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Last Time: $M = L^\infty(X) \rtimes_\alpha \Gamma$

$${}_A L^2(M)_A = \bigoplus_{g \in \Gamma} {}_A \mathcal{J}_{\alpha_g} {}_A \qquad A = L^\infty(X)$$

$L^\infty(X)$ embeds into $\mathcal{J}_e$, Γ into $\xi_0 = 1 \in \mathcal{J}_{\alpha_g}$

In the matrix case

$T_{ij} = e_i T e_j$ except the latter isn't a scalar

minimal projections

In our case

A is the analogue of diagonal matrices

$T \in M \quad \hat{T} \in L^2(M) \quad \hat{T} = \sum \hat{T}_{\alpha_g} \qquad p, q \in A$ projections

$$p \hat{T} q = \sum p \hat{T}_{\alpha_g} q \qquad p = \chi_u \quad q = \chi_v$$

$$p \hat{T}_{\alpha_g} q = \hat{T}_{\alpha_g} \alpha_g(p) q = T_{\alpha_g} \chi_{(\alpha(u) \cap v)}$$

$R_n \quad L^2(R_n) = L^2(M) \qquad T \rightsquigarrow \sum_{\gamma \in \Gamma} a_\gamma \cdot$ (a piece of graph of a_γ)

Observation Let R as above and $\beta : X \to X$, $\beta \mu_* = \mu$
Then graph $(\beta) \subseteq R$ (up to set of 0 measure for μ_R)
$\Leftrightarrow \beta$ "compatible with R" i.e. $\beta(x) \underset{R}{\sim} x$ μ-a.e. in x.

Can do with partially defined stuff: $\beta_* \mu|_u = \mu|_v$, etc.

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Thus any partial automorphism of A gives rise to a vector in $L^2(M) \simeq L^2(R, \mu_R)$

<u>To do</u>

$$\chi_{\text{gr of } \beta} = V_\beta \in M \qquad (V_\beta \in L^2(M) \supseteq M)$$

Let H be an M-module

$\xi \in H \qquad \Theta_\xi = M \longrightarrow H \qquad \Theta_\xi(m) = \xi m$

$$\|\Theta_\xi(m)\|_H \leq \|\xi\|_H \|m\|_\infty$$

We say ξ is <u>bounded</u> if $\|\Theta_\xi(m)\|_H \leq C\|m\|_2$.

I.e. $m \to \Theta_\xi(m)$ is densely defined $M \hookrightarrow H$

Extends to a bounded map $L^2(M) \longrightarrow H$

$\boxed{\text{Example}}$ $\xi \in M \subseteq L^2(M) \qquad \xi = \hat{x}, \ x \in M$

$$\underset{\|\xi m\|_2}{\|m\xi\|_2} \leq \|\xi\|_\infty \|m\|_2$$

$\therefore$ $M \subseteq L^2(M)$ consists of bounded vectors.

Classical analogue: $L^\infty \cap L^2$ is precisely those fns in L^2 which are bounded as multiplication operators on L^2.

<u>Conversely</u> assume ξ bdd

$\Theta_\xi : L^2(M) \longrightarrow L^2(M) \qquad$ extending $\Theta_\xi(m) = m\xi$

$\forall n \in M, \quad \underset{\Theta_\xi(\rho_n(m))}{\Theta_\xi(mn)} = \Theta_\xi(m)n = \rho_n(\Theta_\xi(m)) \Rightarrow \Theta_\xi \in \rho(M)'$

Need $\rho(M)' = M \qquad \supseteq$ clear

other direction omitted for now...

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Back to proving $\chi_{\text{gr. of } \beta} = V_\beta \in M$.

write $h = \sum_i a_i V_{\alpha_i}$ $\qquad R = \bigsqcup_{\alpha_i} \mathcal{G}_{\alpha_i}$

$$V_\beta \cdot h = \sum_i \beta(a_i) V_{\beta \circ \alpha_i}$$

$$\|V_\beta h\|_{L^2(R,\mu_R)} \leq \|h\|_{L^2(R,\mu_R)} \Rightarrow V_\beta \in M.$$

defines a rep of the <u>collection of all partial R-compatible automorphisms</u> by partial isometries on $L^2(R,\mu_R)$

(meaning, a map $\alpha \longmapsto V_\alpha$ where $\alpha \circ \beta \longrightarrow V_\alpha V_\beta$)

$\alpha_1 \circ \alpha_2$ def. as an aut. on

$$\alpha_1 : U_1 \to V_1 \qquad \alpha_2^{-1}(V_2 \cap U_1) = U_2 \cap \alpha_2^{-1}(U_1)$$
$$\alpha_2 : U_2 \to V_2$$

In matrix case:

$$\begin{array}{cc} & \\ e_{ij} = & \begin{bmatrix} & & \\ & 1 & \\ & & \end{bmatrix}_j \\ & \qquad\quad i \end{array}$$

$\shortparallel$

V_α $\qquad \alpha =$ part. aut. w/domain $\{i\}$, range $\{j\}$

$$e_{ij} e_{kl} = \delta_{jk} e_{il}$$

$\boxed{\text{Summary}}$ (1) $\underset{\text{p.i.}}{V_\beta} \in M \quad \forall \beta$ R-compat. part. aut.

(2) Any $x \in M$ has a "matrix"

$$\hat{X} \in L^2(R,\mu_R) = L^2(X \times X; \mu_R)$$

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The product in M corresponds to "a kind of convolution" on $L^2(X \times X, \mu_R)$

$$\widehat{(TS)}(x,y) = \int T(x,z)\,S(z,y)\,d\mu(z)$$

This equals a sum $\sum_{z \in \mathcal{O}_x} T(x,z)\,S(z,y)$

$T, S \in M$

$$= \int_{(x,y) = a \cdot b} T(a)\,S(b)$$

$a = (x_0, y_0)$

$b = (x_1, y_1)$ $\quad a \cdot b = (x_0, y_1)$ if $y_0 = x_1$

$$\alpha : \Gamma \curvearrowright X \rightsquigarrow L^\infty(X) \rtimes_\alpha \Gamma \supseteq L^\infty(X) = A$$

$R \leftarrow - - - - - -$

<u>Th</u> $\mathcal{N}(A)$ in M is exactly $\{ a \cdot V_\beta : \beta : X \to X \ R\text{-compat. aut} \}$

$\shortparallel$

$\{ u : u \in M \text{ unitary}, \ uAu^* = A \}$

$u \in \mathcal{N}(A) \Rightarrow au = u\,\alpha(a) \qquad \alpha(a) = uau^*$

as a vector in ${}_A L^2(R, \mu_R)_A \qquad a\hat{u} = \hat{u}\,\alpha(a)$

$\Rightarrow \operatorname{supp} \hat{u} = \text{graph of } \alpha \Rightarrow \alpha \ R\text{-compatible}$

<u>Consider</u> V_α

then $a V_\alpha^* u = V_\alpha^* \alpha^{-1}(a)\, u = V_\alpha^* u\, a$

$\Rightarrow V_\alpha^* u \in A' \cap M = A$

$\Rightarrow V_\alpha^* u = b$ for some $b \in A$ unitary.

<u>Point:</u> We can canonically recover the equivalence relation R from A.

R = union of graphs of all automorphisms of A induced by elements of $\mathcal{N}(A)$.