Examples

- A is abelian, $\mathcal{A} = L^\infty(X, \mu)$, trace given by $\mu$, $\Gamma \to X$ $\mu$-preserving Murray-von Neumann group-measure space construction

Example $X = L^\infty([0,1])$, $\mathcal{P} = \mathbb{Z}/n\mathbb{Z}$ acting by cyclic permutations

$L^\infty(X) \times \Gamma = M_{n \times n}(\mathbb{C})$

Question: What assumptions guarantee that $\mathcal{A} \times_\alpha \Gamma$ is a factor?

(We write $\mathcal{A} = L^\infty(X, \mu)$)

$\mathcal{A} \subset A \times \Gamma$

$L(\Gamma) \leq A \times_\alpha \Gamma$

$F \in A \cap L(\Gamma)' \iff F \circ \alpha_g = f \quad \forall g \in \Gamma$

$U_g F U_{g^{-1}} = F \circ \alpha_g$

\[\therefore\text{Necessary condition: action is ergodic (only $\Gamma$-invariant $L^\infty$ for constant)}\]

Ex $\Gamma = \mathbb{Z} \times \mathbb{Z}$ acting on $\mathbb{T}$

$(n,m)$ acts by rotating by $n\theta$ where $\frac{\theta}{2\pi} \in \mathbb{Q}$

This action is ergodic but not free.

(An action is called free if $\{g: \alpha_g(x) = x\} = 2\mathbb{Z}$ for a.e. $x$)

Thm: $\alpha$ free and ergodic $\implies A \times_\alpha \Gamma$ is a factor. (Pf later)

Ex $\Gamma$ acting on a point $\mathbb{S}^3$ is not free.
Proof that $A \times \mathbb{F}$ factor for $\mathbb{F}$ ergodic-free.

**Step 1.** $A \subseteq A \times \mathbb{F}$ is a masa $(A^n \subseteq A \times \mathbb{F} = A)$

- Bimodule $A \mathcal{L}^2(A \times_n A)_A$ has a simple rep

\[
A \mathcal{L}^2(A \times_n A)_A \cong \bigoplus_{y \in \mathbb{F}} O y_{x,y}
\]

- $O y_{x,y} = \mathcal{L}^2(X \times X, \mu_{\text{along graph of } x_y})$

Given $f \circ (x, y) \in O y_{x,y}$

\[
(f \circ (x, y))(x, y) = f(x) \circ (x, y) \circ g(y)
\]

Makes this a bimodule over $A = \mathcal{L}^\infty(X).

**Exercise.**

$x = \text{id}$ Support is $\{x = y\}$ so multiplying by $f(x)$ and $f(y)$ are same

$F \circ x = x \circ F \quad F = 1$

\[
\begin{array}{cccc}
\alpha & \longrightarrow & A & \longrightarrow & A(n) \\
\uparrow & & \uparrow & & \uparrow \quad A(x, x) = \alpha(x) \\
\mathcal{L}^2(X, \mu) & \longrightarrow & \mathcal{L}^2(X \times X, \eta_{id}) & \longrightarrow & A(x, y) = 0 \text{ if } x \neq y
\end{array}
\]

So $A \mathcal{G}_{id} A \cong A \mathcal{L}^2(A)_A$.

\[
O x = \mathcal{L}^2(A)_A
\]

denotes the bimodule which as a Hilbert space is $\mathcal{L}^2(A)$

\[
a \cdot \overline{3}_o b = \alpha(a) \overline{3}_o b
\]

$\overline{3}_o = 1$

\[
a \cdot \overline{3}_o (x, y) = a_o(x) \overline{3}_o (x, y) = \overline{3}_o (x, y) a_o(\chi^{-1}(y)) = \overline{3}_o (x, y) a_o(y) = \overline{3}_o \cdot \alpha(a_o)
\]
\[ \sum a_y \mathcal{U}_y \quad \text{isometric} \quad \sum a_y \cdot \overline{\mathcal{F}_y} \]

since
\[ \langle a \mathcal{F}_y, b \mathcal{F}_y \rangle = \int a(x) \overline{b(x)} \, dx \]
and
\[ \langle a \mathcal{U}_y, b \mathcal{U}_y \rangle = \tau(a \mathcal{U}_y \mathcal{U}_y^* b \mathcal{U}_y) = \tau(a b) \]

Now if \( z \in \mathcal{A}' \cap \mathcal{A} \times _\alpha \Gamma \Rightarrow \frac{z}{\lambda} = z \cdot \lambda \) is central for \( \mathcal{A} \).

