Products of Non-Hermitian Random Matrices

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Non-Hermitian random matrices

- C_N is an $N \times N$ real random matrix with i.i.d entries such that
$$\mathbb{E}[C_{ij}] = 0 \quad \mathbb{E}[C_{ij}^2] = 1/N$$
- We study in the large N limit of the empirical spectral measure:
$$\mu_N(z) = \frac{1}{N}\sum_{i=1}^{N} \delta_{\lambda_i}(z)$$

Circular law

- Girko, Bai, . . . , Tao-Vu.
- As $N \to \infty$, $\mu_N(z)$ converges a.s. in distribution to μ_c, the uniform law on the unit disk,

$$\frac{d\mu_c(z)}{dz} = \frac{1}{2\pi}\mathbf{1}_{|z|\leq 1},$$

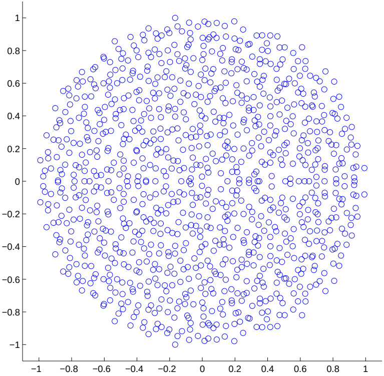


Figure : Eigenvalues of a 1000 × 1000 iid random matrix

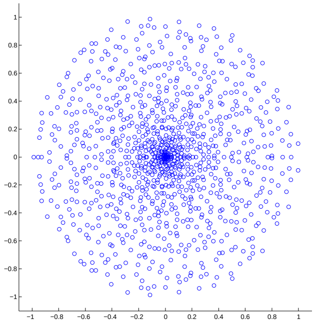
Products of iid random matrices

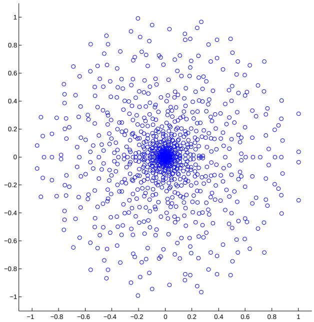
- Let $m \geq 2$ be a fixed integer.
- Let $C_{N,1}, C_{N,2}, \ldots, C_{N,m}$ be an independent family of random matrices each with iid entries.
- Götze-Tikhomirov and O'Rourke-Soshnikov computed the limiting distribution of the product

$$C_{N,1} C_{N,2} \cdots C_{N,m}$$

  as N goes to infinity.
- Limiting density is given by the m^{th} power of the circular law.

$$\frac{d\mu_m(z)}{dz} = \frac{1}{m\pi} |z|^{\frac{2}{m}-2} \mathbf{1}_{|z| \leq 1}.$$





- Left: eigenvalues of the product of two independent 1000×1000 iid random matrices
- Right: eigenvalues of the product of four independent 1000×1000 iid random matrices

Products of iid random matrices

- Studied in physics, either non-rigorously or in Gaussian case.
- Z. Burda, R. A. Janik,and B. Waclaw
- Akemann G, Ipsen J, Kieburg M
- Akemann G, Kieburg M, Wei L

Elliptical Random matrices

- A generalization of the iid model, that interpolates between iid and Wigner.
- X_N is an $N \times N$ real random matrix such that

$$\mathbb{E}[X_{ij}] = 0 \quad \mathbb{E}[X_{ij}^2] = 1/N \quad \mathbb{E}[|X_{ij}|^{2+\epsilon}] < \infty$$

- For $i \neq j$, $-1 \leq \rho \leq 1$

$$\mathbb{E}[X_{ij}X_{ji}] = \rho/N$$

- Entries are otherwise independent.
- Simplest case is weighted sum of GOE and real Ginibre.

$$X_N = \sqrt{\rho}W_N + \sqrt{1-\rho}C_N$$

Elliptical Law

- The limiting distribution of X_N for general ρ is an ellipse. (Girko; Naumov; Nguyen-O'Rourke) and μ_ρ is the uniform probability measure on the ellipsoid

$$\mathcal{E}_\rho = \left\{ z \in \mathbb{C} : \frac{\mathfrak{Re}(z)^2}{(1+\rho)^2} + \frac{\mathfrak{Im}(z)^2}{(1-\rho)^2} < 1 \right\}.$$

