Matrix Models with convex interaction

Alice Guionnet

UMPA, CNRS, Ecole Normale Supérieure de Lyon,France

and

UC Berkeley

Joint works with E. Maurel Segala and D. Shlyakhtenko

UC Berkeley, March 25, 2007

Plan

- From ’t Hooft expansion to some questions in free probability.
- Some arguments from free probability to analyze the first order of ’t Hooft expansion and the associated (planar) combinatorial problem.

Reminder on Jean-Bernard Zuber's talk

Let μ_N be the law of a $N \times N$ complex Gaussian Wigner matrix (**GUE**).
Let $V = \sum_{i=1}^n \beta_i x^i$ be a polynomial.

Then, 't Hooft expansion reads as the equality between formal series

$$F_N(V) := \frac{1}{N^2} \log \int e^{-N\mathrm{tr}(V(X))} d\mu_N(X)$$

$$= \sum_{g \in \mathbb{N}} \frac{1}{N^{2g}} \sum_{k_1, \cdots, k_n \in \mathbb{N}} \prod_{i=1}^n \frac{(-\beta_i)^{k_i}}{k_i!} M_g((k_i, i)_{1 \le i \le n})$$

with $M_g((k_i, i)_{1 \le i \le n})$ the number of maps with genus g (i.e connected graphs embedded in a surface of genus g) with k_i vertices of degree i(all half-edges labelled)

Several matrices generalization

Let μ_N be the law of a $N \times N$ complex Gaussian Wigner matrix (**GUE**). Let $V = \sum_{i=1}^n \beta_i q_i(X_1, \cdots, X_m)$ be a polynomial in m non-commutative variables. q_i monomials.

Then, 't Hooft expansion reads

$$\frac{1}{N^2} \log \int e^{-N\mathrm{tr}(V(X_1,\cdots,X_m))} d\mu_N(X_1) \cdots d\mu_N(X_m)$$

$$= \sum_{g \in \mathbb{N}} \frac{1}{N^{2g}} \sum_{k_1,\cdots,k_n \in \mathbb{N}} \prod_{i=1}^n \frac{(-\beta_i)^{k_i}}{k_i!} M_g((k_i, q_i)_{1 \le i \le n})$$

with $M_g((k_i, q_i)_{1 \le i \le n})$ the number of maps with genus g (i.e connected graphs embedded in a surface of genus g) with k_i stars of type q_i.

Several matrices generalization

Let μ_N be the law of a $N \times N$ complex Gaussian Wigner matrix (**GUE**). Let $V = \sum_{i=1}^n \beta_i q_i(X_1, \cdots, X_m)$ be a polynomial in m non-commutative variables.

Then, 't Hooft expansion reads, for any monomial q

$$\frac{1}{N^2} \log \int e^{-N\mathrm{tr}(V(X_1,\cdots,X_m)+tq(X_1,\cdots,X_m))} d\mu_N(X_1)\cdots d\mu_N(X_m)$$

$$= \sum_{g\in\mathbb{N}} \frac{1}{N^{2g}} \sum_{k_1,\cdots,k_n,k\in\mathbb{N}} \prod_{i=1}^n \frac{(-\beta_i)^{k_i}}{k_i!} \frac{(-t)^k}{k!} M_g((k_i,q_i)_{1\le i\le n},(k,q))$$

with $M_g((k_i,q_i)_{1\le i\le n},(k,q))$ the number of maps with genus g (i.e connected graphs embedded in a surface of genus g) with k_i stars of type q_i and q stars of type q.

Several matrices generalization

Let μ_N be the law of a $N \times N$ complex Gaussian Wigner matrix (**GUE**). Let $V = \sum_{i=1}^{n} \beta_i q_i(X_1, \cdots, X_m)$ be a polynomial in m-non-commutative variables. Let q be a monomial.

