3A Practice Problems for Final Exam

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(1) Find the derivative of the following functions:

(a) \( f(x) = \sin(1-x) \)
(b) \( f(x) = e^{2x+1} \)
(c) \( f(x) = \ln(x^2 + x + 1) \)
(d) \( f(x) = x \cdot 2^x \)
(e) \( f(x) = \tan(x) \) (Hint: \( \tan(x) = \frac{\sin(x)}{\cos(x)} \))
(f) \( f(x) = 9^x \sin(x) \)
(g) \( f(x) = xe^{-x} \)
(h) \( f(x) = x + \frac{1}{x^2} + \frac{x}{\sin(x)} \)
(i) \( f(x) = \sin(\pi x) \cos(e^x) \)

(2) Calculate the following limits (if they exist). You may use whatever rules you want.

(a) \( \lim_{x \to 1} \ln(x) \cdot \cos(\pi x) \)
(b) \( \lim_{x \to 2} \frac{x^3 - 1}{x^2 - 4} \)
(c) \( \lim_{x \to \infty} \frac{x^3 - 1}{x^2 - 2x + 3} \)
(d) \( \lim_{x \to \infty} \frac{x^3 + 1}{x^2 - 4x + 5} \)
(e) \( \lim_{x \to 0^+} \ln(x) \)
(f) \( \lim_{x \to 3} \frac{x^2 - 2x - 3}{x^2 - 9} \)
(g) \( \lim_{x \to 0} \frac{\sin(x)}{x} \)
(h) \( \lim_{x \to 0} \frac{1 - \cos(x)}{\sin(x)} \)
(i) \( \lim_{x \to \infty} \frac{e^x + 1}{e^x + 5} \)
(j) \( \lim_{x \to 0} \frac{3x - 2x}{2x - x} \)

(3) Sketch the graphs of the following functions. Find where \( f(x) \) is increasing/decreasing, and mark any inflection points, and any horizontal or vertical asymptotes:

(a) \( f(x) = x^4 - 4x \)
(b) \( f(x) = x + \frac{1}{x} \)
(c) \( f(x) = x - 3x^{1/3} \)
(d) \( f(x) = \frac{3x}{x^2 - 4} \)
(e) \( f(x) = \frac{x^2 + 4}{x^2 + 3} \)
(f) \( f(x) = e^{-x^2/2} \)

(4) Consider Figure 1, which shows two functions. Identify which is the derivative of the other.
Consider the function
\[ f(x) = \begin{cases} 
    x + 1 & \text{if } x \leq 1 \\
    2 & \text{if } 1 < x \leq 2 \\
    \frac{1}{x^3} & \text{if } x > 2 
\end{cases} \]

(a) Determine whether \( f(x) \) is continuous at \( x = 1 \).
(b) Do the same for \( x = 2 \) and \( x = 3 \).
(c) At what points is \( f(x) \) not differentiable?

Find \( a \) so that
\[ f(x) = \begin{cases} 
    \sin(2x) & \text{if } x \neq 0 \\
    \frac{\sin(2x)}{a} & \text{if } x = 0 
\end{cases} \]
is continuous at \( x = 0 \).

Find the points on \( f(x) = x^3 - 3x - 2 \) where the slope of the tangent line is minimal.

Consider the graph of \( f(x) \) in Figure 2. Which of the following is true?

(a) \( f(-1) < f'(-1) < f''(-1) \)
(b) \( f''(-1) < f'(-1) < f(-1) \)
(c) \( f''(-1) < f(-1) < f'(-1) \)
(d) \( f'(-1) < f(-1) < f''(-1) \).

(a) What is \( 16^{3/4} \)?
(b) Approximate $17^{3/4}$ using linear approximation. Simplify your answer.

(10) A population has been increasing linearly since 1990. In 2000, the population was 100, and in 2005, the population was 500. Find an equation for $P(t)$, where $P$ is the population and $t$ is the number of years since 1990, and use it to approximate the population in 2007.