**Fact** \( \{ z \in \mathcal{G}_x : a z = z a \ \forall a \in \mathcal{A} \} = x \)

\( (a z) (x, y) = (z \cdot a)(x, y) \)

\( z(x, y) a(y) = z(x, y) a(a^{-1}(y)) \ \forall a \quad x = a^{-1}(y) \)

\( \Rightarrow \frac{z}{\lambda} \) as \( z \) is supported on \( \{ x : a(x) = z \} \)

If the action is free, \( \{ x : a(x) = x \} \) is empty for nontrivial \( a \).

So \( \{ 0 \mathcal{G}_x : z \mathcal{G}_x \} \) contains no \( \mathcal{A} \)-central vectors,

\( \Rightarrow \{ \mathcal{A} \text{-central vectors in } L^2(\mathcal{A} \times \Gamma) \} = 0 \).

This proves \( \mathcal{A} \subseteq (\mathcal{A}' \cap \mathcal{A} \times \Gamma) \subseteq L^2(\mathcal{A}) \).

Could there be \( T \in \mathcal{A}' \cap \mathcal{A} \times \Gamma \) with \( T \in L^2(\mathcal{A}) ? \)
Useful Fact \[ \text{conditional expectation i.e. a unital map} \]
\[ E : A \times \mathbb{N} \to A \quad \text{s.t.} \quad E(x) \geq 0 \quad \text{if} \quad x \geq 0 \quad \text{and} \quad E(ax+b) = aE(x) + b \quad \text{s.t.} \]
\[ \forall (E(x)) = \tau(x). \]

Then \[ \tau(ax) = \tau(aE(x)) \] because both equal \[ \tau(E(ax)). \]

\[ \langle T - E(T), a \rangle = 0 \]
but \[ T - E(T) \not\sim 0 \] since \( T \not\in A. \)

\( E(T) \) is ortho-proj. of \( T \) onto \( L^2(A) \),
so \[ E(T) = T \implies E(T) = T \quad \text{since trace faithful and cyclic vector separating.} \]
which is a contradiction. \( \square \)

We can't quite prove this Useful Fact generally yet. Next lecture.

We've shown \( \Gamma \) acts freely \( \implies A \subset A \times \mathbb{N} \) is a MASA.

\( \Gamma \) ergodic \( \implies L(\mathbb{N})' \cap A = C^1 \)

Both \( \implies \) \( (A \times \mathbb{N})' \cap (A \times \mathbb{N}) = C^1 \).

This gives us plenty of examples of factors.

Ex. \( \frac{0}{2\pi} \in \mathbb{R} \implies \) factor.
\[ \mathcal{N}(A) \subseteq A \times \Gamma \]
\[ \{ u \in A \times \Gamma \text{ unitary, } uAu^* = A \} \]

**Def**: If \((X, \mu)\) is a measured space,

- \(R \subseteq X \times X\) is called a **Borel equivalence relation** if \(R\) is an equivalence relation and a Borel subset.
- \(R\) is called **discrete** if orbits are countable.

\(R\) receives a (possibly infinite) measure \(\mu^R\) as follows:

For \(O \subseteq R\) Borel,

\[ \mu^R(O) = \int \# \{ y \text{ s.t. } (x, y) \in O \} \, d\mu(x) \]

**Ex**: \(O_\alpha = \text{graph of an automorphism}\)

\[ \mu^R(O_\alpha) = 1 \] since integrand everywhere 1.

**R** **measurable equivalence relation** if it's taken up to \(\mu^R\)-null sets.

When \(\Gamma \rightarrow X\), \(\Gamma\) induces an equivalence relation of being in the same orbit:

\[ x \sim y \iff x = \alpha_x(y) \text{ for some } x \in \Gamma. \]

\[ R = \bigsqcup_{\gamma \in \Gamma} O_{\gamma} \]