Free probabilities and the large N limit, IV

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Loop equations and all-order asymptotic expansions ...

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Loop equations and all-order asymptotic expansions ...

1. Model and results

2. Schwinger-Dyson equations

3. Sketch of proof of the main result

4. Conclusion

The β ensembles

- Probability measure on $A^N \subseteq \mathbb{R}^N$

$$\mathrm{d}\mu_N^A = \frac{1}{Z_N^A} \exp\Big(N \sum_{i=1}^{N} T(\lambda_i)\Big) \prod_{1 \leq i < j \leq N} |\lambda_i - \lambda_j|^\beta \prod_{i=1}^{N} \mathbf{1}_A(\lambda_i)\mathrm{d}\lambda_i \qquad \beta > 0$$

- It is the measure induced on eigenvalues of a random matrix M

$\mathrm{d}M\, e^{N\,\mathrm{Tr}\,T(M)}$

Wigner, Dyson, Mehta (50s-60s)

$\beta = 1$ real symmetric matrices
$\beta = 2$ hermitian matrices
$\beta = 4$ quaternionic self-dual matrices

M = triagonal — all $\beta > 0$, T polynomial of even degree

Dumitriu, Edelman '02
Krishnapur, Rider, Virág '13

Mean-field models

- Probability measure on $A^N \subseteq \mathbb{R}^N$

$$\mathrm{d}\mu_N = \frac{1}{Z_N} \exp\Big(N^2 \mathcal{T}_0(L_N^{(\lambda)})\Big) \prod_{1 \leq i < j \leq N} |\lambda_i - \lambda_j|^\beta \prod_{i=1}^N \mathbf{1}_A(\lambda_i)\mathrm{d}\lambda_i \qquad \beta > 0$$

where $L_N^{(\lambda)} = \frac{1}{N}\sum_{i=1}^N \delta_{\lambda_i}$ is the (random) empirical measure

- Exemples

in Chern-Simons theory $\quad \mathcal{T}_0(\mu) = \iint \mathrm{d}\mu(x)\mathrm{d}\mu(y) \sum_m \beta_m \ln\left|\frac{\sinh[\alpha_m(x-y)]}{\alpha_m(x-y)}\right|$

O(n) model on random lattices $\quad \mathcal{T}_0(\mu) = -\frac{n}{2}\iint \mathrm{d}\mu(x)\mathrm{d}\mu(y)\,\ln|x+y|$

- Here, we take $\quad \mathcal{T}_0(\mu) = \int T(x_1,\ldots,x_r)\prod_{i=1}^r \mathrm{d}\mu(x_i)$

T real-analytic on A^r

We would like to study when N → ∞ ...

- the (random) empirical measure $L_N^{(\lambda)} = \frac{1}{N}\sum_{i=1}^{N} \delta_{\lambda_i}$

$\rightsquigarrow$ what kind of random variable is $\sum_{i=1}^{N} f(\lambda_i) = N \int f(\xi)\mathrm{d}L_N^{(\lambda)}(\xi)$?

- the partition function

$$Z_N = \int_{A^N} \exp\Big(N^2 \mathcal{T}_0(L_N^{(\lambda)})\Big) \prod_{1\leq i<j\leq N} |\lambda_i - \lambda_j|^{\beta} \prod_{i=1}^{N} \mathrm{d}\lambda_i$$

- the k-point correlators

$$W_k(x_1,\ldots,x_k) = \mathrm{Cumulant}\Big(\int \frac{N\,\mathrm{d}L_N^{(\lambda)}(\xi_1)}{x_1 - \xi_1},\ldots,\int \frac{N\,\mathrm{d}L_N^{(\lambda)}(\xi_k)}{x_k - \xi_k}\Big)$$

The leading order ... is given by a continuous approximation

- Define the energy functional on a proba. measure μ

$$\mathcal{T}(\mu) = \int \Big[\prod_{i=1}^{r} \mathrm{d}\mu(x_i)\Big]\, T(x_1,\ldots,x_r) + \frac{\beta}{2}\iint \mathrm{d}\mu(x_1)\mathrm{d}\mu(x_2)\, \ln|x_1 - x_2|$$

- Assumption 1 : uniqueness of maximizer μ_{eq}

- Characterization : exists a constant C such that $\mathcal{T}'(\mu_{\mathrm{eq}})[\delta_x] \leq C$
  for $x \in A$ μ_{eq}-everywhere

- Assumption 2 : local strict concavity at μ_{eq}

for any ν = finite signed measure of mass 0

$$-\mathcal{T}''(\mu_{\mathrm{eq}})[\nu,\nu] = \mathfrak{D}^2[\nu] \in [0,+\infty]$$

and = 0 iff $\nu = 0$

The leading order ... is given by a continuous approximation

- Define the energy functional on a proba measure μ

$$\mathcal{T}(\mu) = \int \Big[\prod_{i=1}^{r} \mathrm{d}\mu(x_i)\Big]\, T(x_1,\ldots,x_r) + \frac{\beta}{2}\iint \mathrm{d}\mu(x_1)\mathrm{d}\mu(x_2)\,\ln|x_1 - x_2|$$

- Assumption 1 : uniqueness of maximizer μ_{eq}
- Assumption 2 : local strict concavity at μ_{eq}

Lemma

$L_N^{(\lambda)} \longrightarrow \mu_{\mathrm{eq}}$ almost surely and in expectation

