

# A possible way to prove the 4-colour theorem

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Let  $\{\Omega_j\}$  be a "partition" of the unit square  $[0, 1]^2$  into a finite number of open sets. That is, the union of the closure of the  $\Omega_j$ s is the whole unit square. To prove the 4-colour theorem we need to show that a 4-colouring (that is, an assignment of each  $\Omega_j$  to one of 4 colours, such that no two neighboring  $\Omega_j$ s share the same colour) exists no matter what the  $\{\Omega_j\}$ s are. The idea is to set up a minimization problem, such that a minimizer will be a 4-colouring.

Let  $\{\phi_j\}_{j=1}^4$  be a family of smooth functions. Or, we could think of them as a vector valued function  $\phi = (\phi_1, \phi_2, \text{etc})$ . We will colour the set  $\{x | \phi_j(x) > 0\}$  with colour  $j$ . However, for each  $j$ , we would like for every point of  $\Omega_j$  to have the same colour, and we would like for neighboring  $\Omega_j$ s to have distinct colours. So we set up a functional of  $\phi$  (an energy) which penalizes for violating those conditions. Also, every point of the unit square should only be assigned one colour. Then, at every  $x$ , only one  $\phi_j$  should be positive. This can be codified by the condition

$$\sum_j H(\phi_j) = 1$$

( $H$  is the heavyside function.) So we are doing a constrained minimization. Now, suppose that some  $\Omega_i$  is not colored by only one color. That is, there is some subset of  $\Omega_i$  where  $\phi_j > 0$  and another subset where  $\phi_k > 0$ , with  $j \neq k$ . Because of the constraint, this means that  $\phi_j < 0$  where  $\phi_k > 0$ . Then  $\phi_j$  changes sign inside of  $\Omega_i$ . Conversely, if for every  $j$ ,  $\phi_j$  does not change sign inside of  $\Omega_i$ ; then all of  $\Omega_i$  has been assigned to only 1 color. So, we penalize for  $\phi_j = 0$  inside any of the open sets  $\Omega_i$ :

$$\sum_{i,j} \int_{\Omega_i} \delta(\phi_j)$$

(Here  $\delta$  is the Dirac delta function.) Finally, if 2 neighboring cells  $\Omega_i$  and  $\Omega_j$  have been assigned to the same colour  $k$ , then by smoothness,  $\phi_k > 0$  at the shared boundary. Let  $C$  denote the union of the boundaries of the  $\Omega_i$ s, remove the boundary of the unit square. Then we can penalize for colouring neighbors the same with

$$\sum_k \int_C H(\phi_k)$$

Putting it together, then, we want to solve

$$\min_{\phi \text{ s.t. } \sum_j H(\phi_j)=1} \sum_k \int_C H(\phi_k) + \sum_{j,k} \int_{\Omega_j} \delta(\phi_k)$$

(There are standard tricks to turn this into unconstrained optimization. There are some problems with the constraint being bad but we can probably do the same trick as Chan and Vese. Also we can regularize  $H$  and  $\delta$  by approximating them with arctan's and such.) Let's call the energy in the minimization  $F(\phi)$ .  $\phi$  is a 4-colouring of the map  $\{\Omega_j\}$  iff  $F(\phi) = 0$ . Thus to prove the 4-colour theorem, one needs to show that 1) a minimizer of the above exists and further that 2) the minimizer has energy 0.

I think that 1) is doable but the difficulties lie in 2). I think that in 2) one will encounter all the difficulties of trying to prove the 4-colour theorem "normally", so this is an interesting way of looking at the problem, but ultimately useless.