Then, 't Hooft expansion reads

$$\bar{\mu}_N^V(q) := \int \frac{1}{N} \mathrm{tr}(q(X_1, \cdots, X_m)) d\mu_N^V(X_1, \cdots, X_m)$$

$$= \sum_{g \in \mathbb{N}} \frac{1}{N^{2g}} \sum_{k_1, \cdots, k_n \in \mathbb{N}} \prod_{i=1}^{n} \frac{(-\beta_i)^{k_i}}{k_i!} M_g((k_i, q_i)_{1 \le i \le n}, (1, q))$$

with

$$d\mu_N^V(X_1, \cdots, X_m) = \frac{e^{-N\mathrm{tr}(V(X_1, \cdots, X_m))} d\mu_N(X_1) \cdots d\mu_N(X_m)}{\int e^{-N\mathrm{tr}(V(X_1, \cdots, X_m))} d\mu_N(X_1) \cdots d\mu_N(X_m)}.$$

From formal series to large N limit

Let μ_N be the law of a $N \times N$ complex Gaussian Wigner matrix (**GUE**).
Let $V = \sum_{i=1}^n \beta_i q_1(X_1, \cdots, X_m)$ be a polynomial and q be a monomial.

Then, a large N limit of 't Hooft expansion gives, for any monomial q (G-Maurel Segala, Alea 06)

$$\lim_{N\to\infty} \bar{\mu}_N^V(q) = \sum_{k_1,\cdots,k_n \in \mathbb{N}} \prod_{i=1}^n \frac{(-\beta_i)^{k_i}}{k_i!} M_0((k_i, q_i)_{1\le i\le n}, (1, q))$$

It holds if

1. $V = V^*$ with $(zX_{i_1}\cdots X_{i_k})^* = \bar{z}X_{i_k}\cdots X_{i_1}$.
2. $\exists c > 0$, $V + \frac{1-c}{2}\sum_{i=1}^m X_i^2$ is convex in the sense that $X_i^N(kl) \in \mathcal{H}^N, 1 \le i \le m \to \mathrm{tr}(V(X_1^N, \cdots, X_m^N))$ convex $\forall N$.
3. The$\beta_i' s$ are small enough (depending on c).

From formal series to large N limit:removing the convexity hypothesis

Let $\bar{\mu}_N^V$ be the Gibbs measure with potential V wrt **(GUE)**. Let $V = \sum_{i=1}^n \beta_i q_1(X_1, \cdots, X_m)$ be a polynomial in m-non-commutative variables. Let q be a monomial.

Then, a large N limit of 't Hooft expansion reads (G- Maurel Segala 06)

$$\lim_{N\to\infty} \int_{\cap_i \{\|X_i\|_\infty \le L\}} \frac{1}{N} \mathrm{tr}(q(X_1, \cdots, X_m)) d\mu_N^V(X_1, \cdots, X_m)$$

$$= \sum_{k_1, \cdots, k_n \in \mathbb{N}} \prod_{i=1}^n \frac{(-\beta_i)^{k_i}}{k_i!} M_0((k_i, q_i)_{1\le i\le n}, (1, q))$$

It holds if

1. $V = V^*$ with $(zX_{i_1} \cdots X_{i_k})^* = \bar{z} X_{i_k} \cdots X_{i_1}$.
2. There exists $\epsilon_0 > 0$ for all $\epsilon < \epsilon_0$, $\max_i |\beta_i| < \epsilon$ and $L_0(\epsilon) \le L \le L_1(\epsilon)$, $\lim_{\epsilon\to 0} L_0(\epsilon) = 2$ and $\lim_{\epsilon\to 0} L_1(\epsilon) = +\infty$

Idea of the proof.