$$Z_N = \exp\Big\{N^2\big(\mathcal{T}(\mu_{\mathrm{eq}}) + o(1)\big)\Big\}$$

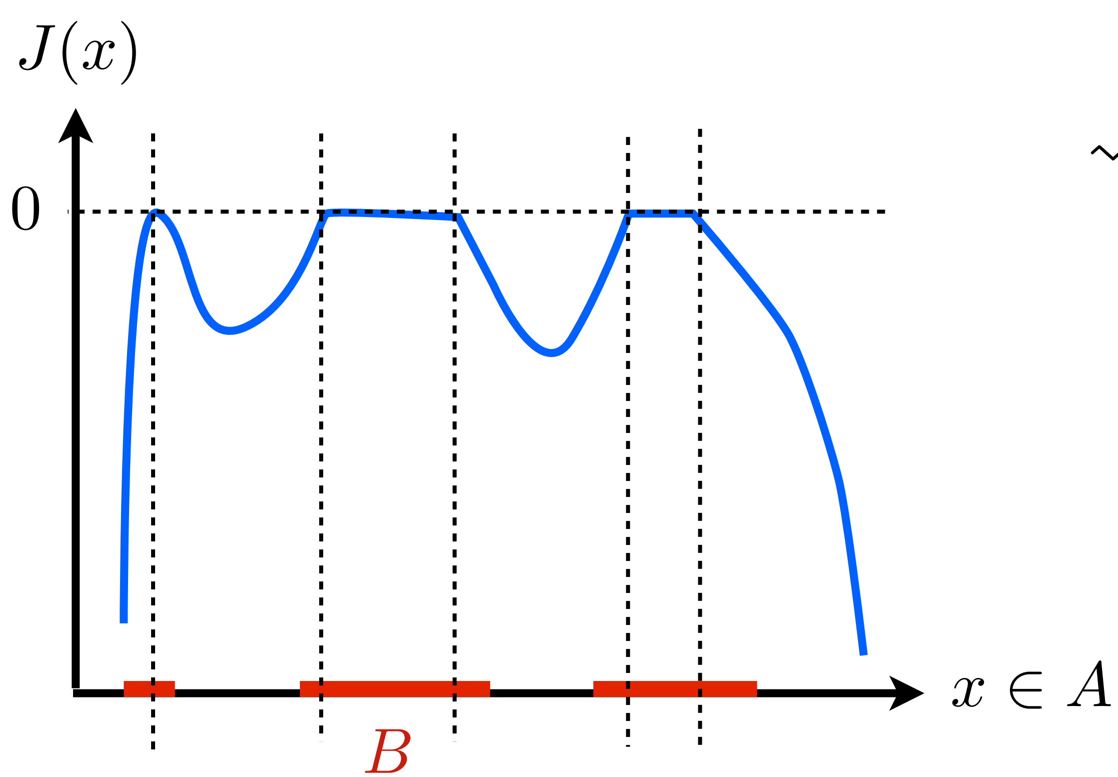
Large deviations for a single particle

- A particle at position x feels the effective potential

$$J(x) = \mathcal{T}'(\mu_{\mathrm{eq}})[\delta_x] - \sup_{\xi \in A} \mathcal{T}'(\mu_{\mathrm{eq}})[\delta_\xi]$$

Lemma

For any closed $F \subseteq A$ $\quad \mathbb{P}\big[\exists i,\ \lambda_i \in F\big] \leq \exp\Big\{N\big(\sup_{x \in F} J(x) + o(1)\big)\Big\}$



$\rightsquigarrow$ One can restrict to a compact $B \subseteq A$ neighborhood of $\{J(x) = 0\}$

$$Z_N^B = Z_N^A(1 + o(e^{-cN}))$$

Large deviations of empirical measure

- Natural "distance" $-\mathcal{T}''(\mu_{\mathrm{eq}})[\nu,\nu] = \mathfrak{D}^2[\nu] \in [0,+\infty]$

  but $\mathfrak{D}[L_N^{(\lambda)} - \mu_{\mathrm{eq}}] = +\infty$ because of atoms and log singularity

- Let us pick a nice regularization **idea from Maïda, Maurel-Segala**

$$L_N^{(\lambda)} = \frac{1}{N}\sum_{i=1}^{N} \delta_{\lambda_i} \rightsquigarrow \widetilde{L}_N^{(\lambda)}$$

Lemma

If T is smooth, we have for N large enough

$$\mathbb{P}_N\big[\mathfrak{D}[\widetilde{L}_N^{(\lambda)} - \mu_{\mathrm{eq}}] > t\big] \leq \exp\big(N\ln N - N^2t^2/2\big)$$

The equilibrium measure

- T real-analytic $\Longrightarrow \begin{cases} \mu_{\mathrm{eq}} \text{ is supported on a finite number of segments} \\ S = \bigcup_{h=0}^{g} [a_h, b_h] \end{cases}$

- $\alpha \in \partial S$ is a hard edge if $\alpha \in \partial A$, is a soft edge otherwise

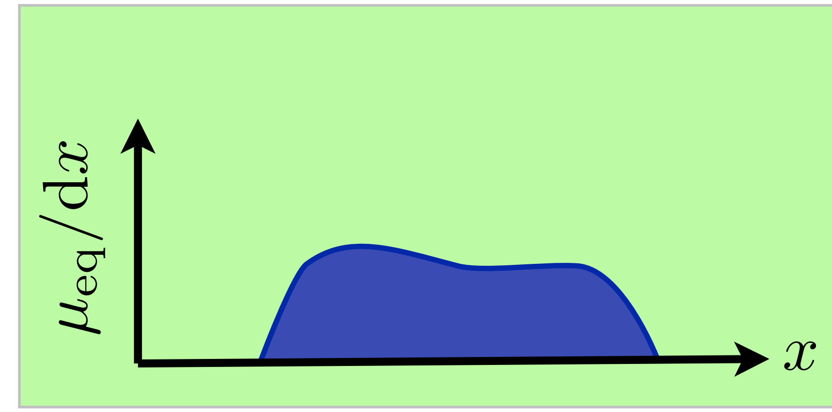
$$\mathrm{d}\mu_{\mathrm{eq}}(x) = \frac{\mathbf{1}_S(x)\mathrm{d}x}{2\pi}\, M(x) \prod_{\alpha \text{ soft}} |x-\alpha|^{1/2} \prod_{\alpha \text{ hard}} |x-\alpha|^{-1/2}$$

- We say that μ_{eq} is off-critical when $M(x) > 0$ on A

Finite size corrections : we assume ...

- Uniqueness of maximizer μ_{eq}
- Local strict concavity at μ_{eq}
- $V = V_0 + (1/N)V_1 + \cdots$ $\begin{cases} V_0 \text{ real analytic on } A \\ V_1 \text{ complex analytic on } A \end{cases}$
- Control of large deviations $J(x) < 0 \quad \text{for} \quad x \in A \setminus S$
- μ_{eq} is off-critical
- f = test function, analytic on A

Result in the 1-cut regime



- 1/N asymptotic expansion

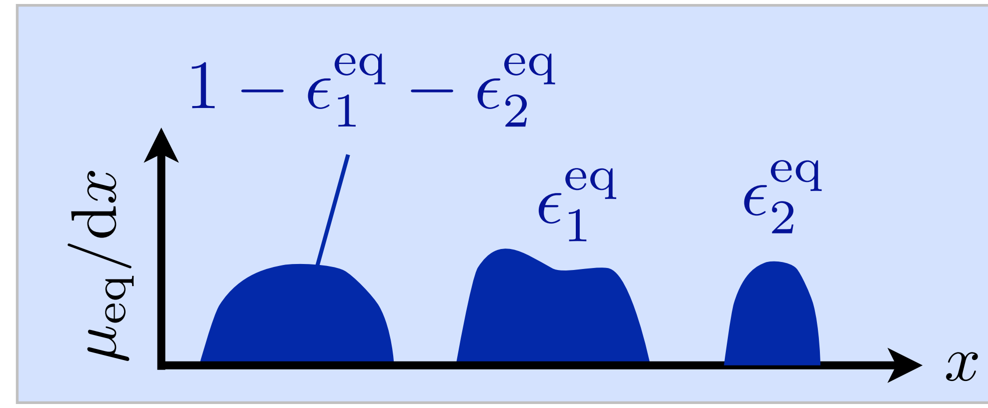
$$Z_N = N^{\gamma N + \gamma'} \exp\Big[\sum_{m \geq -2} N^{-m}\, F^{[m]} + O(N^{-\infty})\Big]$$