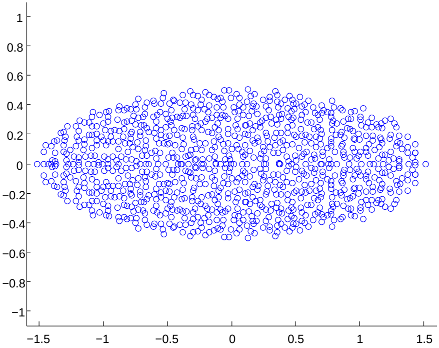


Figure : Eigenvalues of a 1000×1000 Elliptic random matrix, with $\rho = .5$

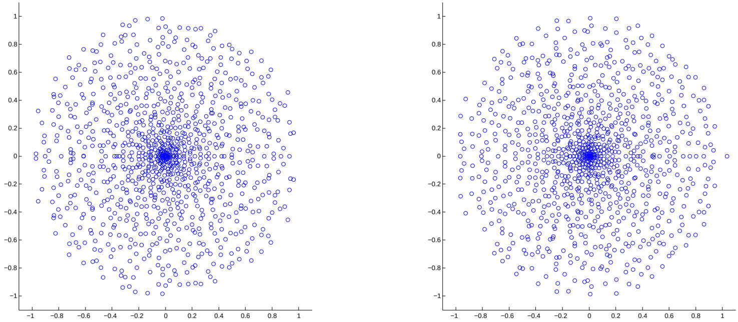
Products of random matrices

Theorem (O'Rourke,R,Soshnikov,Vu)

Let $X_N^1, X_N^2, \ldots, X_N^m$ be independent elliptical random matrices. Each with parameter $-1 < \rho_i < 1$, for $1 \leq i \leq m$. Almost surely the empirical spectral measure of the product

$$X_N^1 X_N^2 \cdots X_N^m$$

converges to μ_m, the m^{th} power of the circular law.



- Left: eigenvalues of the product of two identically distributed elliptic random matrices with Gaussian entries when $\rho_1 = \rho_2 = 1/2$
- Right: eigenvalues of the product of a Wigner matrix and an independent iid random matrix

Linearization

- Let

$$Y_N := \begin{pmatrix} 0 & X_{N,1} & & & 0 \\ 0 & 0 & X_{N,2} & & 0 \\ & & \ddots & \ddots & \\ 0 & & & 0 & X_{N,m-1} \\ X_{N,m} & & & & 0 \end{pmatrix}$$

Linearization

- Note that raising Y_N to the m^{th} power leads to

$$Y_N^m := \begin{pmatrix} Z_{N,1} & 0 & & & 0 \\ 0 & Z_{N,2} & 0 & & 0 \\ & 0 & \ddots & \ddots & \\ & & 0 & Z_{N,m-1} & 0 \\ & & & 0 & Z_{N,m} \end{pmatrix}$$

- Where $Z_{N,k} = X_{N,k} X_{N,k+1} \cdots X_{N,k-1}$
- So λ is an eigenvalue of Y_N iff λ^m is an eigenvalue of $X_{N,1} X_{N,2} \cdots X_{N,m}$.

Hermitization

- The log potential allows one to connect eigenvalues of a non-Hermitian matrix to those of a family of Hermitian matrices.

$$\int \log|z-s|d\mu_N(s) = \frac{1}{N}\log(|\det(Y_N-z)|) = \int_0^\infty \log(x)\nu_{N,z}(x)$$

- Where $\nu_{N,z}(x)$ is the empirical spectral measure of $\begin{pmatrix} 0 & X_N - z \\ (X_N - z)^* & 0 \end{pmatrix}$.
- The spectral measure can be recovered from the log potential.

$$2\pi\mu_N(z) = \Delta \int \log|z-s|d\mu_N(s)$$

Hermitization

- First step is to show $\nu_{N,z} \to \nu_z$
- Show that $\log(x)$ can be integrated by bounding singular values.

- In order to compute $\nu_{N,z}$, we use the Stieltjes transform.

$$a_N(\eta, z) := \int \frac{d\nu_{N,z}(x)}{x - \eta}$$

which is also the normalized trace of the resolvent.

$$R(\eta, z) = \begin{pmatrix} -\eta & C_N - z \\ (C_N - z)^* & -\eta \end{pmatrix}^{-1}$$

- It is useful to keep the block structure of R_N and define

$$\Gamma_N(\eta, z) = (I_2 \otimes \mathrm{tr}_N) R_N(\eta, z) = \begin{pmatrix} a_N(\eta, z) & b_N(\eta, z) \\ c_N(\eta, z) & a_N(\eta, z) \end{pmatrix}$$

Circular law

- The Stietljes transform corresponding to the circular law is characterized as the unique Stieltjes transform that solves the equation

$$a(\eta, z) = \frac{a(\eta, z) + \eta}{|z|^2 - (a(\eta, z) + \eta)^2}$$

  for each $z \in \mathbb{C}$, $\eta \in \mathbb{C}^+$.
- Our goal is to show $a_N(\eta, z)$ approximately satisfies this equation.

