additional relations are imposed by chains of covers \( P'' \to P' \to P \) that we have not already quotiented by. The resulting chain complex is again natural under smooth maps.

A few verifications are in order, all of which are straightforward enough that we state them without proof. First, \( C_{gm,2}^\ast (\alpha; R) \) satisfies the Eilenberg-Steenrod axioms for admissible pairs, with the proof essentially unchanged from Theorem 5.7. Second, the fiber product map
\[
\times_X : C_{gm}^\ast (\alpha; R) \to C_{gm}^\ast (\beta; R)
\]
induced by a pair of maps (with left map a submersion)
\[
\alpha \leftarrow X \to \beta
\]
descends to \( C_{gm,2}^\ast \); this follows because the fiber product with a cover is a cover of the fiber product. Finally, the complex \( C_{gm,2}^\ast (\alpha; R) \) is still a free \( R \)-module: the set of chains naturally forms a partial ordered set with \( \sigma' \succeq \sigma \) if \( \sigma' \) covers \( \sigma \) with fiber of cardinality \( 2^n \); considering this as a graph by forgetting the direction of edges, a choice of basic chain in each path-component of this graph gives a free generating set. Here it was crucial that \( 2 \) is invertible in \( R \).

Now we define the \( SU(2) \) model exactly as suggested above. As a relatively graded \( C_{gm}^\ast (SU(2); R) \)-module, it is given by
\[
\widetilde{CI}^{SU(2)} (Y, E, \pi; R) = \bigoplus_{\alpha \in \mathfrak{e}_+} C_{gm,2}^\ast (\tilde{\alpha}; R),
\]
and the differential is the sum of the usual boundary operator and fiber products with the \( \overline{M}_{\alpha \beta} \) defined above. Note that here the formal definition given above makes sense, with a factor of \( \frac{1}{2} \) when \( \alpha \cong SO(3) \); this means we take the usual fiber product with \( \overline{M}_{\alpha \beta}^{\text{cov}} \) and then multiply by \( \frac{1}{2} \) if \( \alpha \cong SO(3) \). Because the expected decomposition of \( \overline{M}_{\alpha \beta} \) holds, this defines a chain complex (in fact, a dg-module).

**Lemma 6.3.** The projection \( r : C_{gm}^\ast (SU(2); R) \to C_{gm}^\ast (SO(3); R) \) induced by the standard covering homomorphism \( r : SU(2) \to SO(3) \) is a dg-algebra homomorphism which is a quasi-isomorphism. There is a zig-zag of dg-module quasi-isomorphisms between \( \widetilde{CI} \) and \( \widetilde{CI}^{SU(2)} \).
Proof. That $r$ is a dg-algebra homomorphism follows immediately from the fact that the corresponding map $r : SU(2) \to SO(3)$ is a homomorphism of groups. To construct the zig-zag, first consider the $C^{gm}_*(SO(3); R)$-module given by

$$\tilde{C}I' = \bigoplus C^{gm,2}_*(\alpha; R);$$

the natural projection $\tilde{C}I \to \tilde{C}I'$ is a module homomorphism and a quasi-isomorphism, because it is an isomorphism on the $E^1$ page of the corresponding index spectral sequence.

There is a map

$$f : \tilde{C}I^{SU(2)} \to \tilde{C}I';$$

if we denote $\pi_\alpha : \tilde{\alpha} \to \alpha$ for the covering map on each orbit, and $\sigma : P \to \tilde{\alpha}$ is a chain, then we set

$$f(\sigma) = \frac{1}{|\pi_1(\alpha)|} \pi_\alpha \sigma.$$

The map $f$ is clearly equivariant with respect to $r$. Next we should see that $\pi$ gives a chain map. Observe that if $\sigma : P \to \tilde{\alpha}$ is a chain, then

$$\sigma \times_{\tilde{\alpha}} \overline{M}_{\alpha\beta}^{cov} = |\pi_1\alpha|(\pi_\alpha \sigma) \times_{\alpha} \overline{M}_{\alpha\beta}^{cov}$$
as chains equipped with maps to $\tilde{\beta}$; here the factor $|\pi_1\alpha|$ indicates that the former is the disjoint union of that many copies of the latter. Next, we have the canonical covering map

$$(\pi_\alpha \sigma) \times_{\alpha} \overline{M}_{\alpha\beta}^{cov} \to (\pi_\alpha \sigma) \times_{\alpha} \overline{M}_{\alpha\beta},$$

with fibers of cardinality $|\pi_1\alpha| \cdot |\pi_1\beta|$. Combining these with the definition of

$$\overline{M}_{\alpha\beta}' = \frac{1}{|\pi_1\alpha|} \overline{M}_{\alpha\beta}^{cov},$$

we have the relation

$$\sigma \times_{\tilde{\alpha}} \overline{M}_{\alpha\beta}' = \frac{|\pi_1\beta|}{|\pi_1\alpha|} (\pi_\alpha \sigma) \times_{\alpha} \overline{M}_{\alpha\beta}$$
as elements of $C^{gm,2}_*(\beta; R)$.

This is precisely the statement that $f$ is a chain map. $\blacksquare$

In a flagrant abuse of notation, from now on we will write $\tilde{\alpha}$ simply as $\alpha$ and the new moduli space $\overline{M}_{\alpha\beta}'$ simply as $\overline{M}_{\alpha\beta}$, just as before, unless otherwise stated. As we will primarily work in the $SU(2)$-model for the rest of this section, we hope that this will cause no confusion, and will make the distinction clear when both moduli spaces are in play.

To define the map $p$, choose basepoints $q_\alpha \in \alpha$, with the property that each $q_\alpha$ corresponding to an $U(1)$-reducible $\alpha$ has a different stabilizer, which is not the standard $U(1)$.

Recall from Definition 5.7 and Lemma 5.5 that, given a countable family $\mathcal{F}$ of maps from $\delta$-chains to a space $X$, the geometric chain complex has a quasi-isomorphic subcomplex $C^{gm,\mathcal{F}}_*(X; R)$, spanned by those chains which are transverse on every stratum to the countable family of maps. If $X$ is an $SO(3)$-orbit, this is a $\Lambda(u)$-invariant subcomplex, because the evaluation map $SO(3) \to X$ is a submersion. This means that the product of any chain with the fundamental class is transverse to any element of $\mathcal{F}$. 

To define the map 
\[ p : C^\text{gm,F}_\bar{a}(\alpha; R) \to H_\bar{a}(\alpha; R), \]
take \( \mathcal{F} \) to include the basepoint \( q_\alpha \). In degree zero, any 0-dimensional \( \delta \)-chain has an underlying oriented point, and the projection is given by counting these with sign. In top degree, we send a \( \delta \)-chain \( \sigma : P \to \alpha \) to the signed count \( \# \sigma^{-1}(q_\alpha) \).

Note that to define the sign of this signed count, we need to choose an underlying orientation of \( \alpha \), which determines the generator of \( H_{\dim \alpha}(\alpha; R) \) we should send 1 to. The homomorphism we obtain is independent of this choice of orientation of \( \alpha \).

Unfortunately, the fiber products with moduli spaces do not send chains transverse to \( q_\alpha \) to chains transverse to \( q_\beta \). So this submodule of \( \widetilde{CI} \) is not a subcomplex. To fix this, we enlarge the collection \( \mathcal{F} \) that our chains must be transverse to.

Denote \( \mathcal{M}_{\alpha \beta}^j \) for the fiber above \( q_\beta \) of the endpoint map 
\[ \mathcal{M}_{\alpha \beta}^j \to \beta. \]

More generally, given a sequence of orbits \( \gamma_1, \cdots, \gamma_n \), denote \( \mathcal{M}_{\alpha \gamma_1, \cdots, \gamma_n}^j \) denote the fiber above \( q_\beta \) of the map 
\[ \mathcal{M}_{\alpha \gamma_1, \cdots, \gamma_n}^j \to \beta. \]

For the orbit \( \alpha \), we let the collection \( \mathcal{F} \) consist of \( q_\alpha \) and all of the endpoint maps of moduli spaces 
\[ \mathcal{M}_{\alpha \gamma_1, \cdots, \gamma_n}^j \to \alpha. \]

Because this collection is closed under fiber products with moduli spaces on the left, the condition of being transverse to \( \mathcal{F} \) is preserved by \( d_M \). So for this collection \( \mathcal{F} \), we have a \( \Lambda(u) \)-subcomplex
\[ \widetilde{CI}^\mathcal{F}(Y, E, \pi; R) \subset \widetilde{CI}(Y, E, \pi; R) \]
for which the projection map \( p \) above is defined. Running the index spectral sequence, and using that the inclusion of \( C^\text{gm,F}_\bar{a}(\alpha; R) \to C^\text{gm,F}_\bar{a}(\alpha; R) \) is a quasi-isomorphism for each orbit, we see that this subcomplex \( \widetilde{CI}^\mathcal{F} \) is equivariantly quasi-isomorphic to the whole thing. Furthermore, it is a free \( R \)-submodule; its generators are precisely those generators of \( \widetilde{CI} \) for which all strata are transverse to the given family of \( \delta \)-chains.

To ensure that the image of \( i \) is in this subcomplex, we need to demand that \( b_\alpha \) is a regular value of \( \mathcal{M}_{\alpha \gamma_1, \cdots, \gamma_n}^j \to \alpha \); one may symmetrically demand that \( q_\beta \) is the regular value of the endpoint map from the fiber above \( b_\alpha \). It is not difficult to ensure that the \( q_\alpha \) satisfy this, by an application of Sard’s theorem.

We will define the map \( h \) in the course of Lemma 6.5, but for intuition it is convenient to imagine it as follows. In degrees up to \( \dim \alpha - 1 \), one imagines that \( h \) is given by the ‘cone to \( b_\alpha \)’ map, using stereographic projection based at \( q_\alpha \). In degree \( \dim \alpha \), it is chosen arbitrarily to satisfy the requirements that
\[ \text{ip} - 1 = d'h + hd' \]
and \( uh = hu \). The latter condition is automatically satisfied for degree reasons except possibly on 0-chains on an irreducible orbit; in that case, it becomes the demand \( h([SU(2)]) = 0 \). In degree \( \dim \alpha + 1 \), the map \( h \) is again zero for degree reasons.
The issue is that \(h\), as defined here, does not preserve transversality to \(F\) (only to \(q_\alpha\)). We will resolve this when the time comes; for now, we discuss what would happen if \(h\) did indeed preserve transversality to \(F\).

This is enough to apply the homological perturbation lemma and get some result, a differential on \(DCI(Y, E, \pi; R)\) which is \(\Lambda(u)\)-equivariantly homotopy equivalent to \(\widetilde{CI}^F(Y, E, \pi; R)\). Unfortunately, the differential is still more complicated than we would like: it takes the form

\[
p \sum_{n=0}^{\infty} (d_M h)^n d_M i,
\]

where \(d_M\) is the part of the differential coming from fiber products with moduli spaces. It is not impossible that many components of this may be nonzero: one takes a fiber product of the basepoint \(b_\alpha\) with some moduli space, pushes it forward to \(\beta\), cones towards \(b_\beta\), takes another fiber product, and then counts intersections with \(b_\gamma\). One could easily imagine that, starting and ending at irreducibles \(\alpha, \gamma\), one takes the fiber product with a moduli space that increases the dimension by 2, cones towards \(b_\beta\), and then takes a fiber product with a 0-dimensional moduli space and gets a result that has nonzero degree above \(b_\gamma\).

To resolve this, we should reduce the influence of lower-dimensional moduli spaces, an approach suggested by [Don02, Section 7.3.2]. We will be modifying the endpoint maps, as in the following two lemmas. The proofs are somewhat technical, and it is not necessary to understand the arguments to read the rest of the section; we invite the reader to skip them on a first pass.

**Lemma 6.4.** Suppose \(e_{\alpha \beta} : \overline{M}_{\alpha \beta} \to \alpha \times \beta\) are the equivariant endpoint maps associated to moduli spaces of instantons on a weakly admissible bundle \((Y, E)\) with respect a fixed perturbation \(\pi\).

Suppose we choose a collection of \(e^t\alpha \beta : \overline{M}_{\alpha \beta} \to \alpha \times \beta\), homotopies through equivariant maps that are smooth on each stratum, one homotopy for each pair \((\alpha, \beta)\), so that \(e^0\alpha \beta = e_{\alpha \beta}\) above. Furthermore demand that these are compatible in the sense that

\[
e^t_{\alpha \beta} \times \beta e^t_{\beta \gamma} : \overline{M}_{\alpha \beta} \times \beta \overline{M}_{\beta \gamma} \to \alpha \times \gamma
\]

agrees with the restriction of \(e^t_{\alpha \gamma}\) to that stratum of \(\overline{M}_{\alpha \gamma}\).

We may define a chain complex \(\widetilde{CI}_t(Y, E, \pi)\), identical as a \(\text{C}^*_{\text{gm}}(SO(3))\)-module to \(\widetilde{CI}_t\), but whose differential uses the fiber product with the same moduli spaces but endpoint maps \(e^t_{\alpha \beta}\). Then \(\widetilde{CI}_t(Y, E, \pi)\) is \(\Lambda(u)\)-equivariantly quasi-isomorphic to \(\widetilde{CI}_t(Y, E, \pi)\).

This lifts to an equivalence between the \(SU(2)\)-models. In fact, suppose one instead chooses \(SU(2)\)-equivariant homotopies \(e^t_{\alpha \beta} : \overline{M}_{\alpha \beta} \to \hat{\alpha} \times \hat{\beta}\), which are compatible in the sense that the restriction of \(e^t_{\alpha \gamma}\) to the stratum corresponding to factorizations through \(\beta\) is a double cover of \(e_{\alpha \beta} \times \beta e_{\beta \gamma}\). Then one gets an \(SU(2)\)-equivariant homotopy equivalence between the corresponding \(SU(2)\)-models.

**Proof.** We write \(e^t_{\pm} : \overline{M}_{\alpha \beta} \to \alpha\) to be the map \(\pi_\alpha e^t_{\alpha \beta}\); similarly with \(e^t_{\pm}\). Also write \(C = \widetilde{CI}(Y, E, \pi)\), and the differential corresponding to \(\widetilde{CI}_t\) is given as \(\partial + \partial_t\), the first term the boundary operator in the geometric chain complex and the latter term fiber product with moduli spaces whose endpoint maps are given as \(e^t_{\pm}\). In
what follows, we identify the \( n \)-simplex \( \Delta^n \) with the subset of \([0, 1]^n\) consisting of \((t_1, \ldots, t_n)\) with \( t_i \leq t_{i+1} \) for all \( i \).

Consider the maps \( H_n : C \to C \) given by

\[
H_n(\sigma) = \sum_{\beta_1, \ldots, \beta_n} (\sigma \times \Delta^n) \times_{e_1} (\mathcal{M}_{\alpha \beta_1} \times_{e_2} \cdots \times_{e_n} \mathcal{M}_{\beta_{n-1} \beta_n}).
\]

Here, when we write \( \times_{e_i} \), we mean to take fiber products with respect to the maps \( e_{i+1} \) (which should be interpreted as \( \sigma \) in the case \( i = 1 \)) and \( e_i \) over the point \((t_1, \ldots, t_i, \ldots, t_n) \in \Delta^n; \) the map to \( \beta \) is given by \( e_n \).

The claim is that our desired chain map \((C, \partial + \partial_0) \to (C, \partial + \partial_1)\) is given as

\[
f := \sum_{k=0}^5 H_n = \text{Id} + \sum_{k=1}^5 H_n.
\]

Because the basic chains comprising \( H_k(\sigma) \) are of dimension at least \( k \) (we took the product with \( \Delta^k \) in the process), and 5-chains on a 3-dimensional space are degenerate, \( H_k = 0 \) for \( k > 4 \), and this infinite sum makes sense. The map \( f \) is clearly \( C^\infty(SO(3))-\text{equivariant} \), as it is defined using fiber products under equivariant maps. If we see that \( f \) is a chain map, a quick application of the index spectral sequence finishes the job: because \( f \) is the identity modulo higher filtration, it induces the identity on the associated graded chain complexes (the \( E^1(1) \) page of the index spectral sequence), and hence \( f \) is a quasi-isomorphism. The lift to \( \hat{C}_{\text{SU}(2)} \) is obvious: replace \( H \) with fiber products with the lifted moduli spaces \( \hat{\mathcal{M}} \).

What remains to us is to show that \( f \) is a chain map. Let’s start by calculating \( \partial H_n - H_n \partial \). The simplex \( \Delta^n \) has \( n + 1 \) boundary faces; we denote the \( i \)th boundary face by \( \partial_i \), where \( 0 \leq i \leq n \). This is the face in which \( t_i = t_{i+1} \) for \( 1 \leq i \leq n - 1 \); if \( i = 0 \), it’s the face in which \( t_1 = 0 \), and if \( i = n \), it’s the face in which \( t_n = 1 \).

We have

\[
(H_n \partial - \partial H_n)(\sigma) = \sum_{\beta_1, \ldots, \beta_n} (\partial \sigma \times \Delta^n) \times_{e_1} (\mathcal{M}_{\alpha \beta_1} \times_{e_2} \cdots \times_{e_n} \mathcal{M}_{\beta_{n-1} \beta_n})
\]

\[
= \sum_{i=0}^n \sum_{\beta_1, \ldots, \beta_n} (\sigma \times \partial_i \Delta^n) \times_{e_1} (\mathcal{M}_{\alpha \beta_1} \times_{e_2} \cdots \times_{e_n} \mathcal{M}_{\beta_{n-1} \beta_n})
\]

\[
= \sum_{i=1}^n \sum_{\beta_1, \ldots, \beta_n} (\sigma \times \Delta^n) \times_{e_1} (\mathcal{M}_{\alpha \beta_1} \times_{e_2} \cdots \partial \mathcal{M}_{\beta_i \beta_{i+1}} \cdots \times_{e_n} \mathcal{M}_{\beta_{n-1} \beta_n})
\]

\[
- \sum_{\beta_1, \ldots, \beta_n} (\partial \sigma \times \Delta^n) \times_{e_1} (\mathcal{M}_{\alpha \beta_1} \times_{e_2} \cdots \times_{e_n} \mathcal{M}_{\beta_{n-1} \beta_n})
\]

The first and final terms cancel. Of what remains, the first chunk of \( H_n \partial - \partial H_n \) cancels with the second chunk of \( H_{n+1} \partial - \partial H_{n+1} \). More precisely, in the term

\[
\sum_{\beta_1, \ldots, \beta_{n+1}} (\sigma \times \partial_i \Delta^{n+1}) \times_{e_1} (\mathcal{M}_{\alpha \beta_1} \times_{e_2} \cdots \times_{e_n} \mathcal{M}_{\beta_n \beta_{n+1}})
\]

\( \partial_i \Delta^{n+1} \) consists of those sequences \((t_1, \ldots, t_{n+1})\) so that \( t_i = t_{i+1} \). This term is equivalent to

\[
\sum_{\beta_1, \ldots, \beta_{n+1}} (\sigma \Delta^n) \times_{e_1} (\mathcal{M}_{\alpha \beta_1} \times_{e_2} \cdots (\mathcal{M}_{\beta_i \beta_{i+1}} \times_{e_2} \mathcal{M}_{\beta_{i+1} \beta_{i+2}}) \cdots \times_{e_n} \mathcal{M}_{\beta_n \beta_{n+1}})
\]
instead of \( t_{i+1} \) in the fiber product in the middle, we have replaced it with \( t_i = t_{i+1} \). Identifying \( \partial^i \Delta_{n+1} \) with \( \Delta_n \) by

\[
(t_1, \ldots, t_{n+1}) \mapsto (t_1, \ldots, t_i, t_{i+2}, \ldots, t_{n+1}),
\]

and noting that the fiber product we have pointed out in the middle is equivalent to \( \partial \overline{\mathcal{M}}_{\beta_i, \beta_{i+2}} \) (fiber products taken with respect to \( e_{\beta}^i \)), we see that this is the same as

\[
\sum_{\beta_1, \ldots, \beta_i, \beta_{i+2}, \ldots, \beta_{n+1}} (\sigma \times \Delta_n) \times e_{\beta_1} \left( \overline{\mathcal{M}}_{\alpha \beta_1} \times e_{\beta_2} \cdots \partial \overline{\mathcal{M}}_{\beta_i \beta_{i+2}} \cdots \times e_{\beta_{n+1}} \overline{\mathcal{M}}_{\beta_n \beta_{n+1}} \right).
\]

Thus \( \partial \sum_0^{\infty} H_n - \sum_0^{\infty} H_n \partial \) is a telescoping sum, and because \( H_k = 0 \) for \( k \geq 5 \), the leftover terms in the telescope are eventually zero, and this sum is zero, so \( f \) is a chain map, as desired. ■

To be clear, there are no transversality conditions imposed on the above chain complexes. One should choose the basepoints \( p_{i, q} \), and hence the map \( h \), only after having specified the endpoint maps from the moduli spaces. Now what we need is the following.

**Lemma 6.5.** We may find a coherent collection of \( SU(2) \)-equivariant homotopies \( e^i_{\beta} : \overline{\mathcal{M}}_{\alpha \beta} \to \alpha \times \beta \), and basepoints \( b_{i, q} \), so that if \( d_M \) is the map corresponding to fiber products with moduli spaces using the time-1 maps \( e^i_{\beta} \), and \( p, h \) are the maps in the deformation retract data specified by the \( b_i \) and \( q_i \), the map \( pd_M h d_M \) is identically zero.

**Proof.** If we write

\[ X_{\alpha \beta} = (\alpha \times \beta) / SU(2), \]

there is an induced map \( \mathcal{M}_{\alpha \beta} \to X_{\alpha \beta} \): we will soon demand that this map is constant as much as we can force it to be. If one of the orbits (say, \( \alpha \)) is irreducible, then \( X_{\alpha \beta} \cong \beta \); if one of the orbits is trivial, \( X_{\alpha \beta} \) is a point; if both \( \alpha \) and \( \beta \) are \( U(1) \)-reducibles, \( X_{\alpha \beta} \cong [-2, 2] \).

We induct on two different integers: the dimension \( i \) of \( \mathcal{M}_{\alpha \beta} \), and the maximal codimension \( j \) of a stratum of \( \mathcal{M}_{\alpha \beta} \); of course \( i \geq j \).

We will start by explaining the argument in the case of integer homology spheres; the \( U(1) \)-reducibles present substantial new issues that must be addressed, but the idea is largely the same.

Let \( \overline{\mathcal{M}}_{\alpha \beta} \) be a moduli space without any boundary strata, and suppose both \( \alpha \) and \( \beta \) are irreducible, with \( \dim \mathcal{M}_{\alpha \beta} < 3 \). Using the trivialization \( \alpha \cong SU(2) \) given by the basepoint \( b_\alpha \), the left endpoint map to \( \alpha \) gives a trivialization

\[ \overline{\mathcal{M}}_{\alpha \beta} \cong \mathcal{M}_{\alpha \beta} \times SU(2) \]

as well as a canonical isomorphism

\[ \alpha \times \beta \cong X_{\alpha \beta} \times SU(2). \]

Therefore, an equivariant homotopy of the endpoint map is the same as choosing a homotopy of the map

\[ \mathcal{M}_{\alpha \beta} \to X_{\alpha \beta} \cong S^3. \]

Because \( S^3 \) is 2-connected, any such map is null-homotopic; choose a homotopy taking this map to the constant map at \([b_\alpha, b_\beta]\). Do nothing to the higher-dimensional
moduli spaces, and do nothing to the moduli spaces for which one of $\alpha$ or $\beta$ is fully reducible.

We now consider moduli spaces having at most a codimension 1 boundary stratum. Again, we first consider the case where both $\alpha$ and $\beta$ are irreducible. If one of these strata corresponds to a factorization through the full reducible, then $\dim \mathcal{M}_{\alpha\beta} > 3$, and we will not worry about it yet.