γ, γ' depend only on β and the nature of the edges

- Central limit theorem

$$\Big(\sum_{i=1}^{N} f(\lambda_i) - N\int_A f(\xi)\mathrm{d}\mu_{\mathrm{eq}}(\xi)\Big) \longrightarrow \text{ (non-centered) gaussian}$$

Result in the (g + 1)-cuts regime



- Oscillatory asymptotic expansion

$$Z_N = N^{\gamma N+\gamma'}\,(\mathcal{D}_N\Theta_{-N\epsilon^{\mathrm{eq}}})(F^{[-1]'}|F^{[-2]''})\,\exp\Big[\sum_{m\geq -2} N^{-m}\,F^{[m]} + O(N^{-\infty})\Big]$$

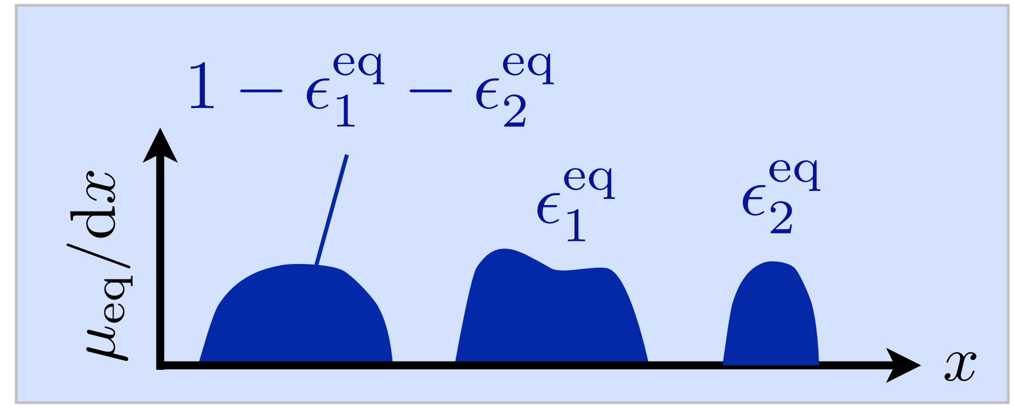
where $\mathcal{D}_N = \sum_{p\geq 0}\frac{1}{p!}\sum_{\substack{\ell_1,\ldots,\ell_p\geq 1\\ m_1,\ldots,m_p\geq -2\\ \sum_i(m_i+\ell_i)>0}} N^{-\sum_i(m_i+\ell_i)}\prod_{i=1}^{p}\frac{F_{\mathrm{eq}}^{[m_i],(\ell_i)}\cdot\nabla_{\mathbf{w}}^{\otimes\ell_i}}{\ell_i!}$

acts as a differential operator on the Siegel theta function

$$\Theta_\mu(\mathbf{w}|\mathbf{Q}) = \sum_{\mathbf{m}\in\mathbb{Z}^g} e^{\mathbf{w}\cdot(\mathbf{m}+\mu)+\frac{1}{2}(\mathbf{m}+\mu)\cdot\mathbf{Q}\cdot(\mathbf{m}+\mu)}$$

- (Pseudo)-periodicity come from $\mu = -N\epsilon_{\mathrm{eq}} \mod \mathbb{Z}^g$

Result in the (g + 1)-cuts regime



- No central limit theorem in general ...

$$\mathbb{E}\big[e^{isX_N[f]}\big] \underset{N\to\infty}{\sim} \mathbb{E}\big[e^{\mathrm{i}s(k[f]G_1+m[f])}\big]\,\mathbb{E}\big[e^{\mathrm{i}su[f]\cdot G_2}\,\big|\,G_2+N\epsilon_{\mathrm{eq}}\in\mathbb{Z}^g\big]$$

gaussian (non-centered) + discrete Gaussian, centered at $\mu = -N\epsilon_{\mathrm{eq}} \bmod \mathbb{Z}^g$

Corollary

$$\Big(\sum_{i=1}^{N} f(\lambda_i) - N\int_A f(\xi)\mathrm{d}\mu_{\mathrm{eq}}(\xi)\Big)$$

converges in law along subsequences

History of β ensembles : 1-cut regime

$\beta = 2$

- If 1/N expansion exists, then $Z_N = N^{\gamma N + \gamma'} \exp\Big[\sum_{m \geq -1} N^{-2m} F^{\{m\}}\Big]$ and $F^{\{m\}}$ can be computed by the moment method
Ambjørn, Chekhov, Kristjansen, Makeenko, 90s

- Rewriting of $F^{\{m\}}$ in terms of a universal topological recursion
Eynard, '04

- Existence of 1/N expansion by
- analysis of SD equations **Albeverio, Pastur, Shcherbina '01**
- RH techniques **Ercolani, McLaughlin '02**
- analysis of int. system **Bleher, Its, '05**

History of β ensembles : 1-cut regime

$\beta > 0$

- if 1/N expansion exists, then $Z_N = N^{\gamma N + \gamma'} \exp\Big[\sum_{m \geq -2} N^{-m}\, F^{[m]}\Big]$ and $F^{[m]}$ computed by a β-topological recursion

  Chekhov, Eynard '06

- Central limit theorem

  Johansson '98

- Existence of 1/N expansion (analysis of SD eqn)

  Borot, Guionnet '11

History of β ensembles : multi-cut regime

$\beta = 2$

- numerous observations of oscillatory behavior
  physicists, '90s
- Riemann-Hilbert techniques up to o(1)
  Deift, Kriecherbauer, McLaughlin, Venakides, Zhou, ...
- heuristic derivation up to o(1)
  Bonnet, David, Eynard '00
- generalization to all orders
  Eynard '07
- observation of "no CLT"
  Pastur '06

$\beta > 0$

- Proof of "no CLT" and asymptotics of Z_N^A up to o(1)
  Shcherbina '12
- General proof
  Borot, Guionnet '13

History of mean-field models

$$\mathrm{d}\mu_N = \frac{1}{Z_N} \exp\left(N^2 \mathcal{T}_0(L_N^{(\lambda)})\right) \prod_{1\leq i<j\leq N} |\lambda_i - \lambda_j|^\beta \prod_{i=1}^{N} \mathbf{1}_A(\lambda_i)\mathrm{d}\lambda_i$$

with r-body interaction $\mathcal{T}_0(\mu) = \int T(x_1, \ldots, x_r) \prod_{i=1}^{r} \mathrm{d}\mu(x_i)$

- same results for mean field models
  Borot, Guionnet, Kozlowski '13
- computation of expansion by a topological recursion
  Borot, '13

Large-N asymptotic expansions in 1-d repulsive particle systems

1. Model and results

2. Schwinger-Dyson equations

3. Sketch of proof of the main result

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What are Schwinger-Dyson equations ?