Assume $V + (1-c)/2 \sum X_i^2$ convex. The limit points τ of $\bar{\mu}_N^V$, as a linear functional on $\mathbb{C}\langle X_1, \cdots, X_m\rangle$, are such that

1. There exists $R = R(c) < \infty$ s.t. $|\tau(X_{i_1} \cdots X_{i_k})| \leq R(c)^k$.

2. τ is solution to Schwinger-Dyson equation : For all $P \in \mathbb{C}\langle X_1, \cdots, X_m\rangle$, all $i \in \{1, \cdots, m\}$,

$$\tau((X_i + D_i V)P) = \tau \otimes \tau(\partial_i P)$$

with $\partial_i P = \sum_{P = P_1 X_i P_2} P_1 \otimes P_2$, $D_i P = \sum_{P = P_1 X_i P_2} P_2 P_1$.

Thm:There exists a unique solution for β_i's small enough. It is such that

$$\tau(q) = \sum_{k_1, \cdots, k_n \in \mathbb{N}} \prod_{i=1}^{n} \frac{(-\beta_i)^{k_i}}{k_i!} M_0((k_i, q_i)_{1 \leq i \leq n}, (1, q))$$

Free probability issues

- Being given $V \in \mathbb{C}\langle X_1, \cdots, X_m\rangle$, is there a unique tracial state τ, i.e $\tau \in \mathbb{C}\langle X_1, \cdots, X_m\rangle'$ such that

$$\tau(PP^*) \geq 0,\ \tau(PQ) = \tau(QP),\ \tau(1) = 1$$

so that for all $P \in \mathbb{C}\langle X_1, \cdots, X_m\rangle$, all $i \in \{1, \cdots, m\}$,

$$\tau(D_i V P) = \tau \otimes \tau(\partial_i P)$$

i.e.$\xi = (D_i V)_{1 \leq i \leq m}$ is the conjuguate variable of τ.

Recall: Voiculescu(00):if ξ is polynomial, then it belongs to the cyclic gradient space, G-Cabanal Duvillard (03): Such τ's are dense.

- If $V = \frac{1}{2} \sum X_i^2 + \sum \beta_i q_i$, $\tau_V(q) = \tau_{(\beta_i)_{1 \leq i \leq m}}(q)$ depends analytically on the parameters $(\beta_i)_{1 \leq i \leq m}$ small enough. When does analyticity breaks down ? How (study of the critical exponents)? What does it mean about the related algebras ?

Result 1:Uniqueness

Thm: Assume that V is 'sufficiently locally convex'. There exists a unique τ, law of m non-commutative variables bounded by b_V, s.t

$$\tau(D_i V P) = \tau \otimes \tau(\partial_i P)$$

Def: Let $*$ be an involution and set

$$X.Y = \frac{1}{2} \sum_{i=1}^{m} (X_i Y_i^* + Y_i X_i^*).$$

V is (c, M)-convex iff for any $X = (X_1, \ldots, X_m)$ and $Y = (Y_1, \ldots, Y_m)$ in some C^*-algebra $(\mathcal{A}, \|\cdot\|_\infty)$ such that $\|X_i\|_\infty, \|Y_i\|_\infty \leq M$, $i = 1, \ldots, m$ we have

$$[DV(X) - DV(Y)].(X - Y) \geq c(X - Y).(X - Y) \tag{1}$$

'sufficiently locally convex'=V (c, M)-convex with $M \geq M(c)$, $c > 0$ ($b_V = b(c)$).

Remarks on Result 1

- We did not assume $V = V^*$
  - If $V = V^*$, τ_V is the law of m self-adjoint non-commutative variables, i.e if $X_i = X_i^*$, $(zX_{i_1} \cdots X_{i_k})^* = \bar{z} X_{i_k} \cdots X_{i_1}$,
$$\tau_V(PP^*) \geq 0,\ \tau(PQ) = \tau(QP),\ \tau(1) = 1$$
  - Otherwise, there exists $\nu = \nu_V$ the law of $(X_i, X_i^*)_{1 \leq i \leq m}$ so that $\tau_V(P) = \nu_V(P(X_1, \cdots, X_m))$.
- If $V = \frac{1}{2} \sum_{i=1}^m X_i^2 + \sum_{i=1}^n \beta_i q_i$, V is $(\frac{1}{2}, M)$ convex for any M provided the β_i's are small enough (depending on M).