Suppose $\dim \mathcal{M}_{\alpha\beta} < 3$. We have already chosen a specified null-homotopy $\partial \mathcal{M}_{\alpha\beta} \to X_{\alpha\beta}$. Again, because $S^3$ is 2-connected and the final stage of the homotopy on the boundary is constant at $[b_\alpha, b_\beta]$, we may extend this to a null-homotopy of the entire moduli space to $[b_\alpha, b_\beta]$.

In every other case (that is, either $\dim \mathcal{M}_{\alpha\beta} \geq 3$ or one of $\alpha$ or $\beta$ is fully reducible), we take the existing homotopy of $B \mathcal{M}_{\alpha\beta} \to X_{\alpha\beta}$ and use the equivariant homotopy extension property to extend it arbitrarily to the whole of $\mathcal{M}_{\alpha\beta}$.

Do the same for those moduli spaces of dimension 2 with codimension 2 strata (between irreducibles) and extend arbitrarily for higher-dimensional moduli spaces. Now inductively extend the existing homotopies on all moduli spaces with strata of the same maximal codimension $j$, for $j \geq 3$.

Having done this, we choose basepoints $q_\alpha$ in each orbit so that the map $\mathcal{M}_{\alpha\beta} \to \alpha \times \beta$ is transverse to the orbit through $(b_\alpha, q_\beta)$ for all $\alpha, \gamma, \beta$, as in the setup before Lemma 6.4. Choose the homotopy $h$ arbitrarily so that $h([b_\alpha]) = 0$ and $h([\alpha]) = 0$: its existence is guaranteed as $i$ and $p$ are quasi-isomorphisms between free $R$-modules whose composite is the identity on homology. We claim that $hd_M i$ is either zero, or of dimension larger than the orbit it lies in. This follows by a case-by-case analysis. The fiber product

$$b_\alpha \times_\alpha \mathcal{M}_{\alpha\beta} \to \beta$$

is in one of the following five cases:

- The orbits $\alpha$ and $\beta$ are both irreducible and $\dim \mathcal{M}_{\alpha\beta} = 0$: in this case, the fiber product is simply some number of copies of $[b_\beta]$, and applying $h$ returns zero.
- The orbits $\alpha$ and $\beta$ are both irreducible, while $0 < \dim \mathcal{M}_{\alpha\beta} < 3$, the fiber product is a constant map to $b_\alpha$ and hence a degenerate chain (its boundary is either also small image, or its boundary is a sum of points whose signed count must be zero), and hence zero at the chain level.
- The orbits $\alpha$ and $\beta$ are both irreducible, while $\dim \mathcal{M}_{\alpha\beta} \geq 3$: taking the fiber product and then applying $h$ leaves a chain of dimension larger than $\beta$.
- The orbit $\alpha$ is fully reducible; in this case $b_\alpha \times_\alpha \mathcal{M}_{\alpha\beta} = \mathcal{M}_{\alpha\beta}$, and hence has dimension at least as large as $\beta$. Applying $h$ gives a chain of dimension larger than $\beta$.
- The orbit $\beta$ is fully reducible; applying $h$ to $d_M i$ gives something of dimension at least 1, which is then automatically degenerate.

Because the projection map $p$ kills any chain of dimension larger than the orbit it lives on, we see that $p(d_M h)^n d_M i = 0$ for $n > 0$, as desired. What remains of the differential simply counts the preimages of $\mathcal{M}_{\alpha\beta} \to X_{\alpha\beta}$ above $[b_\alpha, q_\beta]$. 
We now discuss the changes necessary in the presence of $U(1)$-reducibles. The argument — crush maps down to constant maps, unless the dimensions are too large to ensure this is possible — largely applies, until the case of moduli spaces of dimension 2 between irreducibles $\alpha, \beta$, with a codimension 1 stratum corresponding to factorization through a $U(1)$-reducible $\gamma$. Then, applying the existing homotopies on the boundary, we find that the stratum corresponding to that factorization is (a disjoint union of signed copies of) $U(1)$, and the map from these $U(1)$s to $\beta$ is the embedding of $U(1)$ onto the $U(1)$-orbit through $b_\beta$. This is forced on us by equivariance. We will call these moduli spaces IUI type to indicate that they are between two irreducibles, with a stratum corresponding to a factorization through a $U(1)$-reducible.

It is worth remarking that in the case of 1-dimensional moduli spaces between two $U(1)$-reducibles, we need slightly more argument to see that we may homotope the endpoint maps down to the basepoint. To do this, we use that the $SU(2)$ action on $M$ is free, and hence there is an isomorphism $M \cong M \times SU(2)$. Because $M$ is 1-dimensional, it is a collection of circles and arcs, and the $SU(2)$-equivariant null-homotopy arises from the fact that $\alpha \times \beta \cong S^2 \times S^2$ is simply connected.

To get around the more serious issue with 2-dimensional moduli spaces, we use $h$ to take something that was not previously of small image, and make it small image. To be precise, for each irreducible orbit $\alpha$, choose a disc bounding $b_\alpha \cdot U(1)$. Call this disc $D_\alpha$. In the above situation, what we can do is homotope this moduli space $M_{\alpha,\beta}$, relative to the boundary, so that its image lies inside the disc $D_\alpha$. (Here again we use that $S^2$ is 2-connected.) Once we have done this for all such moduli spaces, we continue the usual pattern of extending arbitrarily to everything whose dimension is 3 or larger. We will continue to use $M_{\alpha,\beta}$ to denote the moduli spaces equipped with this new family of maps.

We will set $h(b_\alpha \cdot U(1)) = D_\alpha$. For convenience, following the isomorphism $\alpha \cong SU(2) \cong S^3$ determined by the basepoint $b_\alpha$, we demand that this is the same disc for all irreducible orbits $\alpha$.

For each moduli space $M_{\alpha,\beta}$ of IUI type, we choose a chain $\sigma_{\alpha,\beta}$ with 

$$\partial \sigma_{\alpha,\beta} = (b_\alpha \times_\alpha M_{\alpha,\beta}) \cup_m D_\alpha,$$

where $m$ is the appropriate number; that is, we choose a chain which cobounds the map from the moduli space and the appropriate number of discs (so that their sum is null-homologous). Furthermore, we demand that $\sigma_{\alpha,\beta}$ factor through the disc $D_\alpha$. That we may do all this follows because $H_2(D^2) = 0$. We will set 

$$h(b_\alpha \times_\alpha M_{\alpha,\beta}) = \sigma_{\alpha,\beta}.$$ 

Lastly, we will let $h(D_\alpha) = 0$.

Next, we choose the basepoints $q_\alpha$ so that the following conditions hold. First, if $b_{\alpha,\gamma,\beta}$ is the fiber product 

$$b_\alpha \times_\alpha M_{\alpha,\gamma_1,\cdots,\gamma_n,\beta},$$

thought of as a chain in $\beta$, we demand that $q_\beta$ is a regular value of $b_{\alpha,\gamma,\beta}$. Similarly, let $b_{\alpha,\gamma,\beta}$ denote the fiber product with $D_\alpha$, where $\alpha$ is an irreducible orbit, and let $\sigma_{\alpha,\gamma,\beta}$ the fiber product with $\sigma_{\xi,\alpha}$ as $\xi$ varies. We further demand that $q_\beta$ is a regular value of these chains in $\beta$. 


Now set $F$ to consist of the chains in $\alpha$ given by

$$\alpha \leftarrow (\mathcal{M}_{\alpha \gamma \beta} \times_{\beta} q_{\beta}) .$$

Then $\tilde{CI}^{SU(2)}$ is a quasi-isomorphic, $\Lambda(u)$-invariant subcomplex of $\tilde{CI}^{SU(2)}$. The first condition above guarantees that $i$ has image inside this subcomplex (that is, $b_{\alpha}$ is transverse to all of the chains in $F$); similarly, $D_{\alpha}$ and $s_{\xi\alpha}$ are all transverse to $F$, and hence our choice of $h(b_{\alpha} \cdot U(1))$ above is valid in this subcomplex, as is our choice of $h(\tilde{\mathcal{M}}_{\alpha \beta})$.

Additionally, we demand the conditions $h([b_{\alpha}]) = 0$ and $h([\alpha]) = 0$. Beyond these we extend $h$ arbitrarily as a chain homotopy on the entire complex between $i \rho$ and 1.

What remains is a case-by-case analysis that $p(d_M h)^n d_M i = 0$ for $n > 0$. Most cases are similar (if not identical) to those discussed above in the integer homology sphere case: automatically zero because the moduli space itself is a degenerate chain, or automatically zero because the dimension counts are wrong. Here we enumerate the cases that are fundamentally new.

- Consider the component of $pd_M h d_M i$ arising by first taking the fiber product with a moduli space $\mathcal{M}_{\alpha \beta}$ of type IOI as above, applying $h$, and then taking the fiber product with a 0-dimensional moduli space between irreducibles $\beta, \gamma$. The output of $h d_M i$ here is the chain $\sigma_{\alpha \beta} \rightarrow \beta$, which is of small image; applying the appropriate component of $d_M$ to this, we again obtain a chain of small image. Projecting, we get zero: the intersection number of any chain of small image with a regular value is always zero.

- Consider the component of $pd_M h d_M i$ arising by first taking the fiber product with a 0-dimensional moduli space $\alpha \rightarrow \beta$, where $\alpha$ is $U(1)$-reducible and $\beta$ is irreducible. The output of $h d_M i$ is some number of copies of the disc $D_{\beta}$. Suppose the moduli space corresponding to the second factor of $d_M$ is a 1-dimensional moduli space $\beta \rightarrow \gamma$, both orbits irreducible. Then the corresponding fiber product is degenerate: it is of small image (its image is the image of the disc $D_{\beta}$), so the corresponding projection is zero.

- Consider the component of $pd_M h d_M i$ arising nearly as above: the first $d_M$ contributes a 0-dimensional moduli space $\alpha \rightarrow \beta$, the next factor contributes a 0-dimensional moduli space $\beta \rightarrow \gamma$, and the last contributes a 0-dimensional moduli space $\gamma \rightarrow \delta$, where all three of $\beta, \gamma, \delta$ are irreducible, while $\alpha$ is $U(1)$-reducible. Then as above, $h d_M i$ outputs some number of copies of the disc $D_{\beta}$; taking another fiber product with a 0-dimensional moduli space outputs $D_{\gamma}$. Applying $h$ to this gives zero.

- Consider the component of $pd_M h d_M i$ arising from a 0-dimensional moduli space $\alpha \rightarrow \beta$, and a 0-dimensional moduli space $\beta \rightarrow \gamma$, where $\alpha, \gamma$ are both $U(1)$-reducible and $\beta$ is irreducible. Because $h d_M i$ is some number of copies of the disc $D_{\beta}$, and the projection $SU(2) \rightarrow S^2$ sends the disc $D_{\beta}$ to a class which generates the second homology of $S^2$, we can not rule these out in the same way as above. However, this would imply that $gr(\alpha, \gamma) = 3$, which contradicts Proposition 5.8 (4), stating that the grading of any two reducibles is the same parity.

Thus, the basepoints $q$, the endpoint maps of the moduli spaces, and the homotopy $h$ have been chosen as desired. 

\[\square\]
Take the complex \( \widetilde{CI}_1(Y, E, \pi; R) \) equipped with the differential \( d + d_M \) induced by the above fiber products with moduli spaces; we described the construction of deformation retract data

\[
(\text{DCI}, 0) \overset{\tilde{\iota}}{\longrightarrow} (\widetilde{CI}_1, d + d_M) \bowtie h
\]

above. Applying the homological perturbation lemma, and using the fact that \( pd_M hd_M \) is identically zero, we arrive at a chain complex \((\text{DCI}, pd_M i)\) with \( \Lambda(u)\)-action, and an equivariant homotopy equivalence to \((\widetilde{CI}_1, d + d_M)\). We may finally describe the differential \( pd_M i \).

Denote by \( C^\text{irr}_* \) the free \( R \)-module on the irreducible orbits \( \alpha \subset \mathcal{C}_\pi \) (graded the same), \( C^{U(1)}_* \) the free \( R \)-module on \( U(1) \)-reducible orbits, and \( C^\theta_* \) the free \( R \)-module on the full reducibles. Set

\[
\text{DCI}(Y, E, \pi) = C^\text{irr}_* \oplus C^{U(1)}_* \oplus C^{U(1)}_* \oplus C^\theta_*. 
\]

As graded \( R \)-modules, \( \text{DCI}(Y, E, \pi) = \oplus H_{\ast-i(\alpha)}(\alpha) \).

The differential \( pd_M i \) that \( \text{DCI} \) inherits is given by the matrix

\[
\hat{\partial}_{\text{CI}} := \begin{pmatrix}
-\hat{\partial}_1 & 0 & 0 & 0 & 0 \\
-U_{V1} & \hat{\partial}_1 & V_4 & V_2 & D_2 \\
-V_1 & 0 & 0 & 0 & \theta V_1 \\
-V_3 & 0 & 0 & 0 & 0 \\
-D_1 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

(We will momentarily define the terms in this matrix.) The action of \( u \) on \( \widetilde{\text{CI}} \) is defined by the matrix whose only nonzero term is \( id : C^\text{irr}_* \rightarrow C^\text{irr}_*[3] \). It is easy to see that \( u \hat{\partial}_{\text{CI}} = \hat{\partial}_{\text{CI}} u = \hat{\partial}_1 \), considered as a map \( C^\text{irr}_* \rightarrow C^\text{irr}_*[3] \).

As before, we denote

\[
\mathcal{M}_{\alpha, \beta} = \mathcal{M}_{\alpha, \beta}/SU(2);
\]

these are the spaces of unframed flowlines between the underlying connections \([\alpha], [\beta] \), having forgotten the framings. Also recall we defined \( X_{\alpha, \beta} = (\alpha \times \beta)/SU(2) \) above. The operators in this matrix all arise of the form \( T(\alpha) = \sum_\beta n(\alpha, \beta) \beta \), where the sum is taken over \( \beta \) in the right degree, and \( n(\alpha, \beta) \) arises either by counting points (with orientation) in \( \mathcal{M}_{\alpha, \beta} \), or degree of the map \( \mathcal{M}_{\alpha, \beta} \rightarrow X_{\alpha, \beta} \), as measured by the number of points lying above \([b_\alpha, q_\beta] \). We’ve chosen \( b_\alpha, q_\beta \) to ensure that this is a regular value (and in the case that both \( \alpha \) and \( \beta \) are \( U(1) \)-reducibles, to have different stabilizer).

- \( \hat{\partial}_1 \) counts points in \( \mathcal{M}_{\alpha, \beta} \), when \( \alpha, \beta \) are both irreducible and \( \text{gr}(\alpha, \beta) = 1 \).
- \( D_1 \) counts points in \( \mathcal{M}_{\alpha, \beta} \) when \( \alpha \) is irreducible, \( \beta \) is fully reducible, and \( \text{gr}(\alpha, \beta) = 1 \).
- \( D_2 \) counts points in \( \mathcal{M}_{\alpha, \beta} \) when \( \alpha \) is fully reducible, \( \beta \) is irreducible, and \( \text{gr}(\alpha, \beta) = 4 \). \(^{16}\)
- \( U_{V1} \) counts the number of points in the fiber above \([b_\alpha, q_\beta] \) of the map \( \mathcal{M}_{\alpha, \beta} \rightarrow X_{\alpha, \beta} \), when \( \alpha \) and \( \beta \) are both irreducible and \( \text{gr}(\alpha, \beta) = 3 \).

\(^{16}\)Recall that this means that \( \overline{\mathcal{M}} \) is of dimension \( \dim \alpha + \text{gr}(\alpha, \beta) - 1 \); because \( \alpha \) is fully reducible, this means that \( \overline{\mathcal{M}} \) is \( 3 \)-dimensional. Because the endpoint map to \( \beta \) is equivariant, and \( \beta \) is irreducible, \( \overline{\mathcal{M}} \) is a finite set of orbits with no stabilizer, and \( \mathcal{M} \) is \( 0 \)-dimensional.
\begin{itemize}
\item $V_1$ counts points in $\mathcal{M}_{\alpha \beta}$ when $\alpha$ is irreducible, $\beta$ is $U(1)$-reducible, and $\text{gr}(\alpha, \beta) = 1$.
\item $V_2$ counts points in $\mathcal{M}_{\alpha \beta}$ when $\alpha$ is $U(1)$-reducible, $\beta$ is irreducible, and $\text{gr}(\alpha, \beta) = 2$.
\item $V_3$ counts points above $[b_\alpha, q_\beta]$ when $\alpha$ is irreducible, $\beta$ is $U(1)$-reducible, and $\text{gr}(\alpha, \beta) = 3$.
\item $V_4$ counts points above $[b_\alpha, q_\beta]$ when $\alpha$ is $U(1)$-reducible, $\beta$ is irreducible, and $\text{gr}(\alpha, \beta) = 4$.
\end{itemize}

The derivation of this matrix is relatively self-explanatory; the signs arise from the factor $(-1)^{\dim \sigma + \dim \alpha}$ in $d_M$. The one small subtlety is why there is no matrix element corresponding to counting points above $[b_\alpha, q_\beta]$ when both $\alpha, \beta$ are $U(1)$-reducible and $\text{gr}(\alpha, \beta) = 3$. By Proposition 5.8 (4), this is impossible: the relative grading of reducible connections is always even!

We summarize the result of the homological perturbation lemma discussion as follows.

**Corollary 6.6.** The $R$-module with differential $(DCI, pd_M)$ described above is a dg-module over $\Lambda(u)$, and is $\Lambda(u)$-equivariantly homotopy equivalent to $\overline{CI}$.

**Remark 6.1.** What we call $DCI$ was described in [Don02, Section 7.3.3] for integer homology spheres, with the notation $\overline{CF}$. Donaldson’s complex includes the data of the filtration\(^{17}\) by degree of an element in $H_+ \alpha$, as well as the action of $u \in \Lambda(u)$. In Section 7.3.2, Donaldson introduces a method for reducing the contribution of small-dimensional moduli spaces, so that this complex makes sense; his homotopy is an example of the many possible homotopies we could have used above. Thus, in particular, $\overline{CI}$ is equivariantly homotopy equivalent to Donaldson’s $\overline{CF}$.

In what follows we use the complex $DCI$ to produce small models which we write as $\overline{DCI}^\pi$ for $CI^+$ and $CI^-$. In the case of integer homology spheres, these are essentially the same as what Donaldson writes $\overline{CF}$ and $\overline{CF}$.

As described in the appendix, the reduced bar construction of an augmented dg-algebra $A$ and a right dg-module $M$ is given, as a graded $R$-module, as

$$B(M, A, R) := \bigoplus_{n=0}^\infty M \otimes \overline{A}[1]^n,$$

where $\overline{A} = \ker \varepsilon$ and $\varepsilon$ is the augmentation. For us, $A = \Lambda(u)$, and $\overline{A}$ is just a copy of $R$ concentrated in degree 3. Therefore, we may write

$$B(M, A, R) = \bigoplus_{n=0}^\infty M[4n].$$

We write this as $M \otimes_R R[U^*]$, where $|U^*| = 4$.\(^{18}\) The differential $d^+$ is seen to be

$$d^+ (m \otimes (U^*)^n) = um \otimes (U^*)^{n-1} + (-1)^n m \otimes (U^*)^n,$$

where $(U^*)^{-1}$ should be interpreted as zero. Write $U^\text{alg}_+$ for the operator with

$$U^\text{alg}_+ (m \otimes (U^*)^n) = (-1)^n um \otimes (U^*)^{n-1};$$

\(^{17}\)Thinking of $DCI$ as a $\mathbb{Z}/8 \times \mathbb{Z}$-graded multicomplex with $d_0 = 0$, this is the filtration corresponding to the $\mathbb{Z}$ factor, while ours is the periodic filtration corresponding to the $\mathbb{Z}/8$ factor.

\(^{18}\)We choose the notation $U^*$ here because $U$ is reserved for the degree $-4$ operation coming from $H^{-4}(BSO(3); R)$.\)
then, we apply this to a \( \mathbb{Z}/8 \)-graded module \( M \) (with finite basis of finite-dimensional \( \Lambda(u) \)-modules degreewise isomorphic to \( R \)) as in Section A.8, we need to complete; in particular,

\[
CI^+ = \prod_{n \geq 0} \widehat{CI}[4n] \cong \widehat{CI} \otimes_R R[U^*] = \widehat{CI} \otimes_R R[U^*].
\]

The action of \( R[U] \subset C_{\Lambda(u)}^- \) is such that \( U \cdot (U^*)^k = (U^*)^{k-1} \); we must restrict to this subalgebra as the action of the entire algebra does not extend to the completion. We denote \( DCI^+ = C_{\Lambda(u)}^+(DCI) \), equipped with its action of \( R[U^*] \).

By convention, if \( A \) is an operator from one summand of \( DCI \) to another, then we use \( A \) to also refer to the corresponding map between summands of \( DCI^+ \) with

\[
A(x \otimes (U^*)^n) = (-1)^n Ax \otimes (U^*)^n.
\]

Then the differential of

\[
DCI^+ \cong (C_{\ast}^{irr} \oplus C_{\ast}^{irr}[3] \oplus C_{\ast}^{U(1)} \oplus C_{\ast}^{U^2(1)}[2] \oplus C_{\ast}^0) \otimes_R R[U^*]
\]

is given by the following matrix.

\[
\partial^+_\text{DCI} := \begin{pmatrix}
-\hat{\partial}_1 & 0 & 0 & 0 & 0 \\
U_{\text{alg}}^+ - U_{\text{Fl}} & \hat{\partial}_1 & V_4 & V_2 & D_2 \\
-\hat{V}_1 & 0 & 0 & 0 & 0 \\
-\hat{V}_3 & 0 & 0 & 0 & 0 \\
-\hat{D}_1 & 0 & 0 & 0 & 0
\end{pmatrix}.
\]

Now we apply the following analogue of [SS10, Lemma 5] to our very similar situation. First, observe that the map \( U_{\text{Fl}} + U_{\text{alg}} \) defines an isomorphism

\[
U^*C_{\ast}^{irr}[U^*] \to C_{\ast}^{irr}[3][U^*];
\]

here we have used the completeness in \( U^* \). The inverse is explicitly given by \( \sum_{n \geq 0} c_n(U^*)^n \to \sum_{n \geq 0} d_n(U^*)^{n+1} \), where

\[
d_n = \sum_{j=0}^{n} (-1)^{j+n(n-1)/2+(n-j)(n-j-1)/2} U_{\text{Fl}}^j c_{n-j}.
\]

We write

\[
P^+ \sum_{n \geq 1} c_n(U^*)^n = \sum_{n \geq 0} d_n(U^*)^{n+1}
\]

for this inverse.

Consider the projection map \( \epsilon_1 : DCI^+ \to C_{\ast}^{irr}[U^*][3] \); we let

\[
\epsilon = (\epsilon_1, \epsilon_1 \partial^+) : DCI^+ \to C_{\ast}^{irr}[U^*][3] \oplus C_{\ast}^{irr}[U^*][2].
\]

The codomain is given the differential \( \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \), and \( \epsilon \) is a chain map; in fact, it is surjective, as \( (x, y) \) is in the image of \( (P^+ (y - \hat{\partial}_1 x), x) \in DCI^+ \). As the codomain is acyclic, inclusion \( \ker(\epsilon) \to DCI^+ \) is a quasi-isomorphism.

Write

\[
\overline{DCI^+} = C_{\ast}^{irr} \oplus \left( C_{\ast}^{U(1)} \oplus C_{\ast}^{U^2(1)}[2] \oplus C_{\ast}^0 \right) \otimes_R R[\mathbb{Z}/8]
\]
as a graded $R$-module. There is also a projection $\pi : DCI^+ \rightarrow DCI^+$, and the composite $\pi : \ker \epsilon \rightarrow DCI^+$ is an isomorphism of $R$-modules. We wish to describe the differential (and $U$-action) on $DCI^+$, pulling back via this isomorphism. The differential is written $(\pi i)\partial^+(\pi i)^{-1}$, and similarly $U$-map (whose degree is $-4$) is $(\pi i)U(\pi i)^{-1}$.