- Let

$$\Gamma(\eta, z) := \begin{pmatrix} -(a(\eta, z) + \eta) & -z \\ -\bar{z} & -(a(\eta, z) + \eta) \end{pmatrix}^{-1}.$$

- By the defining equation of a,

$$\Gamma(\eta, z) = \begin{pmatrix} a(\eta, z) & \frac{z}{(a(\eta,z)+\eta)^2 - |z|^2} \\ \frac{\bar{z}}{(a(\eta,z)+\eta)^2 - |z|^2} & a(\eta, z) \end{pmatrix}.$$

- Letting
$$q := \begin{pmatrix} \eta & z \\ \overline{z} & \eta \end{pmatrix}.$$
and
$$\Sigma(A) := diag(A)$$
- This relationship can compactly be written
$$\Gamma(\eta, z) = -(q + \Sigma(\Gamma(\eta, z)))^{-1}.$$
- So we can instead show Γ_N is close to Γ.

- Schur's complement

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1}_{11} = (A - BD^{-1}C)^{-1}$$

$$\begin{pmatrix} R_{1,1} & R_{1,N+1} \\ R_{N+1,1} & R_{N+1,N+1} \end{pmatrix}$$

$$= -\left(\begin{pmatrix} \eta & z \\ \bar{z} & \eta \end{pmatrix} + \begin{pmatrix} 0 & C^{(1)}_{1\cdot} \\ C^{(1)*}_{1\cdot} & 0 \end{pmatrix} \begin{pmatrix} R^{(1)11} & R^{(1)12} \\ R^{(1)21} & R^{(1)22} \end{pmatrix} \begin{pmatrix} 0 & C^{(1)}_{\cdot 1} \\ C^{(1)*}_{1\cdot} & 0 \end{pmatrix} \right.$$

$$\approx -\left(\begin{pmatrix} \eta & z \\ \bar{z} & \eta \end{pmatrix} + \begin{pmatrix} \mathrm{tr}(R^{22}) & 0 \\ 0 & \mathrm{tr}(R^{11}) \end{pmatrix} \right)^{-1}$$

Resolvent

- So

$$\Gamma_N(\eta, z) \approx -(q + \Sigma(\Gamma_N(\eta, z)))^{-1}$$

- It will suffice to prove the circular law for

$$Y_N = \begin{pmatrix} 0 & X_{N,1} & & & 0 \\ 0 & 0 & X_{N,2} & & 0 \\ & & \ddots & \ddots & \\ 0 & & & 0 & X_{N,m-1} \\ X_{N,m} & & & & 0 \end{pmatrix} \qquad (1)$$

- Let

$$H_N = \begin{pmatrix} 0 & Y_N \\ Y_N^* & 0 \end{pmatrix}$$

- Once again we study the hermitized resolvent

$$R_N(\eta, z) = \left(\begin{pmatrix} 0 & Y_N \\ Y_N^* & 0 \end{pmatrix} - \begin{pmatrix} \eta I_{mN} & z I_{mN} \\ \overline{z} I_{mN} & \eta I_{mN} \end{pmatrix} \right)^{-1}$$

Block Resolvent

- As before we keep the block structure of R_N and let

$$\Gamma_N(\eta, z) = (I_{2m} \otimes \mathrm{tr}_N) R_N(\eta, z)$$

- Let $R_{N;11}$ be the $2m \times 2m$ matrix whose entries are the $(1, 1)$ entry of each block of the resolvent.
- Let $H^{(1)}_{N;1}$ be a $2m \times 2m$ matrix with $N - 1$ dimensional vectors

$$R_{N;11} = -\left(q \otimes I_m + H^{(1)*}_{N;1} R^{(1)}_N H^{(1)}_{N;1}\right)^{-1}$$

$$
H_N = \left(\begin{array}{cccccccccc}
 & & & & & 0 & X_{N,1} & & & 0 \\
 & & & 0 & & & & \ddots & \ddots & \\
 & & & & & 0 & & & 0 & X_{N,m-1} \\
 & & & & & X_{N,m} & & & & 0 \\
0 & 0 & & & X^*_{N,m} & & & & & \\
X^*_{N,1} & & \ddots & \ddots & & & & 0 & & \\
 & & \ddots & 0 & 0 & & & & & \\
0 & & & X^*_{N,m-1} & 0 & & & & &
\end{array}\right)
$$