= relations between expectation values from integration by parts

- In the model $\mathrm{d}\mu_N = \frac{1}{Z_N} \exp\Big(N^2 \mathcal{T}_0(L_N^{(\lambda)})\Big) \prod_{1 \leq i < j \leq N} |\lambda_i - \lambda_j|^{\beta} \prod_{i=1}^{N} \mathbf{1}_A(\lambda_i)\mathrm{d}\lambda_i$

we find for any smooth test function h
and smooth functional $\mathcal{O}$

$$\mathbb{E}\Big[\Big(\sum_i N\, h(\lambda_i)\, \mathcal{T}_0'(L_N^{(\lambda)})[\delta_{\lambda_i}] + \beta \sum_{i<j} \frac{h(\lambda_i) - h(\lambda_j)}{\lambda_i - \lambda_j} + \sum_i h'(\lambda_i)\Big)\mathcal{O}(L_N^{(\lambda)})$$

$$+ \sum_i N^{-1}\, h(\lambda_i)\, \mathcal{O}'(L_N^{(\lambda)})[\delta_{\lambda_i}]\Big] + \mathrm{boundary} = 0$$

What are Schwinger-Dyson equations ?

- Remind the k-points correlators

$$W_k(x_1,\ldots,x_k) = \mathrm{Cumulant}\Big(\int \frac{N\,\mathrm{d}L_N^{(\lambda)}(\xi_1)}{x_1-\xi_1},\ldots,\int \frac{N\,\mathrm{d}L_N^{(\lambda)}(\xi_k)}{x_k-\xi_k}\Big)$$

- Choose $h_z(x) = \dfrac{1}{z-x}$ and $\mathcal{O}_{z_2,\ldots,z_k}(L_N^{(\lambda)}) = \prod_{i=2}^{k}\int \dfrac{\mathrm{d}L_N^{(\lambda)}(\xi_i)}{z_i-\xi_i}$

  for $z, z_i \in \mathbb{C}\setminus A$

$\longrightarrow$ family of functional relations between $W_1,\ldots,W_{r+k-1}$ indexed by $k\geq 1$

The master operator

- Decompose $W_1(z) = N\big(W_{\mathrm{eq}}(z) + \delta_{-1}W_1(z)\big)$

  with $W_{\mathrm{eq}}(z) = \int \frac{\mathrm{d}\mu_{\mathrm{eq}}(\xi)}{z-\xi}$

- Schwinger-Dyson equations can be recast

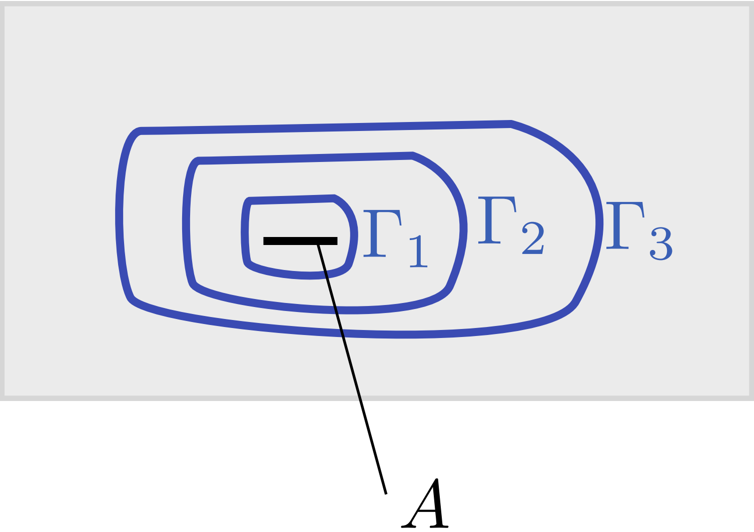
$$(\mathcal{K} + \delta\mathcal{K})[\delta_{-1}W_1](z) = A_1 + \text{boundary}$$

$$(\mathcal{K} + \delta\mathcal{K})[W_n(\cdot, z_2, \ldots, z_n)](z) = A_n + \text{boundary}$$

with : $\mathcal{K}[f](z) = 2W_{\mathrm{eq}}(z)f(z) + \frac{2}{\beta}\,\mathcal{T}_0'(\mu_{\mathrm{eq}})\Big[\frac{f(\lambda)\mathrm{d}\lambda}{z-\lambda}\Big]$

$$\delta\mathcal{K}[f](z) = 2\delta_{-1}W_1(z)f(z) + N^{-1}(1-2/\beta)\partial_z f(z) + \cdots$$

Asymptotic analysis



- Introduce norms $\|f\|_{\Gamma} = \sup_{z \in \mathrm{Ext}(\Gamma)} \big|f(z)\big|$

- Large deviations of empirical measure

$$\|N\delta_{-1}W_1\|_{\Gamma_1} \leq C_1\,(N\ln N)^{1/2}$$
$$\|W_k\|_{\Gamma_k} \leq C_k\,(N\ln N)^{k/2}$$

- Large deviation of single eigenvalue : boundary effects $\in o(e^{-cN})$

- Rigidity of SD equations : if $\mathcal{K}$ invertible and $\|\mathcal{K}^{-1}[f]\|_{\Gamma_{i+1}} \leq c\,\|f\|_{\Gamma_i}$

$$\left\{\begin{array}{rcl} \|N\delta_{-1}W_1\|_{\Gamma_{i_1}} & \leq & c_1\big(\eta_N\kappa_N + 1\big) \\ \|W_k\|_{\Gamma_{i_k}} & \leq & c_k\big(\eta_N^k\kappa_N + N^{2-k}\big) \end{array}\right\}$$

$$\Downarrow$$

$$\left\{\begin{array}{rcl} \|N\delta_{-1}W_1\|_{\Gamma_{i_1+2}} & \leq & c'_1\big(\eta_N(\eta_N/N)\kappa_N + 1\big) \\ \|W_k\|_{\Gamma_{i_k+2}} & \leq & c'_k\big(\eta_N^k(\eta_N/N)\kappa_N + N^{2-k}\big) \end{array}\right\}$$