Explicitly, $(\pi i)^{-1}x$ is the unique element $y \in DCI^+$ so that neither $y$ nor $\partial^+y$ have components in $C_{\ast}^{U}[U][3]$. If
\[x = (x_0, x_1, x_2, x_3) \in C_{\ast}^{\text{irr}} \oplus \left( C_{\ast}^{U(1)} \oplus C_{\ast}^{U(1)}[2] \oplus C_{\ast}^{\text{irr}} \right) \otimes_R R[U^*],\]
then
\[(i\pi)^{-1}(x) = (x_0 - P^+(-U_{\text{Fl}}x_0 + V_4x_1 + V_2x_2 + D_2x_3), 0, x_1, x_2, x_3) \in \left( C_{\ast}^{\text{irr}} \oplus C_{\ast}^{\text{irr}}[3] \oplus C_{\ast}^{U(1)} \oplus C_{\ast}^{U(1)}[2] \oplus C_{\ast}^{\text{irr}} \right) R[U^*].\]

Applying $\partial^+$, projecting onto $DCI^+$, and using that $\partial_1 P^+x$ is in $\ker(\epsilon)$, we find that the desired matrix is
\[\partial^+_{DCI} := \begin{pmatrix}
-\partial_1 & 0 & 0 & 0 \\
n-V_1P^+U_{\text{Fl}} - V_1 & V_1P^+V_4 & V_1P^+V_2 & V_1P^+D_2 \\
n-V_3P^+U_{\text{Fl}} - V_3 & V_3P^+V_4 & V_3P^+V_2 & V_3P^+D_2 \\
n-D_1P^+U_{\text{Fl}} - D_1 & D_1P^+V_4 & D_1P^+V_2 & D_1P^+D_2
\end{pmatrix}\]

A remarkable number of these terms are zero for degree reasons. Recall from Proposition 4.33 that the relative grading of any pair of reducible orbits is even. Now, $D_i$ and $V_i$ have degree of the same parity as $i$, and all terms in the infinite sum $P^+x$ have the same grading as $x$ modulo 4; therefore the entries not in the left column are all of odd degree, and hence identically zero! Therefore, we have

\[\partial^+_{DCI} = \begin{pmatrix}
-\partial_1 & 0 & 0 & 0 \\
n-V_1P^+U_{\text{Fl}} - V_1 & 0 & 0 & 0 \\
n-V_3P^+U_{\text{Fl}} - V_3 & 0 & 0 & 0 \\
n-D_1P^+U_{\text{Fl}} - D_1 & 0 & 0 & 0
\end{pmatrix}\]

Similarly, the action of $U \in H^{-4}(BSO(3); R)$ was previously contraction against $U^*$; using $\sim$ to denote contraction by $U$, the action $U^*$ on $DCI^+$ is given by
\[\begin{pmatrix}
U_{\text{Fl}} & -V_4 & -V_2 & -D_2 \\
0 & U & 0 & 0 \\
0 & 0 & U & 0 \\
0 & 0 & 0 & U
\end{pmatrix} U^* .\]

In the case that $Y$ is an integer homology sphere and $\mathbb{Q} \subset R$, up to a change of basis (just a scaling of each coordinate) the chain complex $DCI^+(Y; R)$ is identical as a $U$-module to Donaldson’s $\overline{CP}(Y; R)$. 
Now we may write
\[
\text{DCI}^- = \left( C_*^{\text{irr}} \oplus C_*^{\text{irr}}[3] \oplus C_*^{U(1)} \oplus C_*^{U(1)}[2] \oplus C_*^0 \right)[U],
\]
with differential
\[
\partial^-_{\text{DCI}} := \begin{pmatrix}
-\partial_1 & 0 & 0 & 0 & 0 \\
U_{\text{alg}}^- - U_{F1} & \partial_1 & V_4 & V_2 & D_2 \\
-V_1 & 0 & 0 & 0 & 0 \\
-V_3 & 0 & 0 & 0 & 0 \\
-D_1 & 0 & 0 & 0 & 0
\end{pmatrix}.
\]

Unlike in the case of \( \text{DCI}^+ \), the operators \( \partial_1, V_i, D_i \) do not pick up an extra sign when acting on \( C[U] \). For \( x \in C_*^{\text{irr}} \), we define the operator
\[
U_{\text{alg}}^- (x \otimes U^k) = (-1)^{|x|+k}x \otimes U^{k+1} \in C_*^{\text{irr}}[3][U];
\]
this is the only sign in the above differential hidden in the notation.

Now
\[
U_{F1} + U_{\text{alg}}^- : C_*^{\text{irr}}[U] \to (C_*^{\text{irr}}[3])[U]/(1 \cdot C_*^{\text{irr}}[3])
\]
is an isomorphism; it is crucial here that we are working with a polynomial ring and not a power series ring. The inverse, \( P^- \), is given on a basis element by
\[
P^- (x \otimes U^{n+1}) = \sum_{i=0}^{n} (-1)^s(n,i,|x|)U_{F1}^{n-i}x \otimes U^i,
\]
where the exponent \( s(n,i,|x|) = n(n+1)/2 - i(i-1)/2 + (n - i + 1)|x| + (n - i) \).

Now we follow [SS10] more closely, and instead of taking the kernel of a map to an acyclic complex, we quotient by an acyclic subcomplex. The subcomplex \( Z \) is spanned by \( C_*^{\text{irr}}[U] \) and its image under \( \partial_{\text{DCI}}^- \), which is an injective map. The quotient \( \text{DCI}^- / Z \) remains a \( U \)-module. If we set
\[
\overline{\text{DCI}}^- = C_*^{\text{irr}}[3] \oplus \left( C_*^{U(1)} \oplus C_*^{U(1)}[2] \oplus C_*^0 \right) \otimes_R \mathbb{R}[U]
\]
we have a natural inclusion map \( i : \overline{\text{DCI}}^- \hookrightarrow \text{DCI}^- \) so that the composite
\[
\pi i : \overline{\text{DCI}}^- \to \text{DCI}^- / Z
\]
is an isomorphism. As above, we may compute the induced differential as
\[
\overline{\partial}_{\text{DCI}}^- = \begin{pmatrix}
-\partial_1 & V_4 + U_{F1}(P-V_4)_{U=0} & V_2 + U_{F1}(P-V_2)_{U=0} & D_2 + U_{F1}(P-D_2)_{U=0} \\
0 & V_1P-V_4 & V_1P-V_2 & V_1P-D_2 \\
0 & V_3P-V_4 & V_3P-V_2 & V_3P-D_2 \\
0 & D_1P-V_4 & D_1P-V_2 & D_1P-D_2
\end{pmatrix}.
\]

Here the term \( V_4 \) in the topmost row, and similarly with \( V_2 \) and \( D_2 \) in that same row, denotes the composite
\[
C_*^{U(1)}[U] \xrightarrow{U=0} C_*^{U(1)} \xrightarrow{V_4} C_*^{\text{irr}}[3];
\]
in the bottom right $3 \times 3$ block matrix, there is no projection component to the terms $V_4, V_2,$ or $D_2$. When we write for instance $(P^{-}V_4)_{U=0}$, we mean the composite

$$ C^{\infty}(1)[U] \xrightarrow{P^{-}V_4} C^{\text{irr}}[3][U] \xrightarrow{U=0} C^{\text{irr}}[3]; $$

that is, apply $P^{-}V_4$ then project to the constant term.

Again, most of these terms are zero for degree reasons, and we may in fact write

$$ \overline{\partial}_{DCI} = \begin{pmatrix}
-\partial_1 & V_4 + U_{\text{Fl}}(P^{-}V_4)_{U=0} & V_2 + U_{\text{Fl}}(P^{-}V_2)_{U=0} & D_2 + U_{\text{Fl}}(P^{-}D_2)_{U=0} \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 
\end{pmatrix}. $$

Finally, the action of $U$ is given on $\overline{\partial}_{DCI}$ by

$$ \begin{pmatrix}
(-1)^{|x|+1} U_{\text{Fl}} & 0 & 0 & 0 \\
(-1)^{|x|+1} V_1 & U & 0 & 0 \\
(-1)^{|x|+1} V_3 & 0 & U & 0 \\
(-1)^{|x|+1} D_1 & 0 & 0 & U 
\end{pmatrix}. $$

Again, when $Y$ is an integer homology sphere and $\mathbb{Q} \subset R$, the chain complex $\overline{\partial}_{DCI}^+(Y; R)$ is identical as a $U$-module to Donaldson’s $\overline{\partial}_{DCI}^-(Y; R)$ up to scaling of basis.

When $(Y, E)$ is equipped with an admissible bundle, there are no reducible connections; in this case, the equivalence of $CI^+(Y, E; R)$ with $\overline{\partial}_{DCI}^+(Y, E; R)$ as a $R[U]$-module (and the same for the minus flavor) immediately gives is the following.

**Corollary 6.7.** When $\frac{1}{2} \in R$, the equivalence $I^\pm(Y, E) \cong I_+(Y, E)$ for an admissible bundle $(Y, E)$ takes the $U$-map to the $U$-map up to sign.

### 6.3. Instanton Tate homology

We have the following theorems for Tate homology. Fix a ground ring $R$ with $\frac{1}{2} \in R$; then we have a canonical isomorphism $H_{SO(3)}(R) \cong R[U]$ and a quasi-isomorphism $C_{\text{ad}}(SO(3); R) \cong \Lambda(u) := \Lambda,$ where $\Lambda(u)$ denotes the exterior algebra on a degree 3 generator $u$.

**Proposition 6.8.** The action of $U^\ast$ is an isomorphism on $I^\ast(Y, E; R)$. In fact, the map $I^-(Y, E; R) \to I^x(Y, E; R)$ may be identified with the localization

$$ I^-(Y, E; R) \to I^-(Y, E; R) \otimes_{R[U]} R[U, U^{-1}]; $$

even more explicitly, we may identify the map $CI^-(Y, E; R) \to CI^x(Y, E; R)$ on the chain level using the reduced Donaldson model as

$$ \overline{\partial}_{DCI}^-(Y, E; R) \to \overline{\partial}_{DCI}^-(Y, E; R) \otimes_{R[U]} R[U, U^{-1}]. $$

**Proof.** We follow much the same lines as in Proposition A.18 in the Z-graded case, and work with the Donaldson model $\overline{\partial}_{DCI}^−$. First, to see that the action of $U$ on $I^x$ is invertible, we look at the spectral sequence corresponding to the index filtration of $CI^x$. The $E^1$ page is identified with a direct sum of copies of

$$ H^x_{\Lambda}(\Lambda), \ H^x_{\Lambda}(R \oplus R[2]); \ H^x_{SO(3)}(R); $$
the first corresponds to irreducible orbits, the second corresponds to $SO(2)$-reducibles and is just a sum of two copies of $H^*_{\Lambda} (R)$, and the last corresponds to full reducibles. In all cases, the action of $U$ is an isomorphism on each factor: that the first group is zero is one of the axioms of Tate homology, and that the second two have invertible $U$ action is precisely the calculation of Lemma A.17. Because the index filtration is complete, a map which is an isomorphism on the $E^1$ page is an isomorphism on homology, and so the action of $U$ is invertible in $I^E(Y, E; R)$.

Given a map $f : M \to M$ of degree $k$ of a dg-module, there is a definition given before Proposition A.18 of a chain complex $M[f^{-1}]$: it is the mapping cone of $M[t] \xrightarrow{1-tf} M[t]$, where $|t| = -k$. This is defined so that $H(M[f^{-1}]) = H(M)[f^{-1}]$, the latter notation meaning the usual module-theoretic sense of inverting an element or map.

If one applies this to define $CI^E(Y, E; R)[U^{-1}]$, one finds that the index filtration is no longer complete, even though the index filtration is complete on $CI^E$. One may then pass to the completion of $CI^E(Y, E; R)[U^{-1}]$: this is the filtered complex that the spectral sequence actually computes the homology of (and the $E^1$ pages of the spectral sequence for a filtered complex and its completion are the same). Precisely, if $F_s M$ is a filtered complex, the completion is

$$
\hat{M} = \lim_{p \to \infty} M/F_{-p}M.
$$

We write this completion as $CI^E(Y, E; R)[U^{-1}]$. There is a natural map

$$
CI^E(Y, E; R) \to CI^E(Y, E; R)[U^{-1}]
$$

which we know to be an isomorphism on the $E^1$ page, because the action of $U^*$ is an isomorphism on the $E^1$ page of $CI^E(Y, E; R)$.

Because completion is natural, we have maps

$$
CI^{-}(Y, E; R)[U^{-1}] \to CI^E(Y, E; R)[U^{-1}];
$$

again, the computations of Lemma A.17 imply that this is an isomorphism on the $E^1$ page, and because these filtrations are complete, the same is true at the level of homology.

To conclude we need to find a quasi-isomorphism

$$
CI^{-}(Y, E; R)[U^{-1}] \simeq DCT^{-}(Y, E; R) \otimes_{R[U]} R[U, U^{-1}].
$$

This follows again via naturality of completion: there is a canonical map

$$
DCT^{-}(Y, E; R)[U^{-1}] \to DCT^{-}(Y, E; R) \otimes_{R[U]} R[U, U^{-1}]
$$

of filtered complexes for which the filtration is not complete (remember that the first term is defined to be the mapping cone of a certain map on $DCI^{-}[t]$). The same argument as before shows this map is an isomorphism on the $E^1$ page, so the corresponding map on completions

$$
DCT^{-}(Y, E; R)[U^{-1}] \to DCT^{-}(Y, E; R) \otimes_{R[U]} R[U, U^{-1}]
$$

is a quasi-isomorphism. We conclude by observing that we may explicitly identify the completion as

$$
DCT^{-}(Y, E; R) \otimes_{R[U]} R[U, U^{-1}] = \overline{DCT^{-}}(Y, E; R) \otimes_{R[U]} R[U, U^{-1}],
$$

using the fact that $\overline{DCT^{-}}$ is a finite direct sum of copies of $R$ and $R[U]$. ■
Corollary 6.9. Let \((Y, E)\) be a rational homology 3-sphere with \(U(2)\)-bundle, and write \(c = c_1(E) \in H^2(Y; \mathbb{Z})\). Then there is a natural isomorphism
\[
I^*(Y, E) \cong R[U^{1/2}, U^{-1/2}] \otimes_{R[\mathbb{Z}/2]} R[H^2Y].
\]
Here the first term is Laurent series in \(U^{-1/2}\), and the action of \(\mathbb{Z}/2\) is \((-1)^k U^{k/2} = (-1)^k U^{k/2}\), while the action on \(R[H^2 Y]\) is given by \((-1) \cdot e^{z} = e^{c-z}\).

Proof. This is immediate from the computation of the differential on \(\mathcal{DCT}^-\) in the previous section: there is no component of the differential going from the reducible part of \(\mathcal{DCT}^-\) back to itself. At the same time, the reducible part is the only thing that survives to \(\mathcal{DCT}^- \otimes_{R[U]} R[U, U^{-1}]\). Therefore the differential on this is identically zero, and the graded group
\[
\mathcal{DCT}^- \otimes_{R[U]} R[U, U^{-1}]
\]

itself calculates the Tate homology.

What we see, then, is that \(I^*(Y, E; R)\) is a direct sum of copies of \(R[U, U^{-1}]\), one for each full reducible and two for each \(SO(2)\)-reducible (one of the copies shifted up by 2). Now, the content of Proposition 5.8 (2) is that pairs \(\{z_1, z_2\}\) with \(z_1 + z_2 = c\) and \(z_1 \neq z_2\) are in bijection with the \(SO(2)\)-reducible critical orbits, and that the pairs with \(z_1 = z_2\) are in bijection with the full reducibles.

Contributing two copies of \(R[U, U^{-1}]\), one shifted up by degree 2, is equivalent to contributing a single copy of
\[
R[U^{1/2}, U^{-1/2}] = R[U, U^{-1}] \oplus U^{-1/2} R[U, U^{-1}],
\]
recalling here that \(|U^{-1/2}| = 2\).

The description in the corollary arises from the observation that we may give this a more succinct description, only in terms of the cohomology group \(H^2(Y)\) and the class \(c_1 E\). The point is that
\[
R[U^{1/2}, U^{-1/2}] \otimes_{R[\mathbb{Z}/2]} R[H^2Y]
\]
may be described explicitly: for a pair \(\{z_1, z_2\}\) with \(z_1 + z_2 = c\) and \(z_1 \neq z_2\), the involution swaps the two corresponding copies of \(R[U^{1/2}, U^{-1/2}]\), and therefore in the quotient we have identified these two towers; so we have a copy of \(R[U^{1/2}, U^{-1/2}]\) for each such pair. If, on the other hand, \(z_1 = z_2\), the action identifies
\[
U^{k/2} \otimes e^{z_1} \sim (-1)^k U^{k/2} \otimes e^{c-z_1} = (-1)^k U^{k/2} \otimes e^{z_1}.
\]
Because 2 is invertible in \(R\), we see that this kills precisely the terms of the form \(U^{k/2}\); so such a pair contributes a copy of \(R[U, U^{-1}]\).

This is precisely the description given above as
\[
\mathcal{DCT}^- (Y, E, \pi; R) \otimes_{R[U]} R[U, U^{-1}].
\]

6.4. Examples of the \(I^*\) and the index spectral sequence.

Example 6.2. Let \(Y = S^3\) equipped with the trivial bundle. Then \(\hat{CI}(Y; R) = R\) concentrated in degree 0. Therefore, \(CI^+(S^3; R)\) is given as the completion \(\hat{C}_k(BSO(3); R)\); in any fixed integer degree, \(\hat{C}_k(BSO(3); R) = C_k(BSO(3); R)\). (It
is only when considering the whole of $\hat{C}_*(BSO(3); R)$ that we notice a difference; this is necessary when discussing $I^+$, which is only $\mathbb{Z}/8$-graded.) In particular,
\[ I_+^+(S^3; R) \cong \hat{H}_*(BSO(3); R). \]
Similarly,
\[ I_-^-(S^3; R) \cong H_{\text{fin}}^-(BSO(3); R), \]
meaning that we take cohomology classes which correspond to functionals of finite support; that is, we do not complete in the direction of negative degrees. Finally,
\[ I^\infty(S^3; R) = \hat{H}_*(BSO(3); R)[3] \oplus H_{\text{fin}}^-(BSO(3); R). \]
The calculations of these groups, and their module structures, is given in Example A.2 for $\frac{1}{2} \in R$ and $R = \mathbb{Z}/2$.

We are presented with a dichotomy. When $\frac{1}{2} \in R$, we see that
\[ I_+^+(S^3; R) \cong R[U^{-1}], \]
\[ I_-^-(S^3; R) \cong R[U], \]
\[ I^\infty(S^3; R) \cong R[U, U^{-1}], \]
all as $R[U]$-modules; this periodicity calculation in Tate homology is the content of Lemma A.17, and the other two are straightforward calculations using the definitions of bar and cobar constructions.

However, when 2 is not invertible, we do not have such a periodicity. The most dramatic case is when $R = \mathbb{Z}/2$; in that case,
\[ I_+^+(S^3; R) = (\mathbb{Z}/2)[u_2^*, u_3^*], \]
and
\[ I_-^-(S^3; R) = (\mathbb{Z}/2)[w_2, w_3]. \]
Then $I^\infty(S^3; R)$ looks like a ‘bi-infinite cone’: the Tate homology $H^\infty_{SO(3)}(\mathbb{Z}/2)$ has rank growing roughly linearly in the degree $|k|$. In fact, the action of $H_{\text{fin}}^-(BSO(3); \mathbb{Z}/2)$ on
\[ \hat{H}_*(BSO(3); \mathbb{Z}/2) \subset H^\infty_{SO(3)}(\mathbb{Z}/2) \]
is nilpotent on each element, as opposed to having an element that induces a periodicity isomorphism $I^\infty(S^3) \to I^\infty(S^3)$!

This suggests that in the case of $R = \mathbb{Z}/2$, it would be more appropriate to localize $\tilde{SI}(BSO(3); \mathbb{Z}/2)$ to some specific exterior algebra $\Lambda(u_1)$ or $\Lambda(u_2)$ in $C_*(SO(3); \mathbb{F}_2)$. Mostly we will content ourselves with the case $\frac{1}{2} \in R$.

**Example 6.3.** If $p > q$ are coprime integers (not necessarily prime), let $L(p, q)$ be the lens space, which by our convention is the quotient of $S^3$ by the $\mathbb{Z}/p$ action generated by $[1] \cdot (z, w) = (e^{2\pi i/p}z, e^{2q\pi i/p}w)$.

First, we work with the trivial $U(2)$-bundle; the set of critical orbits now correspond to
\[ \text{Hom}(\pi_1, SU(2))/\sim = \text{Hom}(\mathbb{Z}/p, SU(2))/\sim, \]
where the equivalence elation is conjugacy in $SU(2)$. Because $\mathbb{Z}/p$ is abelian, and simultaneously commuting matrices are simultaneously diagonalizable, this is the same as $\text{Hom}(\mathbb{Z}/p, S^1)/\text{conj}$; here conj is complex conjugation. Identify $\text{Hom}(\mathbb{Z}/p, S^1)$ with the $p$th roots of unity, and hence with $\mathbb{Z}/p$ again after fixing the generator $e^{2\pi i/p}$; thus reducibles correspond to $(\mathbb{Z}/p)/\pm 1 = [0, p/2]$, and
full reducibles — those fixed by the conjugation action, which is ±1 on \( \mathbb{Z}/p \) — correspond to \([0]\) for any \( p \) and \([p/2]\) when \( p \) is even. The \( SO(2) \)-reducibles correspond to \( 0 < i < p/2 \). Finally, 0 corresponds to the trivial connection.

Austin calculates in [Aus90] the expected dimension of the different components of the moduli space of unframed instantons (before quotienting by the translation action) between two flat connections on \( L(p, q) \), and in particular the expected dimension of \( \mathcal{M}(L(p, q), 0, i) \); the expected dimension of our \( \overline{\mathcal{M}}(L(p, q), \theta, \alpha_i) \) is \( \text{dim } SO(3) - \text{dim } \mathbb{R} = 2 \) dimensions larger; the index \( \text{gr}(\alpha_i) = \text{gr}(\theta, \alpha_i) \) differs from this dimension by subtracting \( \text{dim } \alpha_i \) and adding 1. Thus we should add either 3 or 1 to Austin’s result; we add 3 in the case that \( i \) is either \([0]\) or \([p/2]\), and we add 1 otherwise. Set \( \varepsilon(i) = 1 \) if \( 0 < i < p/2 \) and \( \varepsilon(0) = \varepsilon(p/2) = 0 \), and write \( 0 < q' < p \) for the unique integer with \( qq' = ap + 1 \).

Then the grading function \( \delta(p, q, i) = \text{gr}(L(p, q), i) \in \mathbb{Z}/8 \) is given by

\[
\delta(p, q, i) = \frac{8i^2q'}{p} - \varepsilon(i) + \frac{2}{p} \sum_{j=1}^{p-1} \cot\left(\frac{j\pi}{p}\right) \cot\left(\frac{jq\pi}{p}\right) \sin^2\left(\frac{2i\pi}{p}\right) \pmod{8},
\]

Sasahira [Sas13, Corollary 5.3] has given formulas for the \( \delta(p, q, i) \) in terms of counting solutions to congruences.