Result 2:Analyticity

Let $V = V_\beta = \sum_{i=1}^n \beta_i q_i$ where $(q_i)_{1\le i\le n}$ are monomials. Let $T(c,M) \subset \mathbb{C}^n$ be the interior of the subset of parameters $\beta = (\beta_i)_{1\le i\le n}$ such that V_β is (c,M)-convex. Assume $M \ge M(c)$.

Then , for any $P \in \mathbb{C}\langle X_1, \cdots, X_m\rangle$,

$$\beta \in T(c,M) \to \tau_{V_\beta}(P) \text{ is analytic}.$$

In particular,

$$\beta \to \sum_{k_1,\cdots,k_n} \prod \frac{(-\beta_i)^{k_i}}{k_i!} M_0((k_i, q_i))$$

extends analytically to the interior of the set of β_i's where $\frac{1}{2}\sum_{i=1}^m X_i^2 + \sum \beta_i q_i$ is (c,M)-convex for $M \ge M_0(c)$.

Result 3: Algebras are similar to those generated by semi-circulars

Assume that V is (c, M)-convex with $M \geq M(c)$.

Let Z with law τ_V (or Z, Z^* with law ν_V if $V \neq V*$).

The C^*-algebra generated by Z is exact, projectionless (in particular any $P(Z, Z^*)$ has a connected support).

The von Neumann algebra associated with (Z, Z^*) has the Haagerup approximation property and admits an embedding into the ultrapower of the hyperfinite II_1 factor.

Reminder about Jean-Bernard Zuber's talk

When $m = 1$, to solve explicitly $\tau_{(t_i)_{1\le i\le m}}(x^p)$ it is enough

- To use Schwinger-Dyson equation to find that $G(z) = \tau_{(t_i)_{1\le i\le m}}((z-x)^{-1})$ satisfies

$$G(z) = \frac{1}{2}(W(z) - \sqrt{W(z)^2 - R(z)}) \quad W(z) = z + V'(z)$$

  with R a polynomial of degree smaller to $\deg(V) - 2$.

- To determine R by proving that $\tau_{(t_i)_{1\le i\le m}}$ is a probability measure with a connected compact support in $\mathbb{R}$.

Norm convergence

Haagerup and Thorbjornsen (02) proved

$$\lim_{N\to\infty} \|P(X_1^N, \ldots, X_m^N)\|_\infty = \|P(X_1, \ldots, X_m)\|_\infty \text{ a.s.}$$

if $X_1^N, \cdots, X_m^N$ follows the GUE and $X_1, \cdots, X_m$ are free semi-circular.

Thm: If V is (c, ∞)-convex, $V = V^*$, the limit holds with $X_1^N, \cdots, X_m^N$ with law μ_N^V and $X_1, \cdots, X_m$ with law τ_V.

Idea of the proof: Le coup du Processus

1. See μ_N^V has an invariant measure of

$$dX_t^N = dH_t^N - \frac{1}{2} D_i V(X_t^N) dt$$

   with H^N a Hermitian Brownian motion.

2. See τ_V has an invariant measure of

$$dX_t = dS_t - \frac{1}{2} D_i V(X_t) dt$$

   with S a free Brownian motion.

3. Show that if V is (c, M) convex, $M \geq M(c)$, such a process
   (a) Stays below the treshold M if X_0 has norm below some b.
   (b) Has any solution of Schwinger-Dyson has an invariante measure.
   (c) Has a unique invariante measure uniformly bounded by $B < b$.
   (d) Converges in the uniform norm to Z with law τ_V when $X_0 = 0$.

Conclusion

1. The generating function of maps is given as the solution of Schwinger-Dyson equation which stays sufficiently bounded.
2. It is also given as the invariant measure of a free SDE.
3. What happens at the phase transition ?