Asymptotic analysis

Large deviations of empirical measure

+ Rigidity of SD equations

Corollary

If $\mathcal{K}$ invertible and $\|\mathcal{K}^{-1}[f]\|_{\Gamma_{i+1}} \leq c\,\|f\|_{\Gamma_i}$

we have, for any $M \geq 0$ an asymptotic expansion

$$W_k = \sum_{m=k-2}^{M-1} N^{-m}\, W_k^{[m]} + O(N^{-M}; \Gamma_{M,k})$$

- Remark :

(g + 1) cuts

c = nb. critical conditions $\longrightarrow$ $\dim \mathrm{Ker}\, \mathcal{K} = g + c$

Large-N asymptotic expansions in 1-d repulsive particle systems

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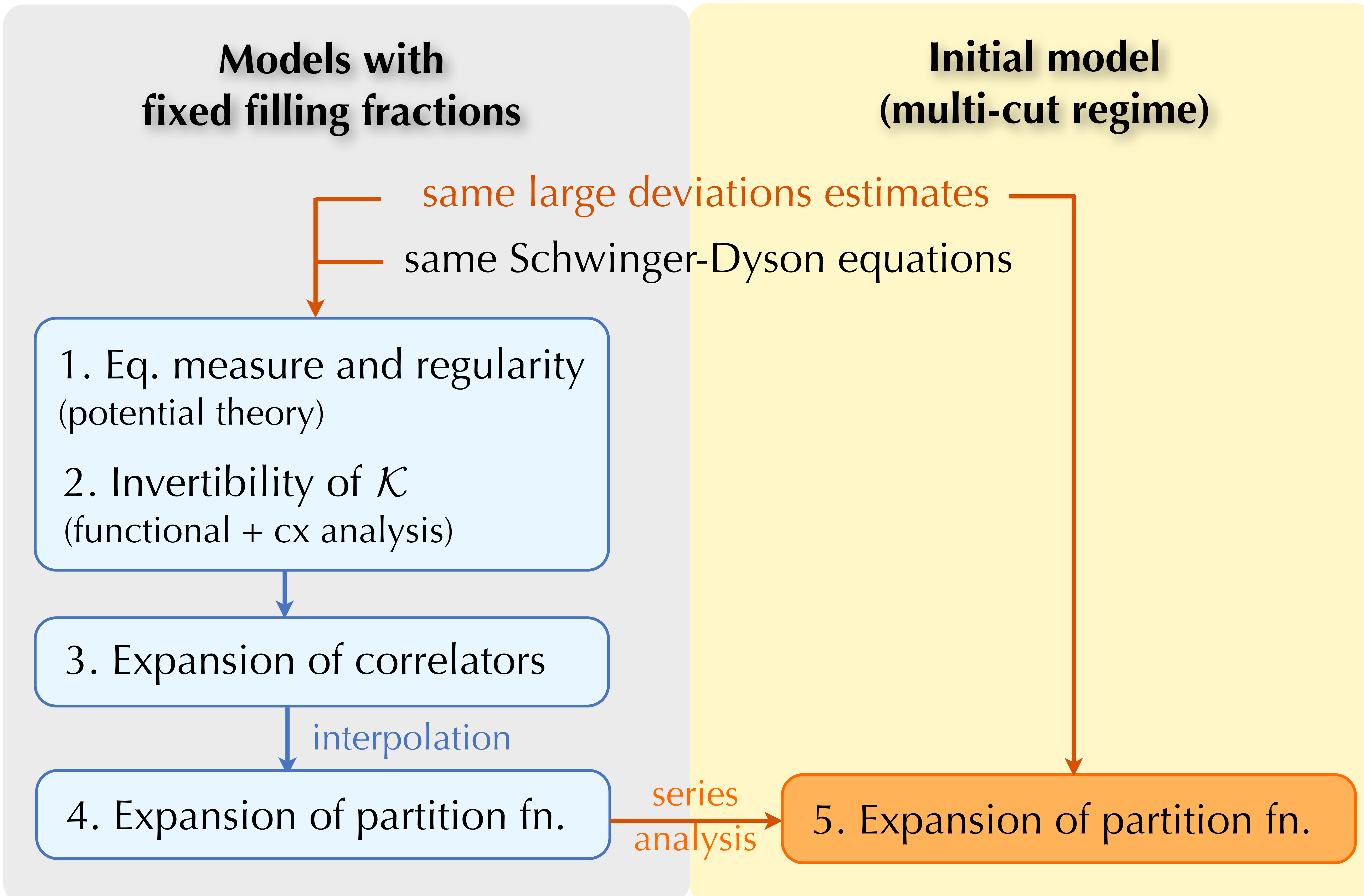
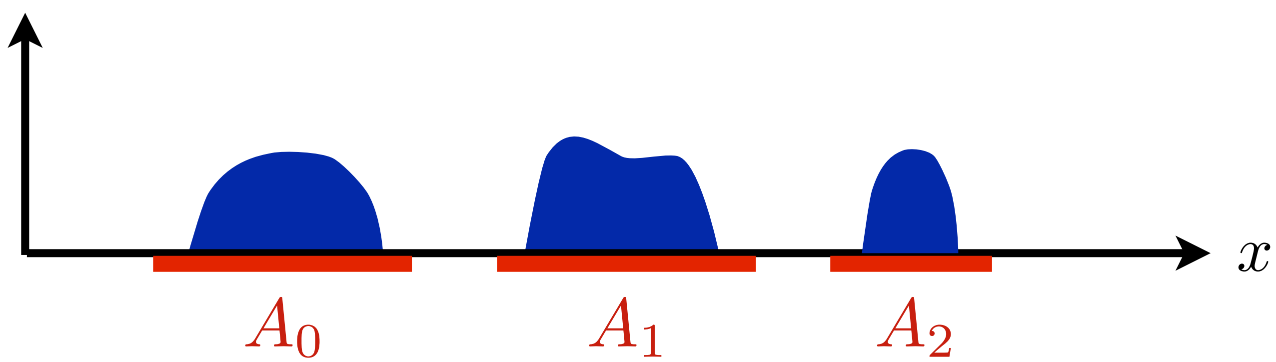
Scheme of the proof



Conditioning on the filling fractions

- From large deviations on single eigenvalue :
  up to $o(e^{-cN})$, we can choose



$$A = \bigcup_{h=0}^{g} A_h$$

- We will study $\mu_{(N_0,\ldots,N_g)}^{(A_0,\ldots,A_g)} = \mu_N^A$ conditioned to have $\begin{cases} N_0 \text{ first } \lambda\text{'s in } A_0 \\ N_1 \text{ next } \lambda\text{'s in } A_1 \\ \text{etc.} \end{cases}$

The partition function decomposes
$$Z_N^A = \sum_{N_0+\cdots N_g=N} \frac{N!}{\prod_{h=0}^{g} N_h!} Z_{(N_0,\ldots,N_g)}^{(A_0,\ldots,A_g)}$$

- $\epsilon_h = N_h/N$ are the filling fractions

Equilibrium measures ...