A few observations about this complicated-looking function are in order:

- We have \( \delta(p, q, 0) = 0 \), as we should.
- If \( p \) is even, \( \delta(p, q, p/2) = 4q' \cdot \frac{p}{2} \); because \( q' \) must be odd, \( \delta(4k, q, 2k) = 0 \) and \( \delta(4k + 2, q, 2k + 1) = 4 \).
- Because the sum only depends on \( q' \)'s value modulo \( p \), and \( \cot \) is odd, when \( 0 < i < p/2 \) we have \( \delta(p, q, i) = -\delta(p, p - q, i) - 2 \). The factor of \(-2\) comes from \(-2\varepsilon(i)\). It is trivially true by the above calculations that \( \delta(p, q, i) = -\delta(p, p - q, i) \) when \( i = 0 \) or \( p/2 \).
- As Austin observes, \( \delta(p, q, i) \) is even for all \( i \).
- If \( qq' = ap + 1 \), then \( \delta(p, q, qi) = \delta(p, q', i) \). Here we are considering \( d \) as a function from the integers, but notice that \( \delta(p, q, i) = -\delta(p, q, i) \) and \( \delta(p, q, i + p) = \delta(p, q, i) \). Then the claimed equality follows because the summation in \( \delta(p, q', i) \) is just the summation in \( \delta(p, q, qi) \), but with index \( jq' \) instead of \( j \); as \( xq' : ((\mathbb{Z}/p)\setminus0) \to ((\mathbb{Z}/p)\setminus0) \) is a bijection, and \( j \)'s value modulo \( p \) is all that’s relevant, the sum is therefore the same.

It seems plausible that \( \delta(p, q, ki) = \delta(p, q', i) \) for all \( i \) if \( k = 1 \) and \( q' = q \) or \( k = q \) and \( q' = q^{-1} \pmod{p} \); calculation shows that this is true at least for \( p \leq 15 \). This would be consistent with the classification of lens spaces up to oriented homeomorphism. However, we do not know a proof.

As for the differential, we saw in Lemma 6.4 that if we coherently homotope the endpoint maps of the compactified moduli spaces, the resulting chain complex is quasi-isomorphic to the original. The discussion following that ensures that between reducible orbits \( \alpha \) and \( \beta \), the endpoint maps may be chosen so that the corresponding fiber product maps \( \times_\alpha \overline{\mathcal{M}}_{\alpha, \beta} \) are identically zero on the chain level. While Lemma 6.5 proved this for the \( SU(2) \)-model (which may only be applied for coefficient rings with \( \frac{1}{2} \in R \)), this passage to a double cover is only necessary for the argument when considering the irreducible orbits; here, we have none.
Thus, as a dg $C_\bullet(SO(3))$-module, and with $L(p, q)$ equipped with the trivial $SO(3)$-bundle, we see that we have a quasi-isomorphism
\[
\tilde{CI}(L(p, q); R) \simeq \begin{cases} R \otimes_{\mathbb{Z}[U]} \tilde{C}_p(S^2; R)[\delta(p, q, i)] & \text{if } p \text{ odd} \\ R \otimes_{\mathbb{Z}[U]} \tilde{C}_p(S^2; R)[\delta(p, q, i)] \oplus R[\delta(p, q, p/2)] & \text{otherwise} \end{cases}
\]

Thus to write down $I^\bullet$ (where $\bullet = +, -, \infty$) we use Theorem A.26, which says that the equivariant homology groups of an orbit are given by the homology groups of the stabilizer (with a degree shift) as long as the stabilizer is connected. So to take $I^\bullet$, we replace every appearance of $C_\bullet(S^2; R)$ with $H_{SO(2)}^\bullet(R)$, with a degree shift of 2 if $\bullet \in \{-, \infty\}$, and every appearance of $R[d]$ with $H_{SO(2)}^\bullet(R)[d]$. When $\frac{1}{2} \in R$, we have isomorphisms as $R[U] = H_{SO(3)}(R)$-modules
\[
H_{SO(2)}^+(R) \cong R[U^{1/2}]/U^{1/2} R[U^{1/2}]
\]
\[
H_{SO(2)}^-(R) \cong R[U^{1/2}]
\]
\[
H_{SO(2)}^0(R) \cong R[U^{1/2}, U^{-1/2}]\]

If $p$ is even, $L(p, q)$ carries a unique nontrivial $SO(3)$-bundle; choose a lift of this to a $U(2)$-bundle. We can identify the classes now with $\{a + \xi \}/ \pm 1$, where $\xi \in \mathbb{Z}/p$ is a choice of odd number (the choice of $U(2)$ lift). This, then, may be identified with the odd numbers in $\mathbb{Z}/2p$ modulo $\pm 1$; no points are fixed, so all are $SO(2)$-reducibles. So we label the reducibles by $i = 1, 3, \cdots, p-1$; there are $p/2$ of them. Their relative grading is also given in [Aus90], now as
\[
\delta(p, q, i, i') = \frac{2(i^2 - (i')^2)q'}{p} - 3 + \frac{2}{p} \sum_{j=1}^{p-1} \cot \left( \frac{\pi j}{p} \right) \cot \left( \frac{j \pi}{p} \right) \cot \left( \frac{jq \pi}{p} \right) \left( \sin^2 \left( \frac{i' \pi}{p} \right) - \sin^2 \left( \frac{i \pi}{p} \right) \right) \mod 8.
\]
Again, this is always even.

The result is here independent of the choice of $U(2)$-lift of the underlying $SO(3)$-bundle because there are no differentials, and thus we may ignore the orientation of the moduli spaces; in fact, we expect this in general (up to noncanonical isomorphism, the noncanonicity coming from sign choices).

**Example 6.4.** Suppose $Y$ is an integer homology sphere and $E$ is the trivial bundle. We may use the results of Section 6.2 to determine the index spectral sequences explicitly.

We write $CI_\bullet(Y)$ for the chain complex $(C^{\text{int}}_\bullet, \partial_1)$, Floer’s original chain complex for integer homology spheres. Donaldson introduced in [Don02, Section 7.1] the complex $\overline{CI}(Y)$, given as $C^{\text{int}}_\bullet \oplus R$ with differential
\[
\overline{\partial} = \begin{pmatrix} \partial_1 & 0 \\ D_1 & 0 \end{pmatrix}
\]
Corollary 6.6 shows that there is an equivariant filtered homotopy equivalence $\overline{CI}(Y) \simeq DCI(Y)$, and so to investigate the index spectral sequence computing $\overline{I}$, we may do the same for the index spectral sequence on the finite-dimensional complex $DCI(Y)$. 
We consider $DCI(Y)$; this is, as an $R$-module, $CI_*(Y) \oplus CI_*(Y)[3] \oplus R$, with differential
\[
\begin{pmatrix}
-\partial_1 & 0 & 0 \\
0 & \partial_1 & D_2 \\
-D_1 & 0 & 0
\end{pmatrix}.
\]
The differential splits as $d_1 + d_4$ into pieces which decrease the filtration the corresponding amount:
\[
d_1 = \begin{pmatrix}
-\partial_1 & 0 & 0 \\
0 & \partial_1 & 0 \\
-D_1 & 0 & 0
\end{pmatrix} \text{ and } d_4 = \begin{pmatrix}
0 & 0 & 0 \\
-U_{F1} & 0 & D_2 \\
0 & 0 & 0
\end{pmatrix}.
\]
Then the $E^2$ page of the index spectral sequence for $DCI(Y)$ is
\[(DCI(Y), d_1) = \mathcal{CI}(Y) \oplus CI(Y)[3] \oplus R,\]
and so the $E^3$ page is $\mathcal{T}(Y) \oplus I(Y)[3]$. The matrix $d_4$ defines a chain map
\[(U, D_2) : \mathcal{CI}(Y) \to CI(Y)[3];\]
writing the induced map $f : \mathcal{T}(Y) \to I(Y)$, we see that the $E^5$ page is
\[(I(Y) \oplus I(Y)[3], f).\]
Therefore, the $E^6$ page (and all successive pages, for degree reasons) is $\ker(f) \oplus \coker(f)$.

If $R$ is a field, there is nothing left to say; there are no extension problems to resolve, and $\mathcal{T}(Y) \cong \ker(f) \oplus \coker(f)$.

Now consider the $CI^+$ spectral sequence by passing through $DCT^+$. The complex $DCT^+$ is, as a graded $R$-module, given by $CI_*(Y) \oplus R[U^*]$, where $|U^*| = 4$, and the differential is given by
\[
\begin{pmatrix}
-\partial_1 & 0 \\
-D_1 - D_1P^+U_{F1} & 0
\end{pmatrix},
\]
where recall that by definition when $x \in CI_*(Y)$, we have
\[P^+x = \sum_{i \geq 0}(-1)^{(i+1)/2}U_{F1}^i x(U^*)^{i+1}.
\]
A similar definition with slightly different signs remains true for
\[x \in CI_*(Y)[3] \otimes (U^*)^k.
\]
We decompose this differential as $d_1 + d_5 + d_9 + \cdots$, where $d_1 = \begin{pmatrix}
-\partial_1 & 0 \\
-D_1 & 0
\end{pmatrix}$
and
\[
d_{4i+1}x = \begin{pmatrix}
0 & 0 \\
(-1)^{1+i(i-1)/2}D_1U_{F1}^i x \otimes (U^*)^i & 0
\end{pmatrix}.
\]
The sign discrepancy arises from the definition $D_1(x \otimes (U^*)^k) = (-1)^kD_1x \otimes (U^*)^k$.
Now this is a multicomplex in the sense of Wall, and we may identify the $E^2$ page as $\mathcal{T} \oplus U^*R[U^*]$.
The map $d_5$ defines a homomorphism $\mathcal{T} \rightarrow U^* R$; write $\mathcal{T}_2$ for its kernel. Inductively, there is a homomorphism
\[ d_{4k+1} : \mathcal{T}_k \rightarrow (U^*)^k R, \]
and we write its kernel as $\mathcal{T}_{k+1}$. We may identify the $q = 0$ line in $E^{4k-2}$ page with $\mathcal{T}_k$ (identifying the successive differentials with zig-zags of the differentials $d_{4k+1}$, and using that any compositions of the differentials are zero). In particular, we identify the $q = 0$ line in the $E^{\infty}$ page with
\[ I_{\infty} = \bigcap \ker(D_1 U_{k+1}) \subset \mathcal{T}. \]

A similar discussion presents itself for the $CI^{-}$ spectral sequence, passing through $DCI^{-}$. As a graded $R$-module,
\[ DCI^{-3}(Y,E) = CI_3(Y,E)[3] \oplus R[U]; \]
thinking of an element of $R[U]$ as a polynomial $f$, we write $f_k$ for the coefficient of $U^k$; this has differential
\[ \partial^- \left( \begin{array}{cc} \partial_1 & D_2 f_0 - \sum_{k \geq 1} (-1)^{k(k-1)/2} U_{k+1} D_2 f_k \\ 0 & 0 \end{array} \right). \]
The top-right term means that $\partial^- (1) = D_2 1$ and
\[ \partial^- (U^k) = (-1)^{k(k-1)/2} U_{k+1} D_2 (1). \]

As before, we may split this into pieces $d_1 + d_5 + d_9 + \cdots$. The $E^1$ page is given as $CI_3[3] \oplus R$ with differential
\[ \left( \begin{array}{cc} -\partial_1 & D_2 \\ 0 & 0 \end{array} \right), \]
which (shifted by 3 and with a sign change) Donaldson calls $CI(Y)$. The homology, written $I$, is the $q = 0$ page of this spectral sequence. The differential $d_5$ (which is, up to sign, $U_{k+1} D_2$) determines a homomorphism $R \rightarrow I_2$; its cokernel might be written $I^{(2)}$, and may be identified with the $E^6$ page. Inductively, the differential $d_{4k+1}$ defines a homomorphism $R \rightarrow I^{(k)}$, and its cokernel is written $I^{(k+1)}$, and may be identified with the $E^{4k+2}$ page.

If $\Lambda$ is the exterior algebra on a generator of degree 3, the norm map is an isomorphism
\[ H^+_\Lambda(\Lambda)[3] \rightarrow H^-_\Lambda(\Lambda) \]
(as a corollary of item (4) of Theorem A.12), and is zero for $H^+_\Lambda(R)[3] \rightarrow H^-_\Lambda(R)$ for degree reasons: the former is supported in degrees no less than 3, and the latter is supported in nonpositive degrees.

Therefore, the induced map of the norm map on the $E^1$ page of the $I^+$ and $I^-$ spectral sequences is the identity on $C^\text{trr}$, and zero on $R[U^*]$. We thus identify the image $I_{(\infty)} \rightarrow I_{(\infty)}$ as the quotient group $I_{(\infty)}/\langle U_{k+1} D_2, 1 \rangle$, the second term in the quotient generating the reducible piece of $I_{(\infty)}$. This is Frøyshov’s reduced Floer group $\hat{I}$; the first hints of this were introduced in the $d^\alpha$ homomorphisms of [Frø95] and used to introduce the $h$-invariant in [Frø02].

\[ \text{19} \] Despite the choice of notation, these groups are not related to the Tate homology groups.
Example 6.5. For the sake of compactness of notation, we write $R_\alpha$ in this example to mean $R[\alpha]$ (a copy of $R$ in degree $\alpha$), and similarly for bigradings $R_{p,q}$. Let us see the above in practice in the explicit case of $\Sigma(2,3,5)$. In this case, we have a reducible generator of $CI$ in degree 0, as well as an irreducible generator in each pair of degrees $(1,4)$ and $(5,0)$, with the action of $u$ sending the lower degree generator to the generator 3 degrees larger. So as a $u$-module our complex is given as

$$R_0 \oplus (R_1 \oplus R_4) \oplus (R_5 \oplus R_0) = R_0 \oplus (R_1 \oplus R_5) \otimes \Lambda(u).$$

For degree reasons, $\delta_1 = 0$, as is $D_2 : \widehat{CI}_0 \to \widehat{CI}_7$. Through either the beautiful and explicit calculations of moduli spaces in [Aus95] or the general theory of [Fro02]20, we may identify that the map $D_1 : \widehat{CI}_1 \to \widehat{CI}_0$ is the identity map $R \to R$, while $U_1 : \widehat{CI}_1 \to \widehat{CI}_0$ and $U_1 : \widehat{CI}_5 \to \widehat{CI}_4$ are multiplication by one of $\pm 8$.

The $E^1$ page of the above spectral sequence is $R_{0,0} \oplus R_{1,0} \oplus R_{1,3} \oplus R_{5,0} \oplus R_{5,3}$, with multiplication-by-$u$ map taking us up 3 vertical degrees. The differential $d_1$ comprises only the map $D_1$, and therefore is the identity $R_{1,0} \to R_{0,0}$.

The $E^2$ page is therefore given by $R_{1,3} \oplus R_{5,0} \oplus R_{5,3}$. There are no differentials until the $E^5$ page, where the next differential is $U_1 : \widehat{CI}_5 \to \widehat{CI}_3$, which we know to be multiplication-by-8. Since we are working over a ground ring with $\frac{1}{2} \in R$, we see that

$$E^\infty(\widehat{CI}(\Sigma(2,3,5))) = R_{5,3};$$

because there can be no extension problems,

$$\widehat{I}(\Sigma(2,3,5)) = R_0,$$

with trivial multiplication-by-$u$ map.

For the $CI^+$ spectral sequence, instead we start with $E^1$ page equal to

$$R_{0,0}[U^*] \oplus R_{1,0} \oplus R_{5,0},$$

where $[U^*] = (0,4)$. Then our first differential $d_1$ is given as the identity map $R_{1,0} \to R_{0,0}$; the $E^5$ page is then identified with $R_{0,4}[U^*] \oplus R_{5,0}$. We may identify the differential $d_3$ with the multiplication-by-8 map $R_{5,0} \to R_{0,4}$. The $E^6$ page, then, is simply $R_{0,8}[U^*]$, and there are no further differentials. As this is concentrated in a single vertical line, there are no extension problems, and we see

$$I^+(\Sigma(2,3,5); R) \cong (U^*)^2 R[U^*],$$

with $U$-action given by contraction against $U^*$. It is clear from this that the above ‘reduced group’ is

$$\widehat{I}(\Sigma(2,3,5)) = 0.$$

In the $CI^-$ spectral sequence, on the other hand, all differentials past the $E^2$ page are identically zero, as they factor through the map $D_2$. In fact,

$$I^- (\Sigma(2,3,5); R) \cong DCT^- (\Sigma(2,3,5)) = R_0[U] \oplus (R_4 \oplus R_0)$$

as a chain complex; by the formula for the $U$-map from Section 6.2, which includes a term corresponding to $D_1$, we see that the $U$-action is given by

$$U : (U^k, x, y) = (U^{k+1} + D_1 x, 0, 0).$$

Therefore we may write $I^- (\Sigma(2,3,5); R) \cong R_4[U] \oplus R_0$ as a $U$-module, where $[U] = -4$, identifying $1 \in R_0[U]$ with $D_1 x$, where $x \in R_4$ is a generator.

20Note that Frøyshov works in cohomology, while we work in homology.
6.5. Orientation reversal and equivariant cohomology. For a dg-$\Lambda$-module $M$ equipped with a periodic filtration whose associated graded complex is bounded, we defined equivariant $\bullet$-cohomology chain complexes $C^\bullet_G(M)$, with an action of the ring $C^-_G(R) = \text{Hom}_{\text{fin}}(C^+_G(R), R)$, the negative chains of finite support.

We define the corresponding $\bullet$-cohomology, written $C^\bullet_G(M)$, to be

$$\text{Hom}(C^\bullet_G(M), R),$$

the literal dual of $C^\bullet_G$. We then define the instanton $\bullet$-cohomology, $I^\bullet$, to be $H^\bullet_{\text{Spin}(3)}(\mathcal{C}T)$, where precisely by $SO(3)$ we mean the dg-algebra $C^\bullet_{\text{spin}}(SO(3); R)$. We have the following duality theorem.

**Theorem 6.10.** Let $\frac{1}{2} \in R$. There is a canonical isomorphism of $\Lambda(u)$-modules $\hat{T}_u(Y, E) \cong \hat{T}_v(\overline{Y}, E)$, and furthermore canonical isomorphisms of $H^{-\bullet}(\text{BSO}(3); R) = R[U]$-modules

$$I^\bullet_u(Y, E) \cong I^\bullet_v(\overline{Y}, E),$$

$$I^\bullet_u(Y, E) \cong I^\bullet_v(\overline{Y}, E),$$

$$I^\bullet_u(Y, E) \cong I^\bullet_v(\overline{Y}, E),$$

**Proof.** We will show that $\text{DCI}(Y)^\vee$ is isomorphic, as a $\Lambda(u)$-module, to $\text{DCI}(\overline{Y})$. This implies the rest of the results, as one may check that at the level of $C^\bullet_A$-modules, we have $(C^+_A(D))^\vee = C^-_A(D^\vee)$ as well as $(C^-_A(D))^\vee = C^+_A(D^\vee)$. These equalities preserve the norm map, so we get the same result for Tate homology.

Before doing this calculation, recall that if $D$ is a chain complex, its dual $D^\vee$ has the differential $\delta f = (-1)^{|f|+1}f d$. This is isomorphic as an $A$-module to the chain complex with $\delta f = fd$, with isomorphism given by $f \mapsto (-1)^{|f|(|f|+1)/2}f$. (Observe that this makes sense for any chain complex graded over $\mathbb{Z}/4N$ for any integer $N \geq 0$.) In what follows, we will use the second differential on the dual.

Recall the definition of the chain complex

$$\text{DCI}(Y) = C^\text{irr}(Y) \oplus C^\text{irr}(Y)[3] \oplus C^U(1)(Y) \oplus C^U(1)(Y)[2] \oplus C^\theta(Y)$$

stated before Corollary 6.6. Passing to the dual (and slightly rewriting), we obtain

$$\text{DCI}(Y)^\vee = C^\text{irr}_{-3}(Y) \oplus C^\text{irr}_{-3}(Y)[3] \oplus C^U_{-2}(Y) \oplus C^U_{-2}(Y)[2] \oplus C^\theta_{-3}(Y),$$

with differential

$$\hat{\delta}_{CI}^\vee := \begin{pmatrix}
-\hat{c}_1 & 0 & 0 & 0 & 0 \\
-U_{V_1} & -\hat{c}_1 & V_3 & V_1 & D_1 \\
-V_2 & 0 & 0 & 0 & 0 \\
-V_4 & 0 & 0 & 0 & 0 \\
-D_2 & 0 & 0 & 0 & 0
\end{pmatrix}.$$

Now if $\pi$ is a regular perturbation on $Y$, then the same perturbation is regular on $\overline{Y}$. Using the time-reversal symmetry $\mathbb{R} \times \overline{Y} \cong \mathbb{R} \times Y$, if $A$ is a connection on the latter going from critical orbits $\alpha$ to $\beta$, then it is sent to a connection $\overline{A}$ going from $\beta$ to $\alpha$ on $\mathbb{R} \times \overline{Y}$; in particular, the indices of the corresponding deformation operators are equal: $\text{ind}(Q_{A,\pi}^\alpha) = \text{ind}(Q_{\overline{A},\pi}^\beta)$. Recalling the definition of grading from Definition 4.5, we see that

$$\text{gr}_Y(\alpha, \beta) + (\dim \alpha - \dim \beta) = \text{gr}_{\overline{Y}}(\beta, \alpha).$$
In particular, if $\beta$ is the trivial connection, we see that $i_Y^*(\alpha) = -i_Y(\alpha) - \dim \alpha$. Thus the graded basis for $DCI(Y)^\vee$ described above is precisely a graded basis for $DCI(\overline{Y})$.

Furthermore, this time-reversal symmetry gives isomorphisms

$$\mathcal{M}_{Y,z}(\alpha, \beta) \cong \mathcal{M}_{\overline{Y},-z}(\beta, \alpha).$$

Following this isomorphism, we may use the same coherent family of homotopies to define $DCI(\overline{Y})$ as for $DCI(Y)$. Once we do this, we have equalities of the operators

$$D_1(Y) \equiv D_2(\overline{Y}),$$
$$V_1(Y) \equiv V_2(\overline{Y}),$$
$$V_3(Y) \equiv V_4(\overline{Y}),$$
$$U_{fl}(Y) \equiv U_{fl}(\overline{Y}).$$

Therefore the isomorphism of graded vector spaces $DCI(Y)^\vee \cong DCI(\overline{Y})$ described above, which also preserves the $\Lambda(u)$-action, is in fact an isomorphism of chain complexes.

**Remark 6.6.** While this theorem is surely true for arbitrary principal ideal domains $R$, there are technical obstructions in proving it. One natural idea is to try to construct an equivariant pairing $\tilde{\mathcal{I}}_q(Y) \otimes \tilde{\mathcal{I}}_q(\overline{Y}) \to R$, or at least a pairing on a quasi-isomorphism $\mathcal{C}_{gm}^*(SO(3); R)$-submodule of that tensor product (one defined by the demand that the chains intersect transversely, as in [McC06]). To do this, one needs to be able to put a module structure on the tensor product. If $A$ is a dg-algebra and we wish to endow the category of $A$-modules with a tensor product structure, we require a comultiplication $\Delta : A \to A \otimes A$. While this is easy in the model of $\mathcal{C}_{gm}(SO(3); R)$ with simplicial chains (given by the Alexander-Whitney map), it seems unlikely this is possible for $\mathcal{C}_{gm}^*(SO(3))$; we have no way to cut a $\delta$-chain into canonical pieces. Perhaps a modification of this model exists that admits the structure of a bialgebra (or even a Hopf algebra), but this is not clear to the author.

A more straightforward approach is to extend the Donaldson model to work with arbitrary principal ideal domains $R$, where instead of $\mathcal{C}_{gm}(SO(3); R) \cong \Lambda(u)$, one would instead use $\mathcal{C}_{gm}(SO(3); R) \cong \mathcal{A}$, where $\mathcal{A}$ is a dg-algebra with a generator $a_i$ in each degree $0 \leq i \leq 3$, with the relations

$$d(a_2) = 2a_1, \quad a_1^2 = 0, \quad \text{and} \quad a_1a_2 = a_3.$$  

We expect that the above proof then generalizes easily, but we leave the details to an interested reader.
Appendix

A. Equivariant homology of dg-modules

Let \( G \) be a finite group acting on a topological space \( X \). We may define its equivariant (co)homology as the (co)homology of the Borel space \((X \times EG)/G\). An algebraic approach to this construction, which works more generally for any \( \mathbb{Z}[G] \)-module, replaces \( C_\ast(X) \) with a resolution in the category of chain complexes over \( \mathbb{Z}[G] \), takes the quotient by \( \mathbb{Z}[G] \), and then computes the homology (or cohomology) of its totalization.