- So

$$\Gamma_N(\eta, z) \approx (q \otimes I_m - \Sigma(\Gamma_N(\eta, z))^{-1}$$

- where Σ being a linear operator on $2m \times 2m$ matrices defined by:

$$\Sigma(A)_{ab} = \sum_{c,d=1}^{2m} \sigma(a, c; d, b) A_{cd}$$

$$\sigma(a, c; d, b) = N\mathbb{E}[H_{12}^{ac} H_{12}^{db}]$$

- $$\Sigma(A)_{ab} = A_{a'a'}\delta_{ab} + \rho_a A_{a'a}\delta_{aa'},$$

Fixed point equation

- In the limit
$$\Gamma = -(q \otimes I_m + \Sigma(\Gamma))^{-1}$$
- This equation has a unique solution that is a matrix valued Stietljes transform (J. Helton, R. Far, R. Speicher)
- As $\eta \to \infty$,
$$\Gamma \sim \frac{-1}{\eta^2 - |z|^2} \begin{pmatrix} \eta I_m & -z I_m \\ -\bar{z} I_m & \eta I_M \end{pmatrix}.$$
- Since Σ leaves main diagonal invariant and sets diagonals of the upper blocks to zero, Γ is of this form.

- So Γ actually satisfies the equation:

$$\Gamma(\eta, z) = -(q \otimes I_m + diag(\Gamma(\eta, z)))^{-1}$$

- This means for $1 \leq i \leq 2m$, the diagonal entries of the matrix valued Stieltjes transform are given by the Stieltjes transform corresponding to the circular law.

$$\Gamma(\eta, z)_{ii} = a(\eta, z)$$

Smallest singular value

- Theorem (Nguyen, O'Rourke) Let X_N be an elliptical random matrix with $-1 < \rho < 1$ and F_N be deterministic matrix, for any $B > 0$, there exists $A > 0$

$$\mathbb{P}\left(\sigma_N(X_N + F_N) \leq N^{-A}\right) = O(N^{-B}).$$

- Theorem (O'Rourke, R, Soshnikov, Vu) Let Y_N be the linearized random matrix and F_N be deterministic matrix, for any $B > 0$, there exists $A > 0$

$$\mathbb{P}\left(\sigma_{mN}(Y_N - zI_{Nm}) \leq N^{-A}\right) = O(N^{-B}).$$

Smallest singular value

- Let $G_N = (Y_N - z)^{-1}$. In suffices to show
$$\mathbb{P}\left(\|G_N\| \geq N^A\right) = O(N^{-B}).$$
- Let G_N^{ab} be the ab^{th} $N \times N$ block of G_N.
-
$$\mathbb{P}\left(\|G_N\| \geq N^A\right)$$
$$\leq \mathbb{P}\left(\text{there exists } a, b \in \{1, \ldots, m\} \text{ with } \|G_N^{ab}\| \geq \frac{1}{m^2} N^A\right).$$

Smallest singular value

- $$G_N^{ab} = z^{\kappa} X_{N,j_1} \cdots X_{N,j_l} \left(X_{N,i_1} \cdots X_{N,i_q} - z^r \right)^{-1},$$
- The second term can be rewritten

$$(X_{N,i_1} \cdots X_{N,i_q} - z^r)^{-1} = X_{N,i_q}^{-1} \cdots X_{N,i_2}^{-1} (X_{N,i_1} - z^r X_{N,i_q}^{-1} \cdots X_{N,i_2}^{-1})^{-1}.$$

- Then the least singular value bound of Nguyen-O'Rourke can be applied.

Free Probability

- In free probability, there are a distinguished set of operators known as R-diagonal operators.
- When they are non-singular, their polar decomposition is

$$uh$$

  where u is a haar unitary operator, h is a positive operator, and u, h are free.
- Additionally, the set of R-diagonal operators is closed under addition and multiplication.

- $$x_1 x_2 = v_1 h_1 v_2 h_2.$$

  We begin by introducing a new free haar unitary u. Then the distribution of $x_1 x_2$ is the same the distribution of

  $$u v_1 h_1 u^* v_2 h_2.$$

  Then $u v_1$ and $u^* v_2$ are haar unitaries, and one can check they are free from each other and h_1 and h_2. Since the product of R-diagonal elements remains R-diagonal $x_1 x_2$ is R-diagonal.

Thank you

Available at arxiv:1403.6080