- Assumption 1 : uniqueness of maximizer (= μ_{eq}) of

$$\mathcal{T}(\mu) = \int \Big[\prod_{i=1}^{r} \mathrm{d}\mu(x_i)\Big]\, T(x_1,\ldots,x_r) + \frac{\beta}{2}\iint \mathrm{d}\mu(x_1)\mathrm{d}\mu(x_2)\, \ln|x_1 - x_2|$$

among all proba. measures

Let $\epsilon_{\mathrm{eq},h} = \mu_{\mathrm{eq}}[A_h]$ be the equilibrium filling fraction

- Assumption 2 : local strict concavity at μ_{eq}

Lemma 1

For ϵ close enough to ϵ_{eq}

$\mathcal{T}$ has a unique maximizer (= $\mu_{\mathrm{eq},\epsilon}$) over proba. measure with $\mu[A_h] = \epsilon_h$

Equilibrium measures ...

- Assumption 1 : uniqueness of maximizer (= μ_{eq}) of

$$\mathcal{T}(\mu) = \int \Big[\prod_{i=1}^{k} \mathrm{d}\mu(x_i)\Big]\, T(x_1,\ldots,x_k) + \frac{\beta}{2}\iint \mathrm{d}\mu(x_1)\mathrm{d}\mu(x_2)\, \ln|x_1 - x_2|$$

among all proba. measures

Let $\epsilon_{\mathrm{eq},h} = \mu_{\mathrm{eq}}[A_h]$ be the equilibrium filling fraction

- Assumption 2 : local strict concavity at μ_{eq}
- Assumption 3 : T is analytic
- Assumption 4 : μ_{eq} has (g + 1) cuts and is off-critical

Lemma 2

For ϵ close enough to ϵ_{eq}

- ☼ $\mu_{\mathrm{eq};\epsilon}$ has (g + 1) cuts and is off-critical
- ☼ The edges depend smoothly on ϵ
- ☼ The density of $\mu_{\mathrm{eq};\epsilon}$ depends smoothly on ϵ away from edges

Equilibrium measures ...

- Assumption 1 : uniqueness of maximizer ($= \mu_{\mathrm{eq}}$) of

$$\mathcal{T}(\mu) = \int \Big[\prod_{i=1}^{k} \mathrm{d}\mu(x_i)\Big]\, T(x_1,\ldots,x_k) + \frac{\beta}{2}\iint \mathrm{d}\mu(x_1)\mathrm{d}\mu(x_2)\,\ln|x_1 - x_2|$$

among all proba. measures

Let $\epsilon_{\mathrm{eq},h} = \mu_{\mathrm{eq}}[A_h]$ be the equilibrium filling fraction

- Assumption 2 : local strict concavity at μ_{eq}
- Assumption 3 : T is analytic
- Assumption 4 : μ_{eq} has (g + 1) cuts and is off-critical

Lemma 3

For ϵ close enough to ϵ_{eq}

the large deviation estimates also holds uniformly
in the conditioned model with filling fractions ϵ

The return of the master operator

- The correlators W_k in the initial model
  $W_{k;\epsilon}$ in the conditioned model

  satisfy the same Schwinger-Dyson equations

- We have $\oint_{A_{h_1}} \cdots \oint_{A_{h_k}} W_{k;\epsilon}(z_1, \ldots, z_k) \prod_{i=1}^{k} \frac{\mathrm{d}z_i}{2\mathrm{i}\pi} = \delta_{k,1}\, N\epsilon_{h_1}$

$\Longrightarrow$ we need the restriction $\mathcal{K}_{0;\epsilon}$ of $\mathcal{K}_\epsilon$ to the codim. = g subspace

$$\{f, \quad \forall h, \quad \oint_{A_h} f(z)\mathrm{d}z = 0\}$$

Lemma 4

For ϵ close enough to ϵ_{eq}

$\mathcal{K}_{0;\epsilon}$ is continuously invertible, and $\mathcal{K}_{0;\epsilon}^{-1}$ depends smoothly on ϵ

Asymptotic expansion of correlators in the conditioned model

Corollary

For ϵ close enough to ϵ_{eq}

we have, for any $M \geq 0$, an asymptotic expansion

$$W_{k;\epsilon} = \sum_{m=k-2}^{M-1} W_{k;\epsilon}^{[m]} + O(N^{-M};\Gamma_{M,k})$$

depending smoothly on ϵ, with remainder uniform in ϵ

Partition function of the conditioned model

$$\frac{Z_{N;\epsilon}^{(T_1)}}{Z_{N;\epsilon}^{(T_0)}} = \exp\Big(N^{2-r}\int \partial_t T_t(x_1,\ldots,x_r)\prod_{i=1}^{r}\mathrm{d}L_N^{(\lambda),T_t}(x_i)\Big)$$

can be expressed in terms of $W_{j;\epsilon}^{T_t}$ for the model with interaction T_t