We will want a generalization of this process that will work for compact Lie groups in some sense “acting on a chain complex” \( C \), which gives equivariant (co)homology in the case of \( G \) acting on \( C_\ast(X) \). We model this by considering dg-modules over the dg algebra \( C_\ast(G) \). After that, we will describe the dual homology theory, called coBorel homology, and a homology theory called Tate homology that compares the two. Our approach to the Borel/bar and coBorel/cobar constructions are strongly inspired by [GM74] and [BMR14], while the approach to Tate homology is essentially that of [Kle02].

We conclude with a discussion of extensions to the case of complexes graded over \( \mathbb{Z}/2N \), equipped with an appropriate object called a periodic filtration, which will be used in the main body of the text to define the equivariant instanton invariants.

A.1. Bar constructions. Here we introduce the basic constructions for modules over a differential graded algebra that we use to define homology and cohomology. From here onwards, \( R \) is a principal ideal domain (PID) which will serve as the ground ring of all of our chain complexes.

Let \( A \) be a homologically graded unital dg-algebra (over the ground ring \( R \)); so \( A \) is a chain complex of \( R \)-modules whose differential decreases grading by 1 and product on homogeneous chains satisfies the graded Leibniz rule \( d(ab) = d(a)b + \) \((-1)^{|a|}a db \). Furthermore, assume \( A \) has an augmentation \( \epsilon: A \to R \) (with \( \epsilon(1) = 1 \)); then we may identify \( \ker(\epsilon) = A/\langle 1 \rangle =: A_\ast \), and give \( R \) the natural structure of an \( A \) (bi)-module. In a differential graded left \( A \)-module \( N \), the same Leibniz rule must hold and the unit should act by the identity; for right \( A \)-modules \( M \), the sign in the Leibniz rule uses the grading of \( N \) instead of the grading of \( A \).

**Definition A.1.** The (two-sided) bar construction \( B(M, A, N) \) is the totalization of the resolution

\[
M \otimes N \leftarrow M \otimes \overline{A} \otimes N \leftarrow M \otimes \overline{A}^\otimes N \leftarrow \cdots
\]

So \( B(M, A, N) \) is given as a graded module by \( \bigoplus_{n=0}^\infty M \otimes \overline{A}[1]^\otimes n \otimes N \) (where \( \overline{A}_k = \overline{A}[1]_{k+1} \), and its differential on a generic element, written as \( m[a_1 | \cdots | a_k]n \) where possibly \( k = 0 \), is given as

\[
(-1)^k \left( dm[a_1, \cdots, a_k]n + \sum_{i=1}^{k} (-1)^{|m|+\varepsilon_i-1} m[a_1 | \cdots | da_i | \cdots | a_k]n + (-1)^{|m|+\varepsilon_k} m[a_1 | \cdots | a_k]dn \right)
+ \left( ma_1[a_2 | \cdots | a_k]n + \sum_{i=1}^{k-1} (-1)^i m[a_1 | \cdots | a_i, a_{i+1} | a_k]n + (-1)^k m[a_1 | \cdots | a_{k-1}]a_kn \right)
\]

Here \( \varepsilon_i = |a_1| + \cdots + |a_i| + i \).
These sign conventions are those of [GM74]. Gugenheim and May define \( \text{Tor}_A(M, N) \) for arbitrary pairs of dg-modules over a dga; their Corollary A.9 shows that
\[
\text{Tor}_A^A(M, N) = H_* B(M, A, N)
\]
as long as \( M, A, \) and \( N \) satisfy appropriate flatness hypotheses. We will state, and then assume, the relevant hypotheses shortly; they are there to ensure that \( B(M, A, N) \) takes quasi-isomorphisms to quasi-isomorphisms.

Most of the time we are interested in the special case \( B(M, A, R) \). If one thinks of \( A = C_*(G) \) as the canonical dg-algebra of interest, acting on the module \( C_*(X) \) for some \( G \)-space \( X \), then \( B(M, A, R) \) models the Borel construction, also known as the ‘homotopy quotient’,
\[
M_{hG} = M \times_G EG.
\]
Note that whenever \( a_k n \) appears in the differential above, this term is zero, as \( a_k \in A = \ker(\epsilon) \), acting on \( n \in R \) via the augmentation \( \epsilon \).

The bar construction is functorial (as an \( A \)-module) under maps \( f : A \rightarrow A', g : M \rightarrow M', h : N \rightarrow N' \) with \( g(ma) = g(m)f(a) \) and similarly for \( h \). It is also functorial under homotopies of such maps. Most commonly one either fixes the modules or \( A \) when using this functoriality; for a map of triples \( (M, A, N) \rightarrow (M', A', N') \), the corresponding map of bar constructions factors as
\[
B(M, A, N) \rightarrow B(M', A, N') \rightarrow B(M', A', N').
\]

There is a canonical map, natural in \( (M, A, N) \),
\[
B(M, A, N) \rightarrow B(M, A, R) \otimes B(R, A, N)
\]
given as
\[
m[a_1 \cdots a_k]n \mapsto \sum_{i=0}^{k} (-1)^{(k-i)(|m|+\varepsilon_i)} m[a_1 \cdots a_i] \otimes [a_{i+1} \cdots a_k]n.
\]
This endows \( BA := B(R, A, R) \) with the structure of a coalgebra, and \( B(M, A, R) \) the structure of a right \( BA \)-comodule.

**Definition A.2.** The cobar construction of \( (N, A, M) \), where \( A \) is a dg-algebra and both \( N \) and \( M \) are right \( A \)-modules, is the chain complex
\[
cB(N, A, M) = \text{Hom}_A(B(N, A, A), M).
\]

This, too, is a special case Gugenheim and May’s constructions; now we have \( H_* cB(N, A, M) = \text{Ext}_A^*(N, M) \) under suitable projectivity hypotheses. We will primarily be interested in the case \( N = R \).

For concreteness, we note that \( cB(N, A, M) \) is isomorphic as a graded \( R \)-module to
\[
\prod_{i=0}^{\infty} N \otimes N \text{Hom}_R(\mathbb{A}[1]^\otimes, M) = \text{Hom}_R(N \otimes BA, M),
\]
graded so that \( |a| + \eta = |\eta(a)| \). This is the negative of the usual grading (which would make \( \text{Hom}_R(C, D) \) into a cochain complex); for \( M \) bounded and \( A \) nonnegatively graded, the chain complex we define here is bounded above but unbounded in the negative direction! Passing through the above isomorphism, the differential of an element \( \eta : B(N, A, R) \rightarrow M \) of degree \( d \) is given as

\[
\sum_{i=0}^{\infty} (-1)^{|\eta(a)|+\varepsilon_i} \eta(a) |a| \varepsilon_i \otimes \eta(b) |b| \varepsilon_i.
\]
The cobar construction is functorial for triples \((M, A, N)\): it is covariant in \(M\), but contravariant in \(A\) and \(N\). The equivariance condition on morphisms is that we demand \(f : A' \to A\) and \(g : M \to M'\) satisfy \(g(mf(a')) = g(m)a'\); the condition on \(A\) and \(N\) is the usual, that \(h(n'a') = h(n')f(a')\).

Dual to equation (4),
\[
cB(R, A, R) = \text{Hom}_R(BA, R)
\]
is naturally an algebra. The chain complex \(cB(R, A, M)\) is naturally a left module over this algebra, following the diagram
\[
\text{Hom}_R(B(R, A, R), M) \otimes \text{Hom}_A(B(R, A, A), M) \to \text{Hom}_A(B(R, A, R) \otimes B(R, A, A), M) \to \text{Hom}_A(B(R, A, M), M) = cB(R, A, M).
\]

Finally, observe that \(B(M, A, R)\) is a left module over \(cB(R, A, R)\), by applying the comodule structure and then the pairing. In the following map, we suggestively rewrite \(cB(R, A, R) = BA^\vee\):
\[
BA^\vee \otimes B(M, A, R) \xrightarrow{\mu \otimes 1} BA^\vee \otimes B(M, A, R) \otimes BA \xrightarrow{\tau \otimes 1} B(M, A, R) \otimes BA^\vee \otimes BA \xrightarrow{\text{eval}} B(M, A, R).
\]

Here, and elsewhere in this text, we write \(M^\vee\) for an \(R\)-module \(M\) to be its dual as an \(R\)-module, \(\text{Hom}_R(M, R)\): if \(M\) was a right \(A\)-module, then \(M^\vee\) carries a left \(A\)-module by \((an)(m) = \eta(ma)\).

Here \(\Delta\) is the comodule structure and \(\tau\) is the swap map \(x \otimes y \mapsto (-1)^{|x||y|} y \otimes x\).

**A.2. Invariance.** Suppose we have a map of pairs \((M, A, N) \to (M', A', N')\) as above inducing a map \(B(M, A, N) \to B(M', A', N')\). When is the induced map an isomorphism on homology? Because the bar construction is functorial under homotopies, this is true if the map of triples is a homotopy equivalence (of triples). More generally, we have the following theorem. These flatness restrictions are harmless for most purposes in topology and for use in this paper, but indicate that the approach taken here is too naive for general modules over a dga. See also [BMR14, Proposition 10.18], where \(B(M, A, N)\) is shown to be a quasi-isomorphism invariant under quasi-isomorphisms of \(A\)-modules that are homotopy equivalences of \(R\)-modules. This result, and much of [GM74], is put into the powerful general framework of model categories, but which we will not need here.

**Theorem A.1.** Suppose we have a map of triples \((g, f, h) : (M, A, N) \to (M', A', N')\) so that \(f\), \(g\) and \(h\) all induce isomorphisms on homology (from here on we will say “are quasi-isomorphisms”). Suppose further that \(N, N', \overline{A}\) and \(\overline{A}'\) are flat (as graded \(R\)-modules). Then the induced map \(B(M, A, N) \to B(M', A', N')\) is a quasi-isomorphism.
Note in particular that we may take $N = R$ in the above theorem. A similar result is true of $cB(N, A, M)$ (making slightly stronger assumptions).

**Theorem A.2.** Suppose $f : A' \to A$, $g : M \to M'$, and $h : N' \to N$ are quasi-isomorphisms that are equivariant as an $(A, N)$-contravariant map of triples. If all of $A$, $A'$, $N$ and $N'$ are $R$-free, then $cB(N, A, M) \to cB(N', A', M')$ is a quasi-isomorphism.

These theorems are proved by appealing to natural spectral sequences associated to the bar and cobar constructions. Now, and later in this appendix, it’s useful to use Boardman’s language of strongly convergent and conditionally convergent spectral sequences from [Boa99]. We have slightly different assumptions and indexing than Boardman, chosen to fit with our preference for homological gradings.

**Definition A.3.** Suppose $C$ is a chain complex equipped with an increasing (possibly unbounded) filtration

$$\cdots \subset F_p C \subset \cdots \subset C.$$

We will call this a ‘filtered complex’. If $\bigcup_p F_p C = C$, then we say that $F_p C$ is an exhaustive filtration. We say that the filtration is Hausdorff if $\bigcap_p F_p C = 0$ and complete if the map $C \to \lim_p C/F_p C$ is an isomorphism. We say that the associated spectral sequence $E^\infty$ converges conditionally to $H^a C$ if the filtration $H^a_\ast (F_p C)$ of $H^a (C)$ is complete and Hausdorff. We say that it converges strongly to $H^a C$ if furthermore we have isomorphisms $E_{p,q}^{\infty} \cong \text{gr}_p H_{p+q}^a C$, where $\text{gr}_p M$ for a filtered module $M$ denotes the associated graded module $F_p M/F_{p-1} M$.

We will only ever care about filtrations that are exhaustive and complete Hausdorff (but it is still important to check these conditions). Completeness in particular is often subtle and delicate.

Conditional convergence does not promise us that we can calculate $H^a C$ from the $E_\infty$ page, but it is both quite powerful and quite flexible. Its relevance comes from the following proposition ([Boa99, Theorems 8.2-8.3]):

**Proposition A.3.** Suppose we have exhaustive filtered complexes $C$ and $C'$ such that associated spectral sequences converge conditionally to $H^a C$ and $H^a C'$. If $f : C \to C'$ is a map of filtered complexes inducing an isomorphism on some finite page $E^\infty (F_p C) \to E^\infty (F_p C')$, then $f$ is a quasi-isomorphism. There are invariants $RE^{\infty}$ and $W(E)$ of conditionally convergent spectral sequences so that if $RE^{\infty} = W(E) = 0$, then the spectral sequence converges strongly.

If $E$ is a half-plane spectral sequence, then $W(E) = 0$. If there are only finitely many differentials leaving a given $E_{s,t}$, then $RE^{\infty} = 0$. If $E$ degenerates on some finite page, then both $RE^{\infty} = W(E) = 0$.

So even though we may not be able to use the $E^\infty$ page of a conditionally convergent spectral sequence to calculate homology, we can still use it to detect quasi-isomorphisms. To apply this, we will make frequent use of [Boa99, Theorems 9.2-9.3], which show that conditionally convergent spectral sequences are relatively common:

**Proposition A.4.** If $C$ is a filtered complex whose filtration is exhaustive and complete Hausdorff, then the associated spectral sequence, whose $E^1$ page given by $H(F_p C/F_{p+1} C)$, is conditionally convergent to $H^a (C)$. If the filtration of $C$ is not exhaustive, complete, or Hausdorff, the associated spectral sequence instead converges conditionally to $\hat{C} = \lim_p \text{colim}_s F_p C/F_{p-1} C$. 
If the filtration is bounded below, it is automatically both complete and Hausdorff; if it is bounded above, it is automatically exhaustive. Another situation that frequently arises is that the filtration is the totalization of a multicomplex; in this case, the completion is easy to describe. We state the definition here (first appearing in [Wal61]).

**Definition A.4.** A multicomplex is a bigraded $R$-module $M_{s,t}$ with differentials $d_r : M_{s,t} \to M_{s-r,t+r-1}$ for $r \geq 0$ so that

$$
\sum_{i+j=n} d_i d_j = 0.
$$

The associated (completed) filtered complex is the subcomplex

$$
\hat{C}_n = \prod_{s \to x} M_{s,n-s} \subset \prod_{n \in \mathbb{Z}} M_{n,n-s}
$$

consisting of those sequences $(x_s)$ with $x_s = 0$ for sufficiently large $s$. (That is to say, the product is only taken in the negative direction.) The differential is given as $\sum_{r \geq 0} d_r$.

**Proposition A.5.** $\hat{C}$ is complete Hausdorff, and we can identify the $E^1$ page of the associated (conditionally converging) spectral sequence with $H(M, d_0)$ equipped with the differential $H(d_1)$.

These tools in hand, we can prove the two invariance results we need.

**Proof of Theorem A.1.** Filter $B(M, A, N)$ by

$$
F_p B(M, A, N) = M \otimes \left( \bigoplus_{i=0}^p \mathcal{A}^{\otimes i} \otimes N \right).
$$

Then we calculate the $E^1$ page of the associated spectral sequence as the (bigraded) complex

$$
H(M \otimes N) \leftarrow H(M \otimes \mathcal{A} \otimes N) \leftarrow \cdots
$$

Because the map $B(M, A, N) \to B(M', A', N')$ preserves the filtration, it induces a map of spectral sequences; if the maps

$$
H(M \otimes \mathcal{A}^{\otimes i} \otimes N) \to H(M' \otimes \mathcal{A}^{\otimes i} \otimes N')
$$

are isomorphisms, then we would have proved that $B(g, f, h)$ induces an isomorphism on the $E^1$ page and hence all pages of the spectral sequence.

We prove this fact inductively on the number of tensor factors.

**Lemma A.6.** If $F : X \to X'$ is a quasi-isomorphism of degreewise $R$-flat complexes, and $G : Y \to Y'$ is a quasi-isomorphism of arbitrary complexes, then $F \otimes G : X \otimes Y \to X' \otimes Y'$ is a quasi-isomorphism.

This follows immediately from comparing the short exact sequences (which exist because $X$, $X'$ are degreewise flat)

$$
0 \to \bigoplus_{p+q=n} H_p(X) \otimes H_q(Y) \to H_n(X \otimes Y) \to \bigoplus_{p+q=n-1} \text{Tor}(H_p X, H_q Y) \to 0,
$$

because the outside terms only depend on $HX$ and $HY$. This is where we need $R$ to be a PID; otherwise we would need to apply the Kunneth spectral sequence, and would need boundedness assumptions. One may use the spectral sequence argument to extend this result to Dedekind domains, as in [KM, Lemma 2.2].
The filtration of $B(M, A, N)$ is trivially complete and Hausdorff, as $F_{-1} = 0$. It is exhaustive because the infinite direct sum is the union of its finite direct summands.

Because $N, N', \overline{A}$, and $\overline{A}$ are degreewise $R$-flat, we have $B(g, f, h)$ is an isomorphism on the $E^2$ pages. Because the filtration is bounded below, the spectral sequences converge strongly by Proposition A.4 and thus $B(g, f, h)$ is a quasi-isomorphism by Proposition A.3.

**Proof of Theorem A.2.** Now the filtration is

$$F_{-p}cB(N, A, M) = \text{Hom}_A(B(N, A, A)/F_p B(N, A, A), M);$$

that is, it consists of those functionals that vanish on $F_p B(N, A, A)$. Let us abbrevi ate $B(N, A, A) =: B$. This filtration is now bounded above, and thus is automatically exhaustive. Note that the intersection

$$\lim_{-p} \text{Hom}_A(B/F_p, M) = \text{Hom}_A\left(\text{colim}_p B/F_p, M\right).$$

The colimit is an exact functor, so

$$\text{colim}_p B/F_p = B/\text{colim}_p F_p = B/B = 0,$$

because the filtration of $B$ was exhaustive, and thus the filtration is Hausdorff. Now, the short exact sequence $0 \to F_p \to B \to B/F_p \to 0$ is split as $A$-modules (not dg), identifying $B/F_p$ with the summand $\oplus_{q \geq p} N \otimes \overline{A}^{\oplus q} \otimes A$ (but ignoring the differential). This implies that $\text{Hom}_A(B, M)/\text{Hom}_A(B/F_p, M) \cong \text{Hom}_A(F_p, M)$ as $R$-modules ($\text{Hom}$ takes split exact sequences to split exact sequences). Applying

$$\lim \text{Hom}_A(B, M)/\text{Hom}_A(B/F_p, M) \cong \lim \text{Hom}_A(F_p, M)$$

$$= \text{Hom}_A(\text{colim}_p F_p, M) = \text{Hom}_A(B, M),$$

we see that the filtration is complete (the last equality because the filtration on $B$ was exhaustive).

We identify the $E^1$ pages (using the isomorphism of $cB(N, A, M)$ as a graded $R$-module to $\text{Hom}(N \otimes B A, M)$) as the totalization of the double complex

$$H(\text{Hom}_R(N, M)) \to H(\text{Hom}_R(N \otimes \overline{A}, M)) \to \cdots$$

We need to see that $\text{Hom}(N \otimes \overline{A}^{\oplus q}, M) \to \text{Hom}(N' \otimes \overline{A}^{\oplus q}, M')$ is a quasi-isomorphism for all $i$, then the theorem will be proved. We write this as a lemma: the only property we will use is that $N \otimes \overline{A}^{\oplus q}$ is free for any $i$ (a tensor product of free modules is free).

**Lemma A.7.** If $X$ and $X'$ are $R$-free, and $F : X' \to X$ is a quasi-isomorphism, then for any quasi-isomorphism $G : Y \to Y'$, the induced map $\text{Hom}_R(X, Y) \to \text{Hom}_R(X', Y')$ is a quasi-isomorphism.

**Proof.** When both $X$ and $d(X) \subset X$ are complexes of projective modules, we have a natural Kunneth short exact sequence

$$0 \to \prod_{p+q=n} \text{Ext}^1(H_p X, H_{-q}Y) \to H_{-n}\text{Hom}(X, Y) \to \prod_{p+q=n} \text{Hom}(H_p X, H_{-q}X) \to 0;$$

from this the theorem is clear, as long as every submodule $d(X)$ of a free module of arbitrary rank is free; this is true for PIDs. For the statement of the Kunneth theorem and that submodules of free modules are projective (and hence free, as projective modules over a PID are free), see [Wei95, Exercises 3.6.1-3.6.2].
We will often be interested in thinking of one-variable versions of the bar and cobar constructions as providing homology theories for $A$-modules; we will introduce new notation for the sake of compactness. Our notation is chosen to fit with both [Jon87] and [OS04].

**Definition A.5.** Suppose that $A$ is an augmented dg-algebra with each $A_n$ free over $R$, and $M$ a right $A$-module. Then we define the positive $A$-chains on $M$ to be $C^+_A(M) := B(M, A, R)$ and the negative $A$-chains on $M$ to be $C^-_A(M) := cB(R, A, M)$. The homology of $C^+_A(M)$ is denoted $H^+_A(M)$, the (positive) $A$-homology of $M$, while the homology of $C^-_A(M)$ is denoted $H^-_A(M)$, the coBorel (or negative) $A$-homology of $M$. Note that both of these are covariant in $M$! When $A$ is clear and we are not interested in varying it, we omit it from the notation.

In the $\mathbb{Z}/2N$ graded setting will need versions of these invariance results for the completed bar constructions and the finitely supported cobar constructions. By the nature of these completions, the spectral sequence arguments no longer work (the filtrations will fail to either be exhaustive or complete, respectively), and so we are left with significantly weaker results. They will, however, be enough for us.

**Definition A.6.** Let $A$ be a dg-algebra, and let $M$ be a right $A$-module. If $N$ is a left $A$-module, the completed bar construction $\bar{B}(M, A, N)$ is given as

$$\bigotimes_{i=0}^{\infty} M \otimes \bar{A}^{\otimes i} \otimes N,$$

with differential defined by the same formula as before, which extends to the completion because every term sends an element of $B_i = M \otimes \bar{A}^{\otimes i} \otimes N$ to either $B_i$ or $B_{i-1}$.

If $N$ is a right $A$-module, the finitely supported cobar construction $\hat{cB}(N, A, M)$ is given as $\text{Hom}_{A}^{\text{fin}}(B(N, A, A), M)$, the set of those $A$-equivariant maps which vanish on

$$\bigoplus_{i \geq k} M \otimes \bar{A}^{\otimes i} \otimes N$$

for some sufficiently large $k$.

Essentially, what we need is the following.

**Lemma A.8.** Suppose $(g, f, h) : (M, A, N) \to (M', A', N')$ satisfy the conditions of Theorem A.1, and additionally, suppose that the algebras $A, A'$ are supported in non-negative degrees, and all of $M, M', N, N'$ are bounded below. Then in any fixed degree, $\left( \bar{B}(M, A, N) \right)_k = B(M, A, N)_k$; in particular, the induced map $\bar{B}(g, f, h)$ is a quasi-isomorphism.

This is self-evident: if $M$ and $N$ are supported in degrees above $m$ and $n$, respectively, then $M \otimes \bar{A}^{\otimes i} \otimes N$ is supported in degrees above $m + i + n$. So elements of $\bar{B}(M, A, N)$ in any fixed degree $k$ in fact lie in $B(M, A, N)$.

We have the dual assertion for the cobar construction.