(11) You have 500 grams of a radioactive substance whose half-life is 20 days. Let $N(t)$ denote the amount of the substance left after $t$ days.
   (a) Find a formula for $N(t)$.
   (b) Find how long it will take for 80% of the substance to decay.

(12) In the following, find $dy/dx$
   (a) $x^3 + y^3 = 4y$
   (b) $-xy + \ln(y) = 1$
   (c) $e^x + ey = y^2$
   (d) $\cos(x) \sin(y) = 2x$
   (e) $y^2 = e^{x^2} + 2x$
   (f) $2\sqrt{y} = x - y$

(13) You must take 400 mg of a certain medicine every 2 hours. For each 2 hour period, the body clears out 60% of the medicine. Let $a_n$ denote the amount of medicine in the body right after you take the $n$-th dose. Assume you have not taken any medicine before your first dose today.
   (a) Write a difference equation for $a_n$.
   (b) Find the equilibrium for this difference equation. Use a sentence and your equilibrium value to answer: In the long term, what is greatest amount of medication in your body.
   (c) You cannot have more than 640 mg of the medicine in your body at any time. After how many hours should you stop taking the medication?
   (d) Let $b_n$ denote the amount in the body right before the $n$-th dose. Write a difference equation for $b_n$ and find the equilibrium.
   (e) Compare the two different equilibrium values from parts (b) and (d). Does it make sense that they are different? What relationship do they have?
Answers:

1. (a) $-\cos(1 - x)$
   (b) $2e^{2x+1}$
   (c) $\frac{2x+1}{x^2+x+1}$
   (d) $x \cdot 2^x \ln(2) + 2^x$
   (e) $\sec^2(x) = \frac{1}{\cos^2(x)}$
   (f) $9^x \sin(x) \ln(9) \left[x \cos(x) + \sin(x)\right]$
   (g) $\frac{(x^2+1)(-xe^{-x}+e^{-x})-2xe^{-x}}{(x^2+1)^2}$
   (h) $f(x) = 1 - \frac{2}{x^2} + \frac{\sin(x) - x \cos(x)}{\sin^2(x)}$
   (i) $\pi \cos(e^\pi) \cos(\pi x)$

2. (a) 0
   (b) 4
   (c) $-1/2$
   (d) 0
   (e) DNE
   (f) 2/3
   (g) 1
   (h) 0
   (i) 0
   (j) $-(\ln(3) - \ln(2))$

3. Check your answers on WolframAlpha or on a graphing calculator.

4. The top function (i.e. the purple one) is the derivative of the bottom (i.e. blue) curve.

5. (a) It is continuous at $x = 1$
   (b) Not continuous at $x = 2$, and not continuous at $x = 3$
   (c) Not differentiable at $x = 2, 3$.

6. $a = 2$

7. (0, -2)

8. d is correct.

9. (a) 8
   (b) $67/8$

10. $P(t) = 100 + 80(t - 10)$, population in 2007 approximately 660.

11. (a) $N(t) = 500(.5)^{t/20}$
    (b) $20 \ln(.8) / \ln(.5)$ days.

12. (a) $y' = \frac{3x^2}{4 - 3y}$
    (b) $y' = \frac{y^2}{4 - xy}$
    (c) $y' = \frac{e^x}{4y - e^y}$
    (d) $y' = \frac{2 + \sin(x) \sin(y)}{\cos(x) \cos(y)}$
(e) \( y' = \frac{2xe^{x^2+2}}{2y} \)

(f) \( y' = \frac{1}{\sqrt{y}+1} \)

\[ (13) \]

(a) \( a_{n+1} = 0.4a_n + 400, \ a_1 = 400 \)

(b) 666.667 grams. In the long run, there will be this amount in the body.

(c) Stop before the fourth treatment.

(d) \( b_{n+1} = 0.4(b_n + 400), \ b_1 = 0, \) equilibrium: 266.667

(e) This is 40\% of the equilibrium from (b), which makes sense since 60\% of the drug will clear out before you measure the amount.