Now we will show that any polynomial in one variable is finitely 2-refinable. (A function $f : \mathbb{R} \mapsto \mathbb{R}$ is finitely 2-refinable if it is in the linear span of $\{f(2x-l) | l \in \mathbb{Z}\}$, i.e., if there exist finitely many scalars $c_l \in \mathbb{R}$ such that $f(x) = \sum_l c_l f(2x-l)$. What we show will also work in $\mathbb{C}$.)

Theorem 1. *Any polynomial in $\mathbb{R}[x]$ is finitely 2-refinable.*

Proof. Let $f(x)$ be a polynomial. Then it is of the form $\sum_{k=0}^n a_k x^k$, where $a_n \neq 0$. Let $l_0, .., l_n$ be $n+1$ arbitrary distinct integers; we will give an explicit computation of coefficients c_l for $l = l_0, .., l_n$ such that

$$f(x) = \sum_{l=l_0}^{l_n} c_l f(2x-l) \qquad (1)$$

This is equivalent to

$$\sum_{j=0}^n a_j x^j = \sum_l c_l \sum_{k=0}^n a_k (2x-l)^k$$

$$= \sum_l c_l \sum_{k=0}^n a_k \sum_{m=0}^k \binom{k}{m} (2x)^m (-l)^{k-m}$$

$$= \sum_{m=0}^n (2x)^m \sum_{k=m}^n \sum_l c_l a_k \binom{k}{m} (-l)^{k-m}$$

Since these are equal as functions, the coefficients of each x^j must match. So, equation (1) is satisfied iff for each $m = 0, .., n$,

$$\frac{a_m}{2^m} = \sum_{k=m}^n \sum_l c_l a_k \binom{k}{m} (-l)^{k-m}$$

$$= \sum_l c_l \sum_{k=0}^{n-m} a_{k+m} \binom{k+m}{m} (-l)^k$$

This can be written as a matrix condition. Let $g_j(x)$ be the polynomial

$$\sum_{k=0}^{n-j} a_{k+j} \binom{k+j}{j} (-1)^k x^k$$

Then the above is equivalent to

$$\begin{bmatrix} g_0(l_0) & g_0(l_1) & ... & g_0(l_n) \\ g_1(l_0) & g_1(l_1) & ... & g_1(l_n) \\ ... & & & \\ g_n(l_0) & g_n(l_1) & ... & g_n(l_n) \end{bmatrix} \begin{bmatrix} c_{l_0} \\ c_{l_1} \\ ... \\ c_{l_n} \end{bmatrix} = \begin{bmatrix} a_0 \\ \frac{a_1}{2} \\ ... \\ \frac{a_n}{2^n} \end{bmatrix}$$

If this $n+1 \times n+1$ matrix with the $(m,j)^{th}$ entry equal to $g_m(l_j)$ (call it A) is invertible, then this system of equations has a solution for some choice of c_ls. In fact, they can be calculated explictly once the parameters l_j have been determined, by applying the inverse of A to the vector $(a_j/2^j)$.

To show that A is invertible, we show by contradiction that its row vectors are linearly independent. Let r_j be the vector in $\mathbb{R}^{n+1}$ corresponding to the j^{th} row of A, and suppose that for some scalars b_j, $\sum b_j r_j$ is the zero vector. This in turn means that for each $l = l_0, .., l_n$,

$$\sum b_j g_j(l) = 0$$

Now, $\sum b_j g_j$ is a polynomial of degree at most n, and it is 0 for $n+1$ distinct values of l, which means that it must be the constant 0 function, because a non-0 polynomial has only as many roots as its degree. Meanwhile, each g_j has degree $n-j$, because the coefficient of x^{n-j} is $a_n\binom{n}{j}(-1)^n$ and therefore not 0 ($a_n \neq 0$). So, we have $\sum b_j g_j =$

$$\begin{array}{llll} b_0\Big(a_n\binom{n}{0}(-1)^n x^n & +a_{n-1}\binom{n}{1}(-1)^{n-1}x^{n-1} & +... & \Big) \\ & b_1\Big(a_n\binom{n}{1}(-1)^{n-1}x^{n-1} & +... & \Big) \\ ... & & & \\ & & & +b_n a_n\binom{n}{n} \end{array}$$

If this is the constant 0 function, then the coefficient of each x^j must be 0. The coefficient of x^n is $b_0 a_n\binom{n}{0}(-1)^n$. If this is 0, then b_0 has to be 0. This clears out the whole top row in the above sum, so that the coefficient of x^{n-1} is $b_1 a_n\binom{n}{1}(-1)^{n-1}$, and so b_1 is 0. Proceeding this way, we see that each b_j must be 0, and so the rows are linearly independent.

$\square$