**Lemma A.9.** Suppose $(g, f, h) : (M', A', N') \to (M, A, N)$ satisfy the conditions of Theorem A.1, and additionally, suppose that the algebras $A, A'$ are supported in non-negative degrees, the modules $N, N'$ are bounded below, and $M$ and $M'$ are bounded above. Then in any fixed degree, $\left( \hat{cB}(N, A, M) \right)_k = cB(N, A, M)_k$; in particular, the induced map $\hat{cB}(h, f, g)$ is a quasi-isomorphism.
Here, because \( N \) is bounded below (by degree, say, \( n \)) and \( A \) is concentrated in non-negative degrees, we see that \( \bigoplus_{i \geq k} N \otimes A[1]^\otimes i \otimes A \) is supported in degrees at least \( n + i \). So an element of \( cB(N, A, M) \) of degree \( d \) is a map \( B(N, A, A) \to M \) which sends an element of \( \bigoplus_{i \geq k} N \otimes A[1]^\otimes i \otimes A \) to some \( M_p \), where \( n + k - d \geq p \). Because \( M \) is bounded above, as \( k \) increases without bound, we see that the target of such a map is zero. So any \( \text{(finite-degree)}^{21} \) element of \( cB(N, A, M) \) is finitely supported in the above sense.

A.3. The dualizing complex and Tate homology. Tate homology, constructed in this section for dg-modules over a dg-algebra and written as \( H^g_A(M) \), has appeared in the literature in many forms. Our approach here is essentially a chain-level interpretation of [Kle02] (which was written in the context of Tate homology of \( G \)-spectra). Tate homology may be viewed, in some sense, as the homology theory that arises when you kill off free objects. The classic reference to Tate homology of spectra is [GM95]; we warn that if \( X \) is a \( G \)-space, our \( H^g_A(C_\bullet^A(X); R) \) is more analogous to what they would call \( t_{HR,G}(HR)(X) \), not \( t_G(HR)(X) \). (We are ‘chainifying’ the group from the start as well.) The idea of Tate homology is beautifully developed in an abstract homotopical setting in [Gre01], which surely includes as a special case the content of this section. A recent approach to Tate homology of \( G \)-spectra via localization of \( \mathcal{C} \)-categories appears in [NS17]; this level of abstraction has the advantage of giving multiplicativity and uniqueness results that are not as easily available otherwise (in some cases, not available at all).

We take our current approach as it seems to minimize input energy, at the cost perhaps of some conceptual clarity and multiplicativity results (we do not construct a product on \( C^\otimes_A(R) \), for instance).

Because \( A \) is a left \( A \)-module, the complex \( cB(R, A, A) \) inherits the structure of a left \( A \)-module, by \( (\alpha)([b_1 | \cdots | b_n]b) = \alpha([b_1 | \cdots | b_n]b) \).

**Definition A.7.** The dualizing complex \( D_A \) of a degreewise \( R \)-free dg-algebra \( A \) is the left \( A \)-module \( cB(R, A, A) \).

**Definition A.8.** Let \( A \) be a degreewise \( R \)-free dg-algebra and \( M \) a left \( A \)-module. The map
\[
N_M : B(M, A, D_A) \to C^\otimes_A(M)
\]
given by
\[
N_M (m[a_1 | \cdots | a_k] \psi) = \begin{cases} 0 & k > 0 \\ m \cdot \psi & k = 0 \end{cases}
\]
is called the norm map of \( M \). \( m \cdot \psi \) makes \( \psi \) into an \( M \)-valued functional by using the fact that \( \psi \) is \( A \)-valued and \( M \) is a right \( A \)-module.) Its mapping cone is denoted \( C^\otimes_A(M) \), the Tate complex of \( M \). Its homology is the Tate homology \( H^g_A(M) \) of \( M \).

The following is important enough to record as a lemma.

**Lemma A.10.** \( D_A \) is degreewise \( R \)-flat.

\(^{21}\)“Not finite degree” makes sense in the context of non-homogeneous chains.
Proof. Recall that

\[ D_A = cB(R, A, A) = \prod_{i=0}^{\infty} (\mathcal{A}^i)^{\otimes_i} \otimes A; \]

\( A \) is degreewise \( R \)-free, and in particular torsion-free; because \( R \) is a PID, \( R \)-flatness is equivalent to \( (R-) \) torsion-freeness. Duals of torsion-free modules are torsion-free, sums of torsion-free modules are torsion free, and products of torsion-free modules are torsion-free, so \( cB(R, A, A) \) is degreewise torsion-free, and hence degreewise flat.

To justify the name “Tate homology”, we show that this satisfies part of the corresponding versions of Klein’s axioms defining Tate cohomology [Kle02], skipping the complete verification that \( H_A^p \) and \( H(B(\cdot, A, D_A)) \) are homology theories: the remaining axioms state that these preserve homotopy pullbacks and filtered homotopy colimits. These are not hard to verify, but we will not use them.

**Theorem A.11.** The functor \( C_A^\varphi(M) \) from left \( A \)-modules to chain complexes satisfies Klein’s four axioms specifying the Tate homology of \( M \):

1. \( C_A^\varphi(M) \) preserves weak equivalences;
2. \( H_A^p(X \otimes A) = 0 \), where \( X \) is a chain complex and the right \( A \)-module structure is given by acting on \( A \);
3. There is a map \( C_A(M) \to C_A^\varphi(M) \), natural in \( M \), whose homotopy fiber preserves weak equivalences.

Proof. In particular, it follows from Theorem A.1 that \( B(M, A, D_A) \) preserves weak equivalences in \( M \), because by Lemma A.10 above, \( D_A \) is flat. The Tate complex is the mapping cone (homotopy fiber) of \( N_M : B(M, A, D_A) \to C_A^\varphi(M) \), and both the domain and the codomain of the norm map preserve weak equivalences, so \( C_A^\varphi(M) \) does as well. This is Axiom 1.

For Axiom 2, note that \( B(X \otimes A, A, D_A) \cong X \otimes B(A, A, D_A) \). Then the norm map is identified with the projection \( X \otimes B(A, A, D_A) \to X \otimes D_A \); but \( B(A, A, N) \to N \) is a homotopy equivalence for any left \( A \)-module \( N \).

The first part of Axiom 3 is obvious (it is the inclusion into a mapping cone), and the second part almost so: the homotopy fiber = mapping cocone is naturally equivalent to \( B(M, A, D_A) \).

We henceforth write \( H_A^{+, \text{tw}}(M) \) for \( H(B(M, A, D_A)) \), and call it the twisted Borel homology. We will investigate its relationship to \( H_A^+(M) \) later.

We won’t prove Klein’s uniqueness theorem that these axioms do uniquely characterize Tate cohomology, but rather use it as motivation that we have the correct definition. (It seems likely that some variation of his argument works in this context.)

In addition to the above, it is important to observe that there is a natural left action of \( C_A^\varphi(R) \) on \( D_A \), and hence on \( B(M, A, D_A) \): the norm map \( N_M \) is easily seen to be \( C_A^\varphi(R) \)-equivariant. Therefore, the mapping cone inherits the structure of a left \( C_A^\varphi(R) \)-module, and the natural map \( C_A(M) \to C_A^\varphi(M) \) is a module homomorphism.

The following theorem summarizes everything we’ve assembled about the three \( A \)-homology functors \( H^+, H^- \), \( H^\varphi \).
**Theorem A.12.** Let $A$ be a dg-algebra over a commutative PID $R$ which is $(R-)\text{ flat in each degree } A_n$. There are functors

$$H_A^{+ \text{tw}}(M), \ H_A^{-}(M) \ H_A^{X}(M),$$

from dg $A$-modules to graded $R$-modules, satisfying the following properties.

1. The functors send short exact sequences of $A$-modules to exact triangles and preserve weak equivalences.
2. $H_A^{-}(R)$ is a ring, and each of these homology theories carry a natural left module structure over $H_A^{-}(R)$.
3. There is an exact triangle of $H_A^{-}(R)$-modules

$$H_A^{+ \text{tw}}(M) \to H_A^{-}(M) \to H_A^{X}(M) \to \cdots$$

4. $H_A^{X}(X \otimes A) = 0$ when $X$ is a chain complex and $X \otimes A$ is given the canonical right action.

We state an invariance theorem for equivariant homology with respect to quasi-isomorphic dgas.

**Proposition A.13.** Suppose $f : A \to A'$ is a quasi-isomorphism of algebras, each degreewise free over $R$. This induces a functor $F : \text{Mod}_A \to \text{Mod}_{A'}$ via restriction of scalars, and there are natural isomorphisms $H_A^\bullet(FM) \cong H_{A'}^\bullet(M)$, for the homology theories $\bullet \in \{+, \text{tw}, -, \infty\}$. These natural isomorphisms induce an $H_A^{-}(R)$-equivariant natural isomorphism of exact triangles.

**Proof.** The fact that $cB(R, A, M) \leftarrow cB(R, A', M)$ is a quasi-isomorphism is Theorem A.2; for that reason, the maps

$$D_A = cB(R, A, A) \to cB(R, A, A') \leftarrow cB(R, A', A') = D_{A'}$$

are quasi-isomorphisms. This is the crucial place we need $A$ and $A'$ to be degreewise free.

To make the diagrams smaller, let us write $B_A$ for $B(D_A, A, FM)$, and $cB_A$ for $cB(R, A, M) = \text{Hom}_A(B(A, A, R), M)$; also write $B' = B(cB(R, A, A'), A', M)$. Then the induced maps on bar constructions $B_A \to B' \leftarrow B_{A'}$ are quasi-isomorphisms by Theorem A.1, using that $cB(R, A, A)$, $cB(R, A, A')$, and $cB(R, A', A')$ are all flat $R$-modules, which is Lemma A.10.

Comparing the Tate homology groups is more complicated. The following diagram commutes, which is an easy check left to the reader.

$$\begin{array}{ccc}
B_A & \xleftarrow{N_A} & B_A' \\
\downarrow^{N_A} & & \downarrow^{N_{A'}} \\
cB_A & \xleftarrow{N'} & cB_{A'}
\end{array}$$

Here $N' : \text{Hom}_A(B(A, A, R), A') \otimes_{A'} B(A', A', M) \to \text{Hom}_A(B(A, A, R), M')$ is obtained from the projection map $B(A', A', M) \to M$. The maps in the bottom row are the natural maps induced on a cone, and we denote the norm maps $N_A$ and $N_{A'}$ instead of the usual $N_M$ to reflect the different choice of algebra $M$ is a module over.
Because all of the maps in the top two rows are quasi-isomorphisms, the maps on the bottom row are also quasi-isomorphisms by an application of the five lemma. In particular, inverting the bottom-right quasi-isomorphism on homology, we have an isomorphism \( H^*_A(FM) \to H^*_A(M) \).

All of the maps in this diagram are module homomorphisms with respect to either \( C_A(R) \) or \( C_{A'}(R) \), as appropriate; this implies that the isomorphisms on homology are \( H_A(R) \)-equivariant.

In what follows we briefly describe the completed version of Tate homology and its comparatively weak invariance properties. The finitely supported dualizing complex, \( p^D A \), is precisely 
\[
\hat x_c B = \text{Hom}_A^\text{fin} B(R, A, A) \subset A.
\]

As before, there is a natural map \( \hat B(M, A, \hat D_A) \to \hat x_c B(R, A, A) \); we write \( \hat C_A^* \) for its mapping cone, and (for now) call this the completed Tate homology.

To begin with, we can compare \( p^D A \) and \( p^D A' \), using Lemma A.9.

**Lemma A.14.** Suppose we are given a quasi-isomorphism \( f : A \to A' \) of nonnegatively graded, \( R \)-free, and bounded above dg-algebras \( A, A' \). Then 
\[
\hat D_A \to \hat x_c B(R, A, A') \leftrightarrow \hat D_{A'}
\]
is a zig-zag of quasi-isomorphisms.

To compare the completed versions of the twisted Borel homology, we need a more subtle argument, as follows. We allow to vary the dg-algebra involved in the dualizing complex to make use of the zig-zag above.

**Lemma A.15.** Suppose \( A \) and \( A' \) are non-negatively graded, \( R \)-free, and bounded above. Let \( f : A \to A' \) be a dg-algebra quasi-isomorphism, and \( f : M \to M' \) be a quasi-isomorphism of bounded below right \( A' \)-modules (thought of also as \( A \)-modules by restriction of scalars), then the maps 
\[
\hat B(M, A, \hat D_A) \to \hat B(M, A, \hat x_c B(R, A, A')) \leftrightarrow \hat B(M, A, \hat D_{A'}) \to \hat B(M', A', \hat D_{A'})
\]
are all quasi-isomorphisms.

**Proof.** All of these will fall prey to the same argument, so we investigate the leftmost one in particular.

The right \( A \)-module \( \hat B(R, A, A) \) (which agrees with \( B(R, A, A) \) in each degree) is the increasing union in each degree of the bounded subcomplexes 
\[
B^{\leq k}(R, A, A) = \bigoplus_{i=0}^k A^\otimes i \otimes A;
\]
this stabilizes in each degree for some finite \( k \). Here we used that \( A \) is bounded to see that these subcomplexes are bounded. Therefore there is also a sequence of bounded quotient complexes 
\[
\hat B^{\leq k}(R, A, A) = \text{Hom}_A^\text{fin} (B^{\leq k}(R, A, A), A),
\]
and similarly with \( \hat B^{\leq k}(A, A') \). We have, in each degree, \( \text{lim}_k \hat B^{\leq k}(R, A, A) = \hat B(R, A, A) \), and this limit stabilizes degreewise at some finite \( k \) (depending on the degree). Further, observe that the map \( \hat B^{\leq k}(R, A, A) \to \hat B^{\leq k}(R, A, A') \) is a
quasi-isomorphism by the argument of Theorem A.2; now the filtration used there is finite.

Now \( \hat{B}(M, A, \hat{D}_A^{\leq k}) \) and \( \hat{B} \left( M, A, \hat{cB}^{\leq k}(R, A, A') \right) \) both satisfy the conditions of Lemma A.8, so the map

\[
\hat{B}(M, A, \hat{D}_A^{\leq k}) \to \hat{B} \left( M, A, \hat{cB}^{\leq k}(R, A, A') \right)
\]

is a quasi-isomorphism for each \( k \). Next, observe that

\[
\hat{B}(M, A, \hat{D}_A) \cong \lim_k \hat{B} \left( M, A, \hat{D}_A^{\leq k} \right)
\]

there is a natural chain map from the leftmost complex to the right, which is an isomorphism at the level of \( R \)-modules because limits commute (the second relevant limit is \( \hat{B} = \lim B^{\leq k} \), which makes sense at the level of \( R \)-modules). Because this limit is over a tower of quotient maps, the tower in particular satisfies the Mittag-Leffler condition, and we may apply [Wei95, Theorem 3.5.8], which says that there is a short exact sequence

\[
0 \to \lim^1 H\hat{B} \left( M, A, \hat{D}_A^{\leq k} \right)[1] \to H\hat{B} \left( M, A, \hat{D}_A \right) \to \lim H\hat{B} \left( M, A, \hat{D}_A^{\leq k} \right) \to 0.
\]

Such a sequence is natural for maps of towers of chain complexes that satisfy the Mittag-Leffler condition.

The given assumptions imply that \( \hat{D}_A^{\leq k} \) is bounded; given the standing assumption that \( M \) is bounded below, we may apply Lemma A.8 to obtain a quasi-isomorphism between \( \hat{B}(M, A, \hat{D}_A^{\leq k}) \) and \( \hat{B}(M, A, \hat{cB}^{\leq k}(R, A, A')) \); then applying the 5-lemma to this exact sequence, we see that the induced map on the non-truncated versions is a quasi-isomorphism.

A similar discussion applies to all of the other chain complexes in the statement of the lemma.

We may thus conclude using the same argument as Proposition A.13 that the completed Tate complexes are quasi-isomorphic (with quasi-isomorphism respecting the \( \hat{C}_A(R) \) action), under the conditions in the previous lemma.

**Corollary A.16.** Suppose \( A \) and \( A' \) are bounded, non-negatively graded, \( R \)-free dg-algebras, for which \( f : M \to M' \) is a quasi-isomorphism of bounded right \( A' \)-modules. Then there is a canonical isomorphism \( \hat{H}_A^\infty(M) \cong \hat{H}_A^\infty(M') \), equivariant under the action of \( \hat{H}_A(R) \cong \hat{H}_A^\infty(R) \).

**A.4. Periodicity in Tate homology.** Tate homology, in some cases, is periodic: there is some class in \( H_A^\infty \) so that its action on \( H_A^\infty(M) \) is an isomorphism when \( M \) is a finite \( A \)-module (in an appropriate sense). A beautiful reference for this phenomenon is [GM95, Section III.16] in the setting of genuine \( G \)-spectra, where \( G \) is a compact Lie group that acts freely on some sphere. Without this assumption, periodicity phenomena often fail; see for instance [BC92].

Here we endeavor only to prove an analogue of it in a simple case which will suffice for our purposes, and in a computational manner. We write \( A = \Lambda := \Lambda(u_n) \), where \( |u_n| = n \), and either \( n \) is odd or the ground ring \( R \) has \( 2 = 0 \). We have \( H_A^\infty = R[U] \), where \( |U| = n - 1 \), and \( H_A^\infty = R[U^*] \) with \( |U| = n + 1 \), where the action of \( U \) is contraction against \( U^* \); these are immediately clear from the definition of bar and cobar construction here, which have no nonzero differentials.
We see from the tautological exact triangle that, as a graded \( R \)-module, we have \( H^1_{\Lambda} = R[U, U^{-1}] \) (where here we have suggestively rewritten \( U^* = U^{-1} \)); we know from the fact that the exact triangle are maps of \( H^1_{\Lambda} \)-modules that the action of \( U \) on these increases the power of \( U \) by 1, except possibly for the action on \( U^{-1} \). The content of the following crucial lemma is that \( U \cdot U^{-1} = 1 \).

**Lemma A.17.** Let \( \Lambda \) be the exterior algebra over \( R \) on an element in degree \( n \), where either \( n \) is odd or \( R \) has characteristic 2. We have an \( H^1_{\Lambda} \)-module isomorphism \( H^1_{\Lambda} \cong R[[U, U^{-1}]] \), with \( U \cdot U^i = U^{i+1} \) for all \( i \in \mathbb{Z} \).

*Proof.* We write this out very explicitly; all of what follows is a transcription of definitions. First, we may write \( D_{\Lambda} = R[[\eta]] \otimes \Lambda \), with differential \( d\eta^k = (-1)^{kn} \eta^{k+1} \otimes u \).

To avoid confusion later, we have used the notation \( \eta \) where previously we wrote \( U \): it is the functional on \( B\Lambda \) which sends \( U^* \) to \( 1 \in R \). As graded \( R \)-modules we have

\[
B(D_{\Lambda}, \Lambda, R) = R[[\eta]] \otimes \Lambda(u) \otimes R[U^*];
\]

under this isomorphism, the differential of \( B(D_{\Lambda}, \Lambda, R) \) is taken to

\[
d(\eta^j \otimes (U^*)^k) = \eta^j \otimes u \otimes (U^*)^{k-1} + (-1)^{kn} \eta^{j+1} \otimes u \otimes (U^*)^k;
\]

and is otherwise zero. The action of \( U \) on this complex is \( U \cdot (\eta^j \otimes p) = \eta^{j+1} \otimes p \) for any \( p \in \Lambda(u) \otimes R[U^*] \). Finally, the norm map \( N_{\Lambda} : B(D_{\Lambda}, \Lambda, R) \to C^1_{\Lambda}(R) \) is given by \( \eta^k \mapsto U^k \) and is otherwise zero.

By definition, the Tate complex

\[
C^\infty_{\Lambda}(R) = B(D_{\Lambda}, \Lambda, R)[1] \oplus C^1_{\Lambda}(R)
\]

is the mapping cone of the norm map; this means that its differential is

\[
d_x = \begin{pmatrix}
-d & 0 \\
N_{\Lambda} & d
\end{pmatrix};
\]

the action of \( U^* \) is the same as before on each component. Now note that

\[
H^1_{\Lambda} B(D_{\Lambda}, \Lambda, R) \cong H^1_{\Lambda}[n]
\]

as an \( H^1_{\Lambda} \)-module, and a chain in \( B(D_{\Lambda}, \Lambda, R) \) representing the degree zero element of \( H^1_{\Lambda} \) is given by \( \eta^0 \otimes u \otimes (U^*)^0 \). Our goal, then, is to show that \( U \cdot (\eta^0 \otimes u \otimes (U^*)^0) \) is homologous to \( U^0 \in C^1_{\Lambda}(R) \). But from this formula we have

\[
d_x(\eta^0 \otimes 1 \otimes (U^*)^0) = -\eta \otimes u \otimes (U^*)^0 + U^0;
\]

this is precisely what we wanted. 

We can use this to prove the following localization theorem.

**Proposition A.18.** Let \( M \) be a dg-module over \( \Lambda \), degreewise free over \( R \); suppose \( M \) has a finite filtration \( 0 = F_{-1}M \subset F_0M \subset \cdots \subset F_nM = M \) so that each piece \( F_kM/F_{k-1}M \) of the associated graded dg-module is quasi-isomorphic to a finite direct sum of copies of \( R \) and \( \Lambda \). Then the action of \( U \in H^1_{\Lambda} \) on \( H^1_{\Lambda}(M) \) is an isomorphism. Therefore the natural map \( H^1_{\Lambda}(M) \to H^1_{\Lambda}(M) \) factors through \( H^1_{\Lambda}(M)[U^{-1}] \), and the map

\[
H^1_{\Lambda}(M)[U^{-1}] \to H^1_{\Lambda}(M)
\]

is an isomorphism, natural for dg-module homomorphisms of such \( M \).
To prove this, we first define the notion of *inverting an endomorphism* of a dg-module with respect to a dg-module homomorphism \( f : M \to M \) of degree \( k \), written \( M[f^{-1}] \). This is NOT the strict notion of inversion of an element familiar in module theory,
\[
\colim \left( M \xrightarrow{f} M \xrightarrow{f} M \xrightarrow{f} \cdots \right),
\]
because this rarely plays well with taking homology. Instead it is a homotopy colimit. The simplest way to phrase this (taken from [HR96, Definition 24.5]) is that \( M[f^{-1}] \) is the mapping cone of the map \( 1 - tf : M[t] \to M[t] \), where \( t \) is a polynomial generator in degree \(-k\). Then immediate from the exact triangle on homology, and the fact that \((1-tf)_* : H(M)[t] \to H(M)[t] \) is injective, we see that \( H(M[t^{-1}]) = H(M)[t^{-1}] \), where on the right side we are inverting an element of a module in the usual sense. (Here we are using that the module-theoretic notion may be defined perfectly well as \( M[t]/(1-tf) \).)

**Proof of Proposition A.18.** First we show that the action of \( U^* \) on \( H^\wedge_\Lambda(R) \) is an isomorphism. This is equivalent to showing that the natural map \( C^\wedge_\Lambda(R) \to C^\wedge_\Lambda(M)[U^{-1}] \) is a quasi-isomorphism. To see this, we use that the filtration \( F_k M \) induces a filtration \( F_k C^\wedge_\Lambda(M) = C^\wedge_\Lambda(F_k M) \), and similarly a filtration on \( C^\wedge_\Lambda(M)[U^{-1}] \); these are complete because the filtration is finite. Now the \( E^1 \) page of the corresponding spectral sequence is given as a direct sum of copies of \( H^\wedge_\Lambda(R) \) and \( H^\wedge_\Lambda(\Lambda) \) (or, respectively, the results of inverting \( U \)), and the action of \( U \) on the \( E^1 \) page is the direct sum of the corresponding actions. But \( H^\wedge_\Lambda(\Lambda) = 0 \), and the fact that \( U^* \) is an isomorphism on \( H^\wedge_\Lambda(R) \) was the content of Lemma A.17. So the natural (filtered) map \( C^\wedge_\Lambda(M) \to C^\wedge_\Lambda(M)[U^{-1}] \) is an isomorphism on the \( E^1 \) page, and therefore a quasi-isomorphism. This proves the first part of the theorem.

