Rank metric completion and L^2-invariants

Andreas Thom

University of Göttingen

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Overview

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5. Pedersen's Theorem and applications

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Homological algebra and derived functors

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The sequence

$$C_* \stackrel{\text{def}}{=} \quad \cdots \to C_n \to \cdots \to C_1 \to C_0 \to 0$$

is called a *free resolution* of the R-module M.

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A functor F is called *right-exact*, if it maps short exact sequences

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$$

to right-exact sequences

$$F(M_1) \longrightarrow F(M_2) \longrightarrow F(M_3) \longrightarrow 0.$$

Corollary

Let F be any right-exact functor from the category of R-modules into some abelian category. The left-derived functors

$$(L_i F)(M) \stackrel{\mathrm{def}}{=} H_i\left(F(C_*)\right)$$

are well defined.

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are well defined.

These functors are very useful and carry a lot of interesting information about the module M and the functor F.

Example

For any extension of modules $0 \to M_1 \to M_2 \to M_3 \to 0$, there exists a long exact sequence

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ending with

$$\cdots \to (L_1 F)(M_3) \to F(M_1) \to F(M_2) \to F(M_3) \to 0.$$

Example

Let K be a right R-module. The functor $M \mapsto K \otimes_R M$ is right-exact. We set:

$$\mathrm{Tor}_k^R(K, M) = (L_k(K \otimes_R ?))(M) = H_k\left(K \otimes_R C_*\right).$$

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Remark

If the functor F is exact, then $(L_i F)(M) = 0$, for $i \geq 1$.

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is exact, then $\dim L_2 = \dim L_1 + \dim L_3$.
3. If $L = \cup_\alpha L_\alpha$, then $\dim L = \sup_\alpha \dim L_\alpha$.

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Theorem

Let F, G be two (right-exact) functors from an abelian category to the category of M-modules. If a natural transformation $H\colon F \to G$ consists of dimension isomorphisms, then so do the induced natural transformations

$$L_k H\colon L_k F \to L_k G.$$

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Example

Let M be an aspherical Riemannian manifold with fundamental group Γ.

$$\beta_k^{(2)}(\Gamma) = \lim_{t\to\infty} \int_F \mathrm{tr}\left(e^{-t\Delta_k}(x,x)\right) dx,$$

where F is a fundamental domain of the Γ, acting on the universal covering.

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- $M \overline{\otimes} M^o$ is both a right $A \otimes A^o$-module and a left $M \overline{\otimes} M^o$-module.
- The dimension is computed with respect to that left module action.

Lemma (Shlyakhtenko-Connes)

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A related quantity is

$$\Delta_k^{(2)}(A, \tau) = \dim \operatorname{Tor}_k^{M \otimes M^o}(M \mathbin{\overline{\otimes}} M^o, M \otimes_A M).$$

It is better suited to approximate $\Delta_k^{(2)}(M, \tau) = \beta_k^{(2)}(M, \tau)$.

Rank metric completion of bi-modules over a finite von Neumann algebra

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$$[\xi] = \inf \left\{ \tau(p) + \tau(q) \colon p, q \in \mathrm{Proj}(K), p^{\perp} \xi q^{\perp} = 0 \right\} \in [0, 1].$$

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defines a pseudo-metric on K.

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The functor of completion is exact.

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is a dimension isomorphism.

Corollary

The induced map

$$\mathrm{Tor}_k^{M \otimes M^o}(M \mathbin{\overline{\otimes}} M^o, K) \to \mathrm{Tor}_k^{M \otimes M^o}(M \mathbin{\overline{\otimes}} M^o, \hat{K}).$$

is a dimension isomorphism for all k.

Pedersen's result and applications

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Corollary

The natural map $\phi \colon M \otimes_A M \to M$ is an isomorphism after completion in the rank metric.

Corollary

Let (M,τ) be as above and let A be a dense C^*-subalgebra. Then,

$$\mathrm{Tor}_k^{M\otimes M^o}(M \mathbin{\overline{\otimes}} M^o, M \otimes_A M) \to \mathrm{Tor}_k^{M\otimes M^o}(M \mathbin{\overline{\otimes}} M^o, M).$$

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$$\Delta_k^{(2)}(A, \tau) = \Delta_k^{(2)}(M, \tau).$$

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Question

Can one make this approach work using $L\Gamma\otimes_{\mathbb{C}\Gamma}L\Gamma$ or $L\Gamma\otimes_{C^\infty\Gamma}L\Gamma$ rather than $L\Gamma\otimes_{C_r\Gamma}L\Gamma$?

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Theorem (Gaboriau)

Orbit equivalent groups have the same L^2-Betti numbers.

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   have isomorphic completions as $L^\infty(X)$-modules.
4. Completing everything with respect to $L^\infty(X)$ preserves the dimension and the result depends only on the equivalence relation.