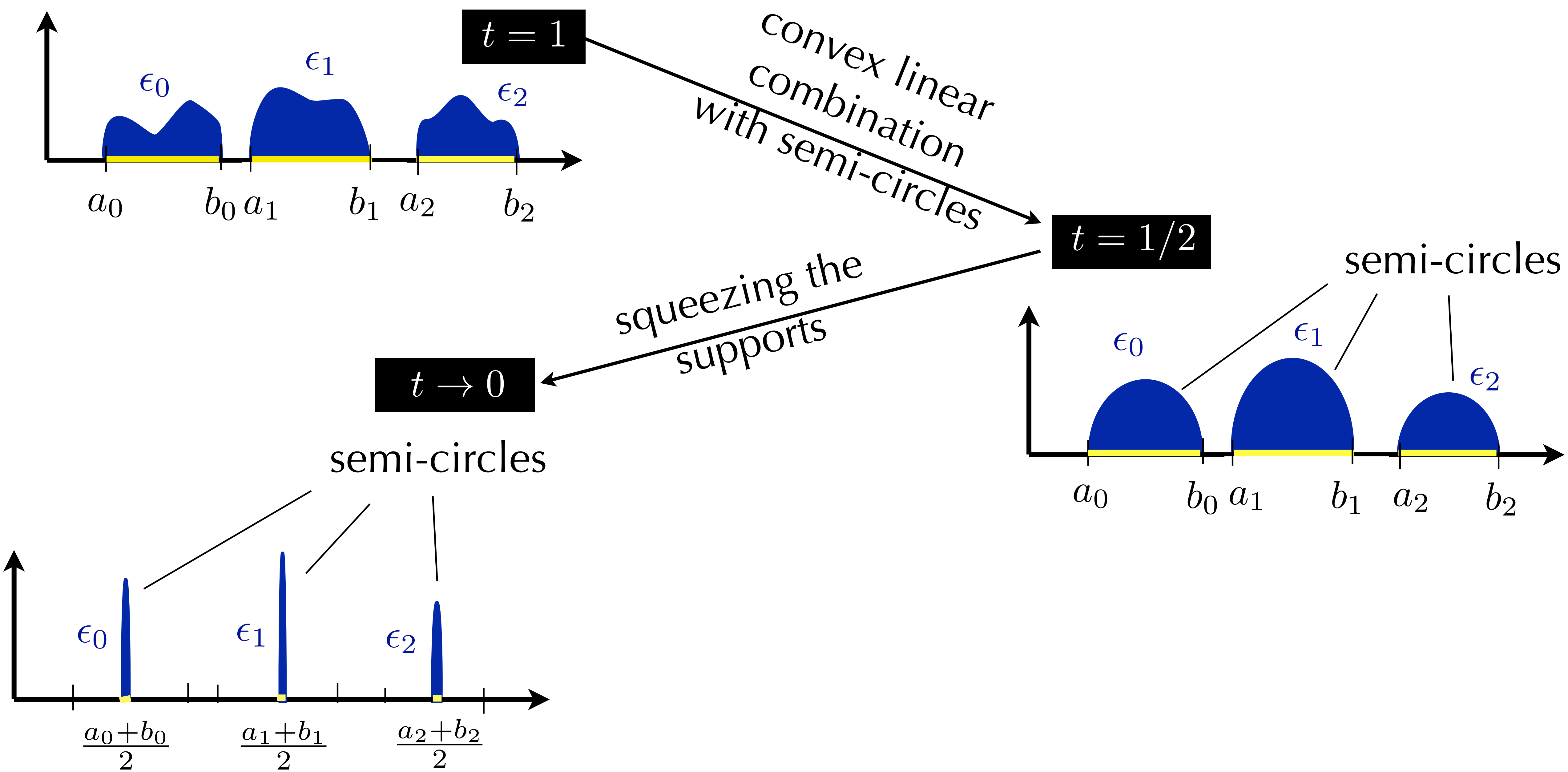
- If we can find a interpolating family $(T_t)_{t\in[0,1]}$
  - ☼ respecting uniformly our assumptions
  - ☼ for which $Z_{N;\epsilon}^{(T_0)}$ is known

we deduce an expansion $Z_{N;\epsilon}^{(T_1)} = Z_{N;\epsilon}^{(T_0)} \times \exp\Big(\sum_{m=-2}^{M-1} N^{-m}\,F_\epsilon^{[m]} + O(N^{-M})\Big)$

- Idea : interpolate in the space of equilibrium measures

$$(\mu_{\mathrm{eq};\epsilon}^{t})_{t\in[0,1]} \longleftrightarrow (T_t)_{t\in[0,1]}$$

An interpolation path ...



$$Z_{N;\epsilon}^{(T_t)} \underset{t\to 0}{\sim} \prod_{0\leq h<h'\leq g} \left|\frac{a_h+b_h-a_{h'}-b_{h'}}{2}\right|^{N^2\epsilon_h\epsilon_{h'}\beta} \prod_{h=0}^{g} \left(\begin{array}{c}\text{Selberg } \beta\text{–Gaussian} \\ \text{integral over } \mathbb{R}^{N_h}\end{array}\right)$$

Sums and interferences - 1/3

We initially wanted to compute $$Z_N = \sum_{N_0+\cdots N_g=N} \frac{N!}{\prod_{h=0}^{g} N_h!}\, Z_{N;(N_0/N,\ldots,N_g/N)}$$

- From large deviations of empirical measures :

$$Z_N = \Big(\sum_{|\mathbf{N}-N\epsilon^\star|\leq \ln N} \frac{N!}{\prod_{h=0}^{g} N_h!}\, Z_{N;\mathbf{N}/N}\Big)\big(1+O(e^{-cN})\big)$$

- For $\mathbf{N} - N\epsilon^\star \in o(N)$, we just proved, with $\epsilon = (N_h/N)_{1\leq h\leq g}$

$$\frac{N!}{\prod_{h=0}^{g} N_h!} Z_{N;\epsilon} = N^{\gamma N+\gamma'}\, \exp\Big[\sum_{m=-2}^{M-1} N^{-m}\, F_\epsilon^{[m]} + O(N^{-M})\Big]$$

where $F_\epsilon^{[m]}$ depend smoothly on $\epsilon \approx \epsilon_{\mathrm{eq}}$

- Extra lemma : $(\nabla_\epsilon F^{[-2]})_{\epsilon_{\mathrm{eq}}} = 0$ and $(\nabla_\epsilon \nabla_\epsilon F^{[-2]})_{\epsilon_{\mathrm{eq}}} < 0$

Sums and interferences - 2/3

We plug the asymptotic formula and use a Taylor expansion at $\epsilon \approx \epsilon_{\mathrm{eq}}$

- E.g. up to o(1) :

$$Z_N = N^{\gamma N + \gamma'} \, e^{N^2 F^{[-2]}_{\mathrm{eq}} + N F^{[-1]}_{\mathrm{eq}} + F^{[0]}_{\mathrm{eq}}}$$

$$\times \Big(\sum_{|\mathbf{N} - N\epsilon_{\mathrm{eq}}| \leq \ln N} e^{\frac{1}{2}(\nabla^{\otimes 2} F^{[-2]})_{\mathrm{eq}} \cdot (\mathbf{N} - N\epsilon_{\mathrm{eq}})^{\otimes 2} + (\nabla F^{[-1]})_{\mathrm{eq}} \cdot (\mathbf{N} - N\epsilon_{\mathrm{eq}})} \Big) \big(1 + O(e^{-c'(\ln N)^3/N})\big)$$