The rest follows similar lines: there is a natural map
\[
C^\wedge_\Lambda(M)[U^{-1}] \to C^\wedge_\Lambda(M)[U^{-1}],
\]
and our goal is to show that this map is a quasi-isomorphism; because the map \( C^\wedge_\Lambda(M) \to C^\wedge_\Lambda(M)[U^{-1}] \) is identified on homology with the result of inverting \( U \), the desired result follows. But using the same filtration as the above, the \( E^1 \) page of the first spectral sequence is a direct sum of copies of \( H^\wedge_\Lambda(R)[U^{-1}] \) and \( H^\wedge_\Lambda(\Lambda)[U^{-1}] \), and the \( E^1 \) page of the second spectral sequence is a direct sum of corresponding copies of \( H^\wedge_\Lambda(R) \) and \( H^\wedge_\Lambda(\Lambda) \). Then the only thing to observe is that what we already know: \( H^\wedge_\Lambda(R)[U^{-1}] \to H^\wedge_\Lambda(R) \) is an isomorphism, and that \( H^\wedge_\Lambda(\Lambda) \) is a copy of \( R \) concentrated in degree zero, so that \( H^\wedge_\Lambda(\Lambda)[U^{-1}] = 0 \). Therefore the above map is an isomorphism on the \( E^1 \) page of the corresponding spectral sequence, and therefore a quasi-isomorphism. \( \blacksquare \)

A.5. **Simplifying the twisted Borel homology.** In this section, we give conditions under which there is a natural isomorphism \( H^{+,tw}_A(M) \cong H^+_A(M)[n] \) for some degree shift \( n \). We would, furthermore, like this isomorphism to preserve the action of \( H^+_A(R) \).

The simplest assumption we can make that guarantees this is that \( A \) satisfies Poincaré duality of degree \( n \), meaning that there is a quasi-isomorphism of right \( A \)-modules \( A \cong A^\vee[n] \). Here (as usual) \( A^\vee := \text{Hom}_R(A, R) \) with the grading making this into a chain complex, with the right \( A \)-action induced by the left \( A \)-action on \( A \). This is enough to get the desired \( R \)-module isomorphism; some slight additional conditions guarantee the \( H^+_A(R) \)-module isomorphism as well.
Remark A.1. This is a very special case of what is called an \textit{Gorenstein condition} in the literature, which usually amounts to something like the condition $\text{Hom}_A(R, A) \cong A[n]$, interpreted in a derived sense. Our condition implies this because of the chain of equivalences

$$\text{Hom}_A(R, A) \simeq \text{Hom}_A(R, \text{Hom}_R(A, R))[n] = \text{Hom}_R(A \otimes_R R, \text{Hom}_A(A, R))[n] = \text{Hom}_R(A, R)[n] \simeq A.$$ 

The second equality uses the tensor-hom adjunction for (bi)modules over two rings, and all appearances of “$\text{Hom}$” are derived. We will not use the Gorenstein condition except through the above Poincaré duality condition; this remark is purely motivational.

A particularly nice reference, which applies to the dg-algebra case, is [DGI06].

\textbf{Theorem A.19.} If $A$ is $R$-flat and satisfies Poincaré duality of degree $n$, then there is a quasi-isomorphism of left $A$-modules $D_A \simeq R[n]$. Thus, there is a natural isomorphism $H_A^+\otimes R(M) \cong H_A^+(M)[n]$. If furthermore $A$ is non-negatively graded and $H_0A \cong R[G]$ for some finite group $G$, this isomorphism preserves the action of $H_A^+(R)$.

\textit{Proof.} We have $D_A = \text{Hom}_A(B(R, A, A), A) \simeq \text{Hom}_A(B(R, A, A), A^\vee)[n]$ by assumption. That $D_A \simeq R[n]$ then follows from the fact that the projection

$$\pi : B(A, A, R) \to R$$

is an $A$-equivariant quasi-isomorphism of $R$-flat modules, and the elementary isomorphism of left $A$-modules

$$\text{Hom}(B(R, A, A), R)) \cong \text{Hom}_A(B(R, A, A), A^\vee)$$

given on an element $\eta \in \text{Hom}(B(R, A, A), R)$ by $\eta \mapsto \eta'$ with $\eta'(b)(a) = \eta(ba)$.

This equivalence $D_A \simeq \text{Hom}_R(B(R, A, A), R)$ is equivariant under the actions of $C_A^-(R)$ (essentially by definition; the product structure uses the left $BA$-comodule structure of $B(R, A, A)$, and all of the maps above only involved the rightmost factor.) We write $B(R, A, A) = EA$ for convenience; this equivalence says $D_A \simeq (EA)^\vee$, this equivalence equivariant under both the left $A$-action and the left $C_A^-(R) = (BA)^\vee$-action.

Now we have a map

$$B(M, A, (EA)^\vee) \to B(M, A, R) \otimes B(R, A, (EA)^\vee) = B(M, A, R) \otimes_R EA \otimes_A (EA)^\vee.$$ 

The pairing $(EA) \otimes (EA)^\vee \to R$ is $A$-equivariant, so it factors through $EA \otimes_A (EA)^\vee$, and thus we have a composite $B(M, A, (EA)^\vee) \to B(M, A, R)$. The claim is that this map is equivariant under the $C_A^-(R) = (BA)^\vee$-action.

We write this computation down at the chain level, following the sign conventions in [Law]; write $\psi$ for an element of $(EA)^\vee$ and $\beta$ for an element of $(BA)^\vee$; then a generic element of $B(M, A, (EA)^\vee) = B(M, A, A) \otimes_A (EA)^\vee$ is written as
m[a_1 \cdots a_n] \psi. Following the given map and then applying \beta, we have
\[
\begin{align*}
\beta \otimes m[a_1 \cdots a_n] \psi &\rightarrow \beta \otimes \sum_{i=0}^{n} (-1)^{s_1} m[a_1] \cdots [a_i] \otimes [a_{i+1}] \cdots [a_n] \psi \\
&\rightarrow \beta \otimes \sum_{i=0}^{n} (-1)^{s_2} m[a_1] \cdots [a_i] \psi([a_{i+1}] \cdots [a_n]) \\
&\rightarrow \beta \otimes \sum_{0 \leq j \leq i \leq n} (-1)^{s_3} m[a_1] \cdots [a_j] \otimes [a_{j+1}] \cdots [a_i] \psi([a_{i+1}] \cdots [a_n]) \\
&\rightarrow \sum_{0 \leq j \leq i \leq n} (-1)^{s_4} m[a_1] \cdots [a_j] \otimes \beta([a_{j+1}] \cdots [a_i]) \psi([a_{i+1}] \cdots [a_n]).
\end{align*}
\]
Here the exponents are
\[
\begin{align*}
s_1 &= (n - i)(|m| + \epsilon_i) \\
s_2 &= (n - i)(|m| + \epsilon_i) + |\psi|(\epsilon_n - \epsilon_i) \\
s_3 &= (i - j)(|m| + \epsilon_j) + (n - i)(|m| + \epsilon_i) + |\psi|(\epsilon_n - \epsilon_i) \\
s_4 &= (|\beta| + i - j)(|m| + \epsilon_j) + (n - i)(|m| + \epsilon_i) + |\psi|(\epsilon_n - \epsilon_i).
\end{align*}
\]
Here recall that \(\epsilon_i := |a_1| + \cdots + |a_i| + i\) is the degree of \([a_1] \cdots [a_j]\). We have used the rule that the swap map is \(a \otimes b \rightarrow (-1)^{|a||b|} b \otimes a\).

On the other hand, if we apply \(\beta\) first to \(\psi\) and then follow the given map, we have
\[
\begin{align*}
\beta \otimes m[a_1] \cdots [a_n] \psi &\rightarrow (-1)^{t_1} m[a_1] \cdots [a_n] (\beta \cdot \psi) \\
&\rightarrow \sum_{0 \leq j \leq n} (-1)^{t_2} m[a_1] \cdots [a_j] \otimes [a_{j+1}] \cdots [a_n] (\beta \cdot \psi) \\
&\rightarrow \sum_{0 \leq j \leq n} (-1)^{t_3} m[a_1] \cdots [a_j] (\beta \cdot \psi) ([a_{j+1}] \cdots [a_i] \otimes [a_{i+1}] \cdots [a_n]) \\
&= \sum_{0 \leq j \leq i \leq n} (-1)^{t_4} m[a_1] \cdots [a_j] (\beta \otimes \psi) ([a_{j+1}] \cdots [a_i] \otimes [a_{i+1}] \cdots [a_n]) \\
&= \sum_{0 \leq j \leq i \leq n} (-1)^{t_5} m[a_1] \cdots [a_j] \beta([a_{j+1}] \cdots [a_i]) \psi([a_{i+1}] \cdots [a_n]),
\end{align*}
\]
where the exponents are
\[
\begin{align*}
t_1 &= |\beta|(m + \epsilon_n) \\
t_2 &= (n - j)(|m| + \epsilon_j) + |\beta|(m + \epsilon_n) \\
t_3 &= (n - j)(|m| + \epsilon_j) + |\beta|(m + \epsilon_n) + (|\beta| + |\psi|)(\epsilon_n - \epsilon_j) \\
t_4 &= (n - j)(|m| + \epsilon_j) + |\beta|(m + \epsilon_n) + (|\beta| + |\psi|)(\epsilon_n - \epsilon_j) + (n - i)(\epsilon_i - \epsilon_j) \\
t_5 &= (n - j)(|m| + \epsilon_j) + |\beta|(m + \epsilon_n) + (|\beta| + |\psi|)(\epsilon_n - \epsilon_j) + (|\psi| + n - i)(\epsilon_i - \epsilon_j).
\end{align*}
\]
The first and second expressions clearly agree, at least up to the signs on each factor. To check that \((-1)^{t_4} = (-1)^{t_5}\), observe the congruences mod 2
\[
\begin{align*}
(n - j)(|m| + \epsilon_j) + |\beta|(m + \epsilon_n) + (|\beta| + |\psi|)(\epsilon_n - \epsilon_j) + (|\psi| + n - i)(\epsilon_i - \epsilon_j)
&\equiv (n - j + |\beta|)|m| + |\psi|\epsilon_n + (|\psi| + n - i)\epsilon_i + (|\beta| + i - j)\epsilon_j \\
&\equiv (|\beta| + i - j)(|m| + \epsilon_j) + (n - i)(|m| + \epsilon_i) + |\psi|(\epsilon_n - \epsilon_i).
\end{align*}
\]
These are easily seen by breaking the top and bottom formulas (which are the relevant exponents of $-1$) into the components labelled by $|m|, \epsilon_n, \epsilon_i,$ and $\epsilon_j$ in the center formula.

The natural isomorphism $H_{\mathcal{A}}^{+,tw}(M) \cong H_{\mathcal{A}}^+(M)[n]$ then follows from the invariance theorem for bar constructions, Theorem A.1, and we have seen this is an isomorphism of $H_{\mathcal{A}}(R)$-modules.

The argument doesn’t change at all after completion, making the usual assumptions that $A$ is non-negatively graded and bounded above, and it follows that if such an algebra satisfies Poincaré duality of degree $n$, we have $\hat{H}_{\mathcal{A}}^{+,tw}(M) \cong H_{\mathcal{A}}^+(M)[n]$ as $H_{\mathcal{A}}(R)$-modules, so long as $M$ is bounded below.

A.6. Spectral sequences. Making some further mild assumptions on the algebras, we have useful spectral sequences for calculating the various flavors of $A$-homology.

**Proposition A.20.** If $A$ is a non-negatively graded dg-algebra ($A_n = 0$ for $n < 0$), there is a projection of dg-algebras $\pi : A \to H_0 A$, through which we can have $A$ act on $H_q M$. Then for any $\bullet \in \{+, -, \infty\}$, there is a conditionally convergent spectral sequence of $H_{\mathcal{A}}(R)$-modules

$$H^*_{\mathcal{A}}(A, H_q M) \to H^*_{p+q}(A, M).$$

If $H_0 A = R$, observe that the action of $A$ on $H(M)$ is trivial; if $H(M)$ is flat over $R$, we may identify $E^2_{p,q} \cong H^*_{\mathcal{A}}(A) \otimes H_q(M)$.

The spectral sequence for $H^+$ is always strongly convergent, the spectral sequence for $H^-$ and $H^{\infty}_+$ are strongly convergent if $H_\bullet(M)$ is bounded below.

**Proof.** Recalling that $B(M, A, R) \cong M \otimes BA$ as graded modules, we filter

$$F_p B(M, A, R) = \oplus_{i \leq p} (M)_{n-i} \otimes BA_i;$$

that is, $F_p B(M, A, R)$ consists of elements with total $BA$-degree at most $p$. The differential on $B(M, A, R)$ can be written (ignoring signs) as three terms:

$$d(m[a_1 | \cdots | a_n]) = (dm[a_1 | \cdots | a_n]) + (ma_1[a_2 | \cdots | a_n]) + (md([a_1 | \cdots | a_n])).$$

The first and last term clearly preserve the filtration, and the second does because $A_n = 0$ for $n < -1$. The first differential of the associated spectral sequence is the differential on the associated graded complex, which is $d_0 = d_M \otimes 1_{BA}$. We can thus identify the $E^1$ page with $BA \otimes HM$ (remember $BA$ is degreewise $R$-flat). The piece that decreases filtration by exactly 1 is $d_1$, given by the differential in $BA$ and multiplication by elements of $A_0$. Therefore we identify that the $E^1$ page is given as $B(HM, A, R)$, where a positive degree element of $A$ acts trivially (these components of the differential decrease the filtration by at least 2) and the action of $A_0$ factors through $A_0 \to H_0 A$. (This filtration is a pleasant example of the filtration on the totalization of a multicomplex; see [Boa99, Theorem 11.3].) Because this is a complete exhaustive Hausdorff filtration, it gives rise to a conditionally convergent spectral sequence by Proposition A.4.
In Boardman’s language this is a spectral sequence with exiting differentials because \((BA)_i = 0\) for negative \(i\), and strong convergence is given by A.3.

Observe that the action of \(C^{-}(R)\), which contracts against \(BA\), is filtered; so this is a spectral sequence of \(H_{\underline{A}}^{-}(R)\)-modules.

The proof for the \(H^{-}\) and \(H^{\infty}\) spectral sequences follow mostly as before. As graded modules, we may write
\[
cB(R, A, M)_n = \prod_{i = -\infty}^{0} cB(R, A, R)_i \otimes M_{n-i},
\]
where the symbol \(\prod_{i = -\infty}^{0}\) is as in Definition A.4, and filter as usual
\[
F_p cB(R, A, M)_n = \prod_{i = p}^{0} cB(R, A, R)_i \otimes M_{n-i}.
\]
(We again need \(A\) to be nonnegatively graded for this to be a filtration, and identify the \(E^2\) page as before.) Because this filtration is complete, exhaustive, and Hausdorff, we thus get a conditionally convergent spectral sequence of \(H_{\underline{A}}^{-}(R)\)-modules by Proposition A.4, because the action of \(C^{-}(R)\) is again filtered. As for \(C^{\infty}(M)\), as a graded \(R\)-module we have
\[
C^{\infty}_{\underline{A}}(M)_n = \prod_{i = -\infty}^{0} C^{\infty}_{\underline{A}}(R)_i \otimes M.
\]
The same filtration applies and is complete, giving a conditionally convergent spectral sequence of \(H_{\underline{A}}^{-}(R)\)-modules with the expected \(E^2\) page.

If \(H_{\ast}(M)\) is not bounded below, then in principle there could be infinitely many nonzero differentials leaving a given point in the spectral sequence for \(H_{\underline{A}}^{-}(M)\) or \(H^{\infty}_{\underline{A}}(M)\); but if \(H_{\ast}(M)\) is bounded below, then each \(E^{p,t}_n\) stabilizes at some finite \(p\), and thus \(RE^{\infty}_n = 0\). Because \(H_{\ast}(M)\) is bounded below, this is a half-plane spectral sequence, and so \(W(E) = 0\) as well, so these spectral sequences are strongly convergent.

\[\square\]

A.7. Group algebras. The most important application of the \(A\)-homology functors is for \(A = C_{\ast}(G; R)\), where \(G\) is a topological group (but most importantly a compact Lie group). The product structure is given as the composite
\[
C_{\ast}(G) \otimes C_{\ast}(G) \xrightarrow{EZ} C_{\ast}(G \times G) \xrightarrow{\gamma} C_{\ast}(G),
\]
where \(EZ\) is the Eilenberg-Zilber map. The unit is \([e] \in C_0(G)\) and the augmentation \(C_0(G) \rightarrow R\) is the natural augmentation (add up points). These makes \(C_{\ast}(G; R)\) into an associative augmented algebra. For the rest of this section, we frequently abuse notation and write \(G\) for \(C_{\ast}(G; R)\) unless there is danger of confusion.

We begin with the following well-known fact to justify our definition of group homology; a more detailed proof may be found in [GM74, Theorem 3.9]. Here \(EG\) denotes a left \(G\)-space that \(G\) acts freely on, and for a right \(G\)-space \(X\) we denote \(X_{hG} = (X \times EG)/G\) for the Borel construction on \(X\).

**Lemma A.21.** Let \(G\) be a compact Lie group and \(M = C_{\ast}(X; R)\) for \(X\) a right \(G\)-space. Then \(C^+_G(M) \simeq C_{\ast}(X \times EG)/G; R)\). We also have that \(C_G^{-}(M^\ast) = C_G^{-}(C_{\ast}^{-}(X))\) is the dual of \(C^{-}_{G}(M)\), and in particular
\[
H_{-G}(C_{\ast}^{-}(X; R)) \simeq H_{G}^{-}(X_{hG}; R).
\]
Product structures are compatible with this: the action of $H^*_G(R)$ on $H^*_G(M)$ is that of $H^{-*}(BG; R)$ on $H_*(X_{BG}; R)$.

The first fact follows because $(EG \times X)/G$ can be given as the bar construction $B(G, X)$ in topological spaces, and the natural map

$$B(C_*(X), C_*(G), R) \to C_*(B(X, G, \ast); R)$$

given levelwise by Eilenberg-Zilber maps is a homotopy-equivalence (with homotopy inverse given by a map built out of Alexander-Whitney maps). The second statement, that $C^*_A(M^\vee)$ is the dual of $C^*_A(M)$, is true for any algebra $A$. There is a slight subtlety in making sense of this: while $M$ is a right $A$-module, instead $M^\vee$ is a left $A$-module. So when we write $C^*_A(M^\vee)$, we mean $\text{Hom}_A(B(R, A, A), M^\vee)$.

Our first goal is to calculate what we need to make Theorem A.12 practically useful. The following is our first goal, which we prove in pieces.

**Theorem A.22.** Let $G$ be a compact Lie group $G$ of dimension $n$, and suppose that the homomorphism $\det \text{Ad} : \pi_0 G \to \mathbb{Z}/2$ is trivial, where $\text{Ad} : G \to \text{GL}(g)$ is the adjoint homomorphism. Then $C_*(G; R)$ is a Poincaré duality algebra of degree $n$, and in particular by Theorem A.19 we have $H^*_G\text{tw}(M) \cong H^*_G(M)[n]$.

**Lemma A.23.** If $G$ is a compact Lie group, then $D_G \cong \tilde{C}_*(S^{Ad}) =: \tilde{S}^{Ad}$ as left $G$-modules, where $S^{Ad}$ is the one-point compactification of the adjoint representation on $g$ and the $\tilde{C}$ indicates we take reduced chains.

**Proof.** As in [Kle01, Theorem 10.1], Atiyah duality gives a quasi-isomorphism of $C_*(G)$-bimodules

$$C_*(G) \to \text{Hom}_R(C_*(G), \tilde{C}_*(S^{Ad})),$$

where the right module structure comes from the left action on $C_*(G)$ and the left module structure from the left $G$-module structure on $\tilde{S}^{Ad}$; we thus have quasi-isomorphisms of left $G$-modules

$$D_G \to cB(R, G, \text{Hom}(G, \tilde{S}^{Ad})) \leftarrow cB(R, G, \text{Hom}(G, R)) \otimes \tilde{S}^{Ad}.$$

As in the proof of Theorem A.19, we conclude using the contractibility of $B(R, A, A)$ and the quasi-isomorphism of left $A$-modules

$$\text{Hom}(B(R, A, A), R) \cong \text{Hom}_A(B(R, A, A), \text{Hom}(A, R)).$$

The above lemma is useful because $H_*(\tilde{S}^{Ad})$ is a copy of $R$ concentrated in degree $d = \dim G$, and the action of $H_0 G = R[\pi_0 G]$ is given by $\det \text{Ad}$. Indeed, writing this $G$-module as $\lambda$, we have

**Lemma A.24.** If $G$ is a compact Lie group, then $\tilde{S}^{Ad}$ is related to $\lambda$ by a zig-zag of quasi-isomorphisms of left $G$-modules. Hence $C_*(G; R) \cong C_*(G; R)^\vee \otimes \lambda[d]$.

**Proof.** Let $B^\tau$ be the $G$-subcomplex of $\tilde{S}^{Ad}$ given by

$$B^\tau_k = \begin{cases} 0 & k < d \\ \text{Im}(d_{k+1}) \subset C_k(S^{Ad}) & k = d \\ C_k(S^{Ad}) & k > d \end{cases}$$

Clearly $B^\tau$ is acyclic, so $\tilde{S}^{Ad} \to \tilde{S}^{Ad}/B^\tau$ is a quasi-isomorphism of left $G$-modules. Then there is a natural inclusion $\lambda \to \tilde{S}^{Ad}/B^\tau$ picking out the generator.
of homology (because $\lambda = Z_d(\tilde{S}^{Ad})/B_d(\tilde{S}^{Ad})$). This is clearly a quasi-isomorphism of left $G$-modules.

This, combined with the Atiyah duality equivalence of the previous lemma, then gives us an equivalence

$$C_\ast(G) \simeq C_\ast(G)^\ast \otimes \tilde{S}^{Ad} \simeq C_\ast(G)^\ast \otimes \lambda[d].$$

With this, we can assemble the last of basic groups $H^G_G$, $H_G^\ast$, and $H_G^\ast$.

**Corollary A.25.** Let $G$ be a compact Lie group of dimension $d$, and suppose that $\det Ad$ is trivial. Then $H_G B(D_G; G, R) = H_{G-d}(BG; R)$, Thus, by Theorem A.12 (4),

$$H^G_G(G; R) = \begin{cases} H_{k-d-1}(BG; R) & k > d \\ H^{-k}(BG; R) & k \leq 0 \end{cases}$$

If $\det Ad$ was not trivial, instead the result would be $H_{k-d-1}(BG; \lambda)$ in degrees $k > d$.

In particular, the long exact sequence of Theorem A.12 (4) reduces to

$$\cdots \to H_{k-d}^+(M) \to H_k^\ast(M) \to H_k^\ast(M) \to \cdots$$

In general, the first term would be replaced by $H^G_{k-d}(M \otimes \lambda)$ (the degree shift reflected in the fact that $\lambda$ is concentrated in degree $d$).

We can generalize this calculation to that of the equivariant homologies of orbits $G/H$.

**Theorem A.26.** If $G$ is a compact Lie group and $H$ is a closed subgroup, let $G/H$ be the orbit of right cosets of $H$ (which is thus a right $G$-space). Then there is an $H$-module $\lambda_{NH}$, given by a copy of $R$ in degree $\dim G - \dim H$ with action given by a map $\pi_0 H \to \{\pm 1\}$ so that we have the following isomorphisms:

1. $H^G_{k-h}^+(G/H) \cong H^H_{k-h}^+(R)$
2. $H^G_{k-h}^-(G/H) \cong H^H_{k-h}^-(\lambda_{NH})$
3. $H^G_{k-h}^\ast(G/H) \cong H^H_{k-h}^\ast(\lambda_{NH})$.

**Proof.** The map $C_\ast(G) \to C_{\ast}(G/H)$ is $C_\ast(H)$-invariant, inducing a quasi-isomorphism of right $G$-modules $B(R, H, G) \to C_\ast(G/H)$ (that this is a quasi-isomorphism follows from [GM74, Theorem 3.9]). There is further a canonical isomorphism of chain complexes

$$B(B(R, H, G), G, R) \cong B(R, H, B(G, G, R)).$$

(1) then follows because there is an equivalence $B(G, G, R) \to R$ of left $G$-modules (induced by the augmentation $\varepsilon : C_\ast(G) \to R$). The same argument in general identifies $H^G_G(G \otimes_H M)$ with $H^H_{k-h}^+(M)$.

