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Last time: $A \rtimes_\alpha \Gamma$ — A tracial VNA, α trace-preserving action of Γ on A

Examples

- A abelian $A = L^\infty(X,\mu)$ trace given by μ $\Gamma \curvearrowright X$ μ-preserving
  Murray - von Neumann group - measure space construction

Example $X = \ell^\infty(\{1,\dots,n\})$ $\Gamma = \mathbb{Z}/n\mathbb{Z}$ acting by cyclic permutations

$$L^\infty(X) \rtimes \Gamma = M_{n\times n}(\mathbb{C})$$

Question What assumptions guarantee that $A \rtimes_\alpha \Gamma$ is a factor?
(We write $A = L^\infty(X,\mu)$)

$$A \subseteq A \rtimes_\alpha \Gamma \qquad L(\Gamma) \subseteq A \rtimes_\alpha \Gamma$$

$$f \in A \cap L(\Gamma)' \iff f\circ\alpha_g = f \quad \forall g \in \Gamma$$

$$u_g f u_{g^{-1}} = f \circ \alpha_g$$

$\therefore$ Necessary condition: action is ergodic (only Γ-invariant L^∞ fns constant)

Ex $\Gamma = \mathbb{Z}\times\mathbb{Z}$ acting on $\mathbb{T}$
(n,m) acts by rotating by $n\theta$ where $\frac{\theta}{2\pi} \notin \mathbb{Q}$

This action is ergodic but not free.

(An action is called free if $\{g : \alpha_g(x) = x\} = \{e\}$ for a.a. x)

Thm α free and ergodic $\Rightarrow A \rtimes_\alpha \Gamma$ is a factor. (Pf later)

Ex Γ acting on a point $\{*\}$ is not free.

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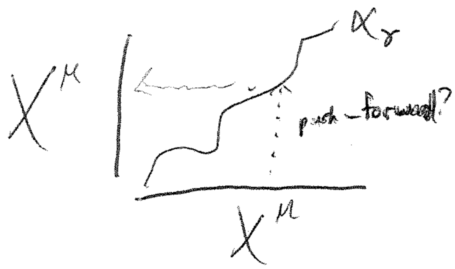
PROOF THAT $A \rtimes_\alpha \Gamma$ FACTOR FOR $\Gamma \curvearrowright A$ ERGODIC + FREE

Step 1 $A \subseteq A \rtimes \Gamma$ is a MASA $\quad (A' \cap A \rtimes_\alpha \Gamma = A)$

- Bimodule ${}_A L^2(A \rtimes A)_A$ has a simple rep

$$ {}_A L^2(A \rtimes_\alpha A)_A \simeq \bigoplus_{\gamma \in \Gamma} \mathcal{J}_{\alpha_\gamma} $$

$$ \mathcal{J}_{\alpha_\gamma} = L^2(X \times X, \underbrace{\mu \text{ along graph of } \alpha_\gamma}_{\eta_\alpha}) $$



Given $\xi(x,y) \in \mathcal{J}_{\alpha_\gamma}$

$$ (f \xi g)(x,y) = f(x)\, \xi(x,y)\, g(y) $$

makes this a bimodule over $A = L^\infty(X)$.

- Ex $\quad \alpha = id \quad$ Support is $\{x=y\}$ so multiplying by $f(x)$ and $f(y)$ are same

$$ f \cdot \xi = \xi \cdot f \qquad \xi = 1 $$

not the same A from before!

$$ a \longmapsto A \qquad A(x,x) = a(x) $$
$$ A(x,y) = 0 \text{ if } x \neq y $$

$a \in L^2(X,\mu) \qquad A \in L^2(X \times X, \eta_{id})$

so $\quad {}_A \mathcal{J}^{id}{}_A \simeq {}_A L^2(A)_A$.

$$ \mathcal{J}_\alpha = {}_{\alpha(A)} L^2(A)_A $$

denotes the bimodule which as a Hilbert space is $L^2(A)$

$$ a \cdot \xi \cdot b = \alpha(a)\, \xi\, b $$

- $\xi_0 = 1 \qquad a \cdot \xi_0(x,y) = a_0(x)\, \xi_0(x,y) = \xi_0(x,y)\, a_0(\alpha^{-1}(y))$ $\quad (y = \alpha(x))$
$$ = \xi_0 \cdot \alpha(a_0) $$

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$$\sum \alpha_\gamma u_\gamma \xleftarrow{\text{isometric}} \sum a_\gamma \cdot \xi_0^{(\gamma)}$$ $\swarrow \xi_0 \in \mathcal{J}_{\alpha_\gamma}$

since $\langle a\xi_0^{(\gamma)}, b\xi_0^{(\gamma)}\rangle = \int a(x)\overline{b(x)}dx$

and $\langle au_\gamma, bu_\gamma\rangle = \tau(au_\gamma u_\gamma^* b^*) = \tau(ab^*)$.

Now if $z \in A' \cap A \rtimes_\alpha \Gamma \Rightarrow \hat{z} = z \cdot 1$ is central for A.

Fact $\{\xi \in \mathcal{g}_\alpha : a\xi = \xi a \;\; \forall a \in A\} = \chi_{\{x : \alpha(x) = x\}} \cdot \mathcal{g}_\alpha$

$(a\xi)(x,y) = (\xi \cdot a)(x,y)$

$\xi(x,y)a(y) = \xi(x,y)a(\alpha^{-1}(y)) \quad \forall a \qquad x = \alpha^{-1}(y)$

$\therefore \; \xi(x,y)$ as fn of y is supported on $\{x : \alpha(x) = x\}$.

If the action is free, $\{x : \alpha(x) = x\}$ is empty for nontrivial α.

So $\{\mathcal{J}_{\alpha_\gamma} : \gamma \neq e\}$ contains no A-central vectors.

$\Rightarrow \{A\text{-central vectors in } {}_A L^2(A \rtimes \Gamma)_A\} = L^2(A) = \mathcal{J}_e$.

This proves $A \subseteq (A' \cap A \rtimes \Gamma) \subset L^2(A)$.

Could there be $T \notin A$, $T \in A' \cap A \rtimes \Gamma$ with $T \in L^2(A)$?

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Useful Fact $\exists$ conditional expectation i.e. a unital map

$E : A \rtimes \Gamma \longrightarrow A$ s.t. $E(x) \geq 0$ if $x \geq 0$ and
$E(axb) = a\, E(x)\, b$ s.t.
$\tau(E(x)) = \tau(x)$.

Then $\tau(ax) = \tau(a\,E(x))$ because both equal $\tau(E(ax))$.

$\langle T - E(T), a \rangle = 0$
but $T - E(T) \neq 0$ since $T \notin A$.

$\widehat{E(T)}$ is orth-proj. of $\hat{T}$ onto $L^2(A)$.

so $\widehat{E(T)} = \hat{T} \Rightarrow E(T) = T$ since trace faithful and cyclic vector separating

which is a contradiction. $\square$

We can't quite prove this Useful Fact generally yet. Next lecture.

We've shown Γ acts freely $\Rightarrow A \subseteq A \rtimes \Gamma$ is a MASA.
Γ ergodic $\Rightarrow L(\Gamma)' \cap A = \mathbb{C}1$

Both $\Rightarrow$ $(A \rtimes \Gamma)' \cap (A \rtimes \Gamma) = \mathbb{C}1$.

This gives us plenty of examples of factors.
Ex $\frac{\theta}{2\pi} \notin \mathbb{Q} \Rightarrow$ factor.

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$\mathcal{N}(A) \subseteq A \rtimes \Gamma$

$