It is the general term of a super-exponentially fast converging series :

$$Z_N = N^{\gamma N + \gamma'} \, e^{N^2 F^{[-2]}_{\mathrm{eq}} + N F^{[-1]}_{\mathrm{eq}} + F^{[0]}_{\mathrm{eq}}}$$

$$\times \Big(\sum_{\mathbf{N} \in \mathbb{Z}^g} e^{\frac{1}{2}(\nabla^{\otimes 2} F^{[-2]})_{\mathrm{eq}} \cdot (\mathbf{N} - N\epsilon_{\mathrm{eq}})^{\otimes 2} + (\nabla F^{[-1]})_{\mathrm{eq}} \cdot (\mathbf{N} - N\epsilon_{\mathrm{eq}})} \Big) \big(1 + O(e^{-c''(\ln N)^3/N})\big)$$

- We recognize $\Theta_{-N\epsilon_{\mathrm{eq}}}\big((\nabla F^{[-1]})_{\mathrm{eq}} \,\big|\, (\nabla^{\otimes 2} F^{[-2]})_{\mathrm{eq}}\big)$

Sums and interferences - 3/3

- Including higher orders yields terms of the form

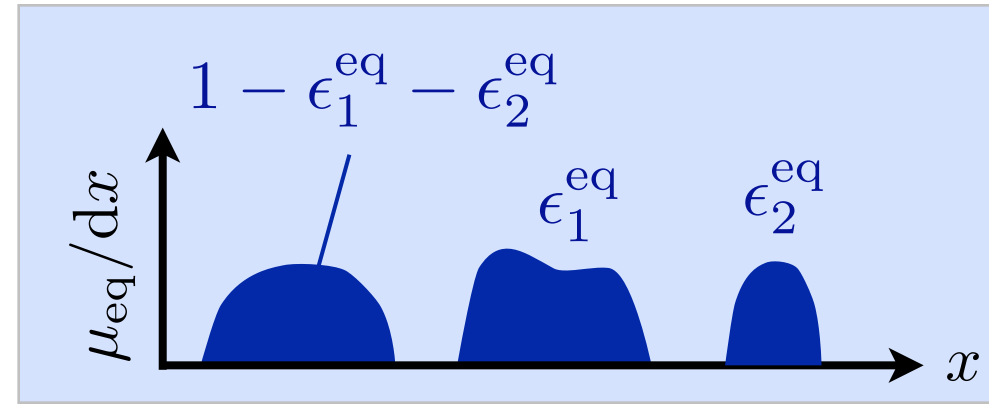
$$\sum_{\mathbf{N}\in\mathbb{Z}^g} \frac{1}{p!}\Big(\prod_{i=1}^{p} \frac{(\nabla^{\otimes \ell_i} F^{[m_i]})_{\mathrm{eq}}}{\ell_i!}\Big) \cdot (\mathbf{N} - N\epsilon_{\mathrm{eq}})^{\otimes(\sum_i \ell_i)}\, e^{\frac{1}{2}\mathbf{Q}\cdot(\mathbf{N}-N\epsilon_{\mathrm{eq}})^{\otimes 2}+\mathbf{w}\cdot(\mathbf{N}-N\epsilon_{\mathrm{eq}})}$$

We recognize $\displaystyle\sum_{\mathbf{N}\in\mathbb{Z}^g} \frac{1}{p!}\Big(\prod_{i=1}^{p} \frac{(\nabla^{\otimes \ell_i} F^{[m_i]})_{\mathrm{eq}}}{\ell_i!}\Big) \cdot (\nabla_{\mathbf{w}}^{\otimes(\sum_i \ell_i)}\Theta_{-N\epsilon_{\mathrm{eq}}})(\mathbf{w}|\mathbf{Q})$

Here $\mathbf{Q} = (\nabla^{\otimes 2} F^{[-2]})_{\mathrm{eq}}$ and $\mathbf{w} = (\nabla F^{[-1]})_{\mathrm{eq}}$

- We justified step by step the heuristics of **Bonnet, David, Eynard '00, Eynard '07**

Summary : the (g + 1)-cuts regime



- Oscillatory asymptotic expansion

$$Z_N = N^{\gamma N+\gamma'}(\mathcal{D}_N\Theta_{-N\epsilon_{\mathrm{eq}}})\big((\nabla F^{[-1]})_{\mathrm{eq}}\,\big|\,(\nabla^{\otimes 2}F^{[-2]})_{\mathrm{eq}}\big)\exp\Big[\sum_{m\geq -2} N^{-m}\,F^{[m]} + O(N^{-\infty})\Big]$$

where $$\mathcal{D}_N = \sum_{p\geq 0}\frac{1}{p!}\sum_{\substack{\ell_1,\ldots,\ell_p\geq 1\\ m_1,\ldots,m_p\geq -2\\ \sum_i(m_i+\ell_i)>0}} N^{-\sum_i(m_i+\ell_i)}\prod_{i=1}^{p}\frac{(\nabla^{\otimes \ell_i}F^{[m_i]})_{\mathrm{eq}}\cdot\nabla_{\mathbf{w}}^{\otimes \ell_i}}{\ell_i!}$$

acts as a differential operator on the Siegel theta function

$$\Theta_{\mu}(\mathbf{w}|\mathbf{Q}) = \sum_{\mathbf{m}\in\mathbb{Z}^g} e^{\mathbf{w}\cdot(\mathbf{m}+\mu)+\frac{1}{2}(\mathbf{m}+\mu)\cdot\mathbf{Q}\cdot(\mathbf{m}+\mu)}$$

- Moving characteristics $\mu = -N\epsilon_{\mathrm{eq}} \bmod \mathbb{Z}^g$

  Quadratic form $\mathbf{Q} = -\mathrm{Hessian}_{\epsilon=\epsilon_{\mathrm{eq}}}[\mathcal{T}(\mu_{\mathrm{eq};\epsilon})]$

All order asymptotics for β-ensembles in the multi-cut regime

1. Beta-ensembles and random matrices

2. Applications to orthogonal polynomials

3. Sketch of the proof of the main result

4. Conclusion

In progress

- A toy model for XXZ spin correlation functions (two-scale problem)

$$Z_N = \prod_{1 \leq i < j \leq N} \sinh\big[N^\alpha c_1(\lambda_i - \lambda_j)\big] \sinh\big[N^\alpha c_2(\lambda_i - \lambda_j)\big] \prod_{i=1}^{N} e^{-N^{1+\alpha} V(\lambda_i)} \, \mathrm{d}\lambda_i$$

Open problems

- Same questions for $\lambda_i \in \mathbb{Z}$?
  no Schwinger-Dyson equations ...

- Same (non-perturbative) questions for multi-matrix models ?
  more complicated Schwinger-Dyson equations and convexity issues ...

- Universality from Schwinger-Dyson equations ?