(2) follows from a combination of this idea and the argument of Lemma A.23: exactly analogous to the argument of [Kle01, Theorem 10.1], there is an Atiyah duality quasi-isomorphism $C_\ast(G/H) \to \text{Hom}_{C_\ast H}(C_\ast G, \tilde{C}_\ast S^{NH})$ of right $C_\ast(G)$-modules. Then we may show
\[
\text{Map}_G \left( B(R, G, G), G/H \right) \approx \text{Map}_G \left( B(R, G, G), \text{Map}_H (G, \tilde{S}^{NH}) \right) \\
\approx \text{Map}_H \left( B(R, G, G), \tilde{S}^{NH} \right) \approx \text{Map}_H \left( B(H, H, R), \tilde{S}^{NH} \right) = cB(R, H, \tilde{S}^{NH}).
\]

The second equivalence is given just as in Lemma A.23, via \( \eta(g) \to \eta(bg) \). The second-to-last equivalence follows from [GM74, Page 11] and verifying that, in that language, \( B(G, G, R) \) is a proper split Kunneth resolution of \( R \) as an \( H \)-module, which follows similar lines as verifying that \( C_\bullet(G) \) is split as an \( H \)-module, implicit in the proof of (1). The rest of the statement follows as in Lemma A.24, reducing \( \tilde{S}^{NH} \) to a 1-dimensional representation \( \lambda_{NH} \).

We can combine the previous two parts and the long exact sequence of Theorem A.12 (4) to verify the isomorphism on Tate homology. This follows from the following homotopy commutative diagram, where every vertical arrow is a quasi-isomorphism and horizontal arrow is an appropriate modification of the norm map.

\[
\begin{array}{ccc}
B(G/H, G, D_G) & \longrightarrow & cB(R, G, G/H) \\
\downarrow & & \downarrow \\
B(R, H, D_G) & \longrightarrow & cB(R, G, G/H) \\
\downarrow & & \downarrow \\
B(R, H, D_H \otimes \tilde{S}^{NH}) & \longrightarrow & cB(R, H, \tilde{S}^{NH})
\end{array}
\]

The bottom-left vertical map is induced by the map of pairs (recall that \( cB \) is contravariant in the algebra and covariant in the module) \( (G, G) \to (H, H \otimes \tilde{S}^{NH}) \), the first map inclusion \( H \hookrightarrow G \) and the second map a logarithm map

\[
C_\bullet(G) \to C_\bullet(\text{Th}(NH)) \xrightarrow{AW} C_\bullet(H) \otimes \tilde{S}^{NH}.
\]

This is the explicit form Klein gives of the Atiyah duality isomorphism. As it is a quasi-isomorphism, the map on bar constructions is a quasi-isomorphism.

Finally, we replace the bottom left term with \( B(D_H \otimes \lambda_{NH}, H, *) \) and commute \( \lambda_{NH} \) across with the following lemma to identify the bottom row with the norm map for the \( H \)-module \( \lambda_{NH} \).

**Lemma A.27.** Let \( \rho \) be a \( G \)-bimodule with action (on both sides) induced by some group homomorphism \( f : \pi_0G \to R^\times \). Then \( B(\rho, G, M) \cong B(G, \rho \otimes M) \).

**Proof.** If \( \pi : C_\bullet(G; R) \to H_0G \to R \) is the composition of the projection and the map \( [g] \mapsto f(g) \), there is an isomorphism \( B(R, G, \rho \otimes M) \to B(\rho, G, M) \) given by

\[
\pi(a_1 \cdots a_k)[a_1 | \cdots | a_k]m \mapsto [a_1 | \cdots | a_k]m;
\]

the inverse uses the inverse in \( \pi_0G \) (giving rise to the antipode in \( H_0G = R[\pi_0G] \)). This is part of a general isomorphism for moving factors across two-sided bar construction over a Hopf algebra, but the simple form here follows because the \( G \)-action factors through the action of a central algebra concentrated in degree 0.

Observe that everywhere thusfar we have twisted by some orientation character, if we had worked with \( R \) a ring of characteristic 2, the action of \( \pi_0G \) on \( R \) would in fact be trivial.
We conclude this section with some related results which we use in the main text.

**Example A.2.** We may apply this to calculate the three cases relevant to us in this text: \( G = SO(3) \) and \( H \) one of the three subgroups \( \{e\}, SO(2), \) and \( SO(3) \). In every case \( H \) is connected, and so the representation \( \lambda_{NH} \) of \( \pi_0 H \) is trivial and is precisely \( R[\dim G - \dim H] \). Thus \( H^\bullet_{SO(3)}(SO(3)/H; R) = H^\bullet_H(R) \), with a dimension shift if appropriate.

When \( H = \{e\} \), the Tate homology is trivial, and \( H^+_\{e\}(R) = H^-\{e\}(R) = R \) concentrated in degree zero. (This is just the axiomatic property of Tate homology.)

When \( H = SO(2) \), we are left with \( H^-_{SO(2)}(R) = H_*(\mathbb{C}P^\infty) \) and \( H^+_{SO(2)}(R) = H^*(\mathbb{C}P^\infty) \) by Lemma A.21, and the Tate homology is a splicing of these. By Proposition A.18, as modules over \( H^-_{SO(2)}(R) \), we may write the Borel homology, coBorel homology, and Tate homology respectively as

\[
R[V], R[V^{-1}], R[V, V^{-1}],
\]

where \( |V| = 2 \). Then, because \( H^-_{SO(3)}(R) \rightarrow H^-_{SO(2)}(R) \) sends

\[
p_1 \mapsto V^2 \in H^-(BSO(2); R),
\]

we learn the module structure of these over \( \langle p_1 \rangle \in H^-(BSO(3); R) \). When \( \frac{1}{2} \in R \), we have \( H^-(BSO(2); R) = H^-(BSU(2); R)R[p_1] \), so this determines the module structure over \( H^-_{SO(3)}(R) \).

On the other hand, we have that

\[
H^-(BSO(3); \mathbb{Z}/2) = (\mathbb{Z}/2)[w_2, w_3]
\]

is a power series ring in two generators (where here one takes \( H^* = \prod H^{-k} \) as opposed to a direct sum; the latter would give a polynomial ring). This is standard, and proved in [Hat, Proposition 3.12]. A calculation with integral coefficients is given in [BJ82], and in particular shows that \( p_1 \) restricts to \( e^2 \) on oriented 2-plane bundles.

**Proposition A.28.** There is a chain of algebra quasi-isomorphisms

\[
C^\text{gm}_\ast(SO(3); R) \cong C_\ast(SO(3); R).
\]

**Proof.** We neglect to mention the coefficients \( R \) throughout, as they play no major role.

First we should replace singular chains with something more easily comparable with the degeneracy relations involved in \( C^\text{gm}_\ast \). We define the chain complex of smooth singular chains \( C^\text{sm}_\ast(M) \) to be the set of smooth maps from \( \Delta^n \rightarrow M \); then it has a quotient \( C^\text{smd}_\ast(M; R) \) after we carry out the previous identifications under orientation-preserving diffeomorphisms and quotienting by the same degeneracies. There are, in this situation, two forgetful maps \( C^\text{smd}_\ast(M) \leftarrow C^\text{sm}_\ast(M) \rightarrow C_\ast(M) \). The easiest way to see that all of these maps are quasi-isomorphisms is to prove that all of these theories are in fact homology theories on smooth manifolds \( M \), and that the induced map on \( M = \text{pt} \) is an isomorphism on homology. In fact the rightmost chain complexes are identical for \( M = \text{pt} \). On the other hand, \( C^\text{smd}_\ast(\text{pt}) = R \), a copy of the ground ring concentrated in degree zero. All higher-dimensional chains are degenerate. Because \( H_\ast(\text{pt}) = R_{(0)} \), and the given maps do the obvious things to points (which are cycles generating \( H_\ast(\text{pt}) \)), they are all homology isomorphisms. A proof that these are homology theories follows similar lines as Theorem 5.7.
It is also possible to show these quasi-isomorphisms extremely explicitly, using e.g. smooth approximation to find cycles in \( C^\text{sm}_* \) homologous to any given cycle in \( C_* \).

So we now need to relate the chain complex of smooth simplices modulo degeneracy, \( C^\text{sm}_*(M) \), to \( C^\text{gm}_*(M) \).

The theory interpolating between these is the chain complex of \emph{triangulated geometric chains} on a smooth manifold \( M \). This chain complex \( C^\text{tgm}_*(M) \) is functorial under smooth maps, and its homology groups define a homology theory (as before).

Precisely, a \emph{triangulated basic chain} on \( M \) is a compact smooth oriented manifold with corners \( P \) equipped with a smooth triangulation (that is, a homeomorphism \( f : |X| \to P \), where \( |X| \) is the realization of a simplicial complex, so that \( f \) is a diffeomorphism from each closed simplex onto its image), and a smooth map \( \sigma : P \to M \). Two triangulated basic chains are isomorphic if there is an orientation-preserving diffeomorphism \( \phi : P \to P' \) and an isomorphism of simplicial complexes \( \psi : X \to X' \) so that \( \phi f = f'\psi \) and \( \phi \sigma = f'\sigma \). The triangulated geometric chain complex \( C^\text{tgm}_*(M) \) is defined following the same procedure as for \( C^\text{gm}_*(M) \): identify orientation-reversals with their negative, and quotient by the subcomplex of degenerate chains (triangulated basic chains for which the images of \( \sigma|_{\Delta_k} \) and \( \sigma|_{\Delta^n_k} \) are both contained in the image of some smooth manifold of smaller dimension than \( k \), resp \( k - 1 \), for each component simplex \( \Delta_k \).

Now observe that there are natural chain maps

\[
C^\text{sm}_*(M) \leftarrow C^\text{tgm}_*(M) \to C^\text{sm}_*(M).
\]

The left map is given by forgetting the triangulation and thinking of a smooth manifold with corners as a very special kind of \( \delta \)-chain, and the right map is given by sending a triangulated \( n \)-chain to the sum of its component simplices: \( \sigma : P \to M \) to

\[
\sum_{\Delta^n \subseteq X} \sigma|_{\Delta^n}.
\]

As before, because these are homology theories, to show that these are quasi-isomorphisms in general it suffices to check that these maps are isomorphisms on \( H_*(\text{pt}) \). Each of these chain complexes (because of the nondegeneracy requirements) are simply a copy of \( R \) concentrated in degree zero, and the induced map between them is the identity.

Now, if \( G \) is a Lie group, \( C^\text{sm}_*(G) \) is a dg-algebra using the Eilenberg-Zilber product (multiply the chains \( \sigma \times \eta \) using the group structure and triangulate the result according to a standard triangulation of \( \Delta^k \times \Delta^n \)). So is \( C^\text{tgm}_*(M) \): now if \( \sigma : P \to M \) and \( \eta : Q \to M \) are triangulated basic chains, their product is \( \sigma \times \eta : P \times Q \to M \) equipped with the product triangulation (again, triangulate each component \( \Delta_k \times \Delta^n \) in a standard way). With these algebra structures, the maps

\[
C^\text{sm}_*(M) \leftarrow C^\text{tgm}_*(M) \to C^\text{gm}_*(M)
\]

are in fact dg-algebra homomorphisms. It is even easier to see that the map \( C^\text{sm}_*(M) \leftarrow C^\text{sm}_*(M) \to C_*(M) \) are dg-algebra homomorphisms, giving us the desired zig-zag of dg-algebra equivalences.

A.8. Periodic homological algebra. In this section, we set up a version of the machinery before, built to work for complexes graded over \( \mathbb{Z}/2\mathbb{N} \) which are finite in a suitable sense. In the main text, we will frequently want to use a filtration
resembling the index filtration on the Morse-Bott complex of a finite-dimensional compact smooth manifold, but because the index is only defined in \( \mathbb{Z}/8 \), this doesn’t make sense! So if we want to use spectral sequences to check that maps are quasi-isomorphisms, we must do something at least slightly more subtle.

We resolve this by passing instead to the unrolled complex \( \tilde{C}_q \) of \( C \), defined as \( \tilde{C}_k = C_{k \mod 2N} \), with the obvious differential. A 2\( \mathbb{Z} \)-periodic complex is a complex \( \tilde{C} \) equipped with a periodicity isomorphism \( \varphi : \tilde{C} \rightarrow \tilde{C}[2N] \). From here, we recover the \( \mathbb{Z}/2N \)-graded complex by picking an interval \([i, i + 2N - 1] \subseteq \mathbb{Z} \), and for \( k \in [i, i + 2N - 1] \), define

\[
C_k \mod 2N = \tilde{C}_k;
\]

the differential is defined in the obvious way except on \( C_{i \mod 2N} \), where it is defined as \( dx = \varphi(d\tilde{x}) \); this makes sense as \( \varphi(d\tilde{x}) \) is in degree \( i - 1 + 2N = i + 2N - 1 \). The definitions immediately imply that the \( \mathbb{Z}/2N \)-graded complex \( C \) did not depend on the choice of representative interval, and that \( H_k(\tilde{C}) = H_{k \mod 2N}(C) \).

This implies that if we’re trying to show that a map \( C \rightarrow C' \) of \( \mathbb{Z}/2N \)-graded complexes is a quasi-isomorphism, this is equivalent to showing that the same is true for the map \( \tilde{f} : \tilde{C} \rightarrow \tilde{C}' \) of 2\( \mathbb{Z} \)-periodic complexes. We may more or less pass freely between these notions. To avoid notational irritation, we ignore the periodicity isomorphism \( \varphi \) — up to isomorphism of 2\( \mathbb{Z} \)-periodic complexes, we can take \( \tilde{C}_k = \tilde{C}_{k+2N} \) on the nose, and \( \varphi = \text{Id}[2N] \).

Instead of attempting to filter a \( \mathbb{Z}/2N \)-graded complex, we find that the appropriate notion seems to the following.

**Definition A.9.** Let \( \tilde{C} \) be a 2\( \mathbb{Z} \)-periodic complex. A periodic filtration on \( \tilde{C} \) is a filtration \( \cdots \subset F_s \tilde{C} \subset F_{s+1} \tilde{C} \subset \cdots \), with

\[
\bigcup_s (F_s \tilde{C}) = \tilde{C}; \quad \bigcap_s F_s \tilde{C} = 0
\]

and \( F_{s+2N} \tilde{C} = (F_s \tilde{C})[2N] \); that is,

\[
F_{s+2N} \tilde{C}_{t+2N} = F_s \tilde{C}_t.
\]

So while the filtration on each of the individual abelian groups \( F_s C_k \) may stabilize (so \( F_N C_k = C_k \) for large \( N \)) — as indeed is frequently the case for us — the filtration itself is infinite in both directions (i.e., neither is the complex \( F_{-N} \tilde{C} \) equal to zero for any \( N \geq 0 \), nor is \( F_N \tilde{C} = \tilde{C} \) for any \( N \)).

We now pass to the spectral sequence \( E^r_{s,t} \) of the filtered complex \( \tilde{C} \). The associated graded complex is

\[
E^r_{s,t} \tilde{C} = F_s \tilde{C}_{s+t}/F_{s-1} \tilde{C}_{s+t},
\]

and the fact that the filtration is periodic implies that \( E^r_{s,t} = E^r_{s+2N,t} \). This is the map induced by the periodicity isomorphism \( \tilde{C} \rightarrow \tilde{C}[2N] \), which is a filtered chain map; because it induces an isomorphism on the \( E^0 \) page, the same is true for all pages \( E^r \) of the spectral sequence: there is an isomorphism \( E^r_{s,t} \rightarrow E^r_{s+2N,t} \) preserving the differentials.

If so desired, we may thus view this filtration as inducing a \((\mathbb{Z}/2N, \mathbb{Z})\)-bigraded spectral sequence \( E^r_{[s],t} \), with \([s] \in \mathbb{Z}/2N\). This may be pictured as a cylinder, where differentials wrap around.
If we have a map \( f : \tilde{C} \to \tilde{C}' \) of unrolled complexes, compatible with a periodic filtration of each, it induces a map of the spectral sequences \( E_{s,t}^r \). We would like to know when we can check that the map \( f \) is a quasi-isomorphism from the corresponding fact about the \( E^2 \) page of this spectral sequence. Because the ‘unrolled’ spectral sequence \( E_{s,t}^r \) (considered with \((\mathbb{Z},\mathbb{Z})\)-bigrading, \(8\mathbb{Z}\)-periodic in the first grading) is a whole-plane spectral sequence, it is difficult to prove that the \( E^\infty \) page actually calculates the homology groups. However, as long as the periodic filtration is complete, we at least still know from Theorem A.4 that we may detect quasi-isomorphisms from isomorphisms on the \( E^2 \) page; further, if the spectral sequence degenerates on some finite page, we may indeed calculate the associated graded homology groups from the \( E^\infty \) page.

Now let \( A \) be a bounded, non-negatively graded dg-algebra satisfying Poincaré duality of degree \( n \), as in Section A.5. Let \( M \) be a right \( dg\)-\( A \)-module, graded over \( \mathbb{Z}/2N \), equipped with a complete periodic filtration; suppose that the associated graded \( A \)-module \( F_pM/F_{p-1}M \) is bounded.

We would like to apply the constructions of the previous sections to construct ‘equivariant homology’ complexes \( C^*_A(M) \), with corresponding periodic filtrations. The clear thing to try is to pass to the unfolded complex \( \tilde{M} \) and consider the corresponding complexes \( C^*_A(M) \).

For a right \( A \)-module \( M \) with periodic filtration as above, we thus write

\[
\begin{align*}
C^+_A(M) &:= \hat{B}(\tilde{M}, A, R), \\
C^-_A(M) &:= \hat{cB}(A, \tilde{M}) = \text{Hom}^{\hat{R}}(B(R, A, A), \tilde{M}) \\
C_A^0(M) &:= \text{Cone} \left( \hat{B}(M, A, \hat{D}_A) \to \hat{cB}(A, \tilde{M}) \right).
\end{align*}
\]

The filtrations \( F_s C^+_A(M) = \hat{C}_A^s(F_s \tilde{M}) \) are now complete, exhaustive, and Hausdorff; these homology groups are what the spectral sequences induced by the filtration on \( C^*_A(M) \) actually converge to.

We may assemble this into the following homology theories for periodically graded \( A \)-modules.

**Theorem A.29.** Let \( A \) be a non-negatively graded, bounded above, and \( R \)-free \( dg\)-algebra satisfying Poincaré duality of degree \( n \). If \( M \) is a \( \mathbb{Z}/2N \)-graded right \( A \)-module, equipped with a periodic filtration whose associated graded complex \( \text{gr}_p M \) is bounded for each \( p \), there are \( \mathbb{Z}/2N \)-graded complexes \( C^*_A(M) \), for \( \bullet \in \{+, -, \infty\} \), satisfying the following properties.

1. Each \( C^*_A(M) \) is equipped with a left action of \( \hat{C}_A^\infty(R) \);
2. The \( C^*_A(M) \) are functorial under \( A \)-module maps \( f : M \to M' \) of \( \mathbb{Z}/2N \)-graded complexes, which are filtered in the sense that \( \hat{f}(F_s \tilde{M}) \subset F_{s+d} \tilde{M}' \) for all \( s \) and a fixed constant \( d \).
(3) There is a strongly convergent spectral sequence of $H(A)$-modules,
$$\text{gr}_p(H_q(M)) \to H_{p+q}(M),$$
which degenerates at some finite page;

(4) There is a conditionally convergent spectral sequence of $H_A(R)$-modules
$$H^\bullet_A(\text{gr}(\hat{M})) \to H^\bullet_A(M),$$
which is strongly convergent for $H^\bullet_A(M)$.

(5) A filtered $A$-module map $M \to M'$ which is a quasi-isomorphism on the associated graded complexes induces a quasi-isomorphism $C^\bullet_A(M) \to C^\bullet_A(M')$;

(6) If the associated graded is free over $A$, in the sense that $\text{gr}_p\hat{M} \cong X_p \otimes A$ for each $p$ and some $R$-free complex $X_p$, then $H^\bullet_{\hat{A}}(M) = 0$;

(7) There is a long exact sequence of $H_{\hat{A}}(R)$-modules
$$\cdots \to H^\bullet_{\hat{A}}(M) \xrightarrow{[n]} H^\bullet_{\hat{A}}(M) \to H^\bullet_{\hat{A}}(\hat{M}) \xrightarrow{[-n-1]} H^\bullet_{\hat{A}}(\hat{M}) \to \cdots$$
natural under filtered $A$-module maps.

Proof. The actions of $C^\bullet_A(R)$ on $C^\bullet_A(M)$ were detailed in the sections the completed complexes were introduced. An $A$-module map $f : M \to M'$ induces an $A$-module map $f : M \to M'$, and therefore a map $C^\bullet_A(M) \to C^\bullet_A(M')$, so it induces a map on their completions — which are the unwrapped, $\mathbb{Z}$-graded complexes of $C^\bullet_A(M)$ and $C^\bullet_A(M')$.

By assumption, each $\text{gr}_p(M_k) \cong \text{gr}_{p+2N}(M_{k+2N})$ is bounded in $k$, which has the same bound for $k \equiv k' \pmod{2N}$. Because there are only finitely many equivalence classes modulo $2N$, there is a uniform bound in $k$ on $\text{gr}_p(M_{p+k})$, independent of $p$. In particular, the filtration is finite in each degree, and the associated spectral sequence is contained in some finite strip; if the strip has width $\ell$, then the differentials on any $E_{s,t}$ must leave the strip if $r > \ell + 1$, and so the spectral sequence degenerates at $E^{s+2}_r$. This gives (3).

The existence of the spectral sequence was outlined above. If we also know that $\hat{H}^\bullet_A(\text{gr}_p\hat{M})$ takes quasi-isomorphisms of $A$-modules to isomorphisms of $\hat{H}^\bullet_A(R)$-modules, we may conclude item (5), as we have detected an isomorphism on the $E^1$ page. That this is true follows because $A$ is $R$-free, non-negatively graded, and bounded above, while each $\text{gr}_p\hat{M}$ is bounded: for $\hat{H}^+$ this is Lemma A.8, for $\hat{H}^-$ this is Lemma A.9, and for $\hat{H}^\infty$ this is Corollary A.16.

The vanishing theorem A.12 (4) implies that if $\text{gr}\hat{M} \cong X \otimes A$, we have
$$H^\infty_{\hat{A}}(\text{gr}\hat{M}) = 0,$$
and so the spectral sequence $H^\infty_{\hat{A}}(\text{gr}\hat{M}) \to H^\infty_{\hat{A}}(\hat{M})$ collapses at the $E^1$ page, giving $H^\infty_{\hat{A}}(\hat{M}) = 0$.

The long exact sequence is clear from the existence of a short exact sequence
$$0 \to \hat{B}(\hat{M}, A, \hat{D}_{\hat{A}}) \to C_{\hat{A}}(\hat{N}_{\hat{M}}) \to C^\bullet_{\hat{A}}(M) \to 0,$$
where $C_{\hat{A}}(\hat{N}_{\hat{M}})$ is the mapping cylinder of the norm map, and so is naturally equivalent as $C^\bullet_{\hat{A}}(R)$-modules to $C^\bullet_{\hat{A}}(M)$. In particular, by the completed analogue of Theorem A.19 (which is proved in an essentially identical manner), we have that the homology of the first term is naturally equivalent to $H^\infty_{\hat{A}}(M)[n]$ as an $H^\bullet_{\hat{A}}(R)$-module. The discussion there guarantees this for $M$ bounded below, but if
$M' \subset \tilde{M}$ is some bounded below $A$-submodule which equals $M$ in all sufficiently large degrees, one may identify $\hat{H}_A^{+,tw}(M) = \lim_{n \to -\infty} \hat{H}_A^{+,tw}(M'[−2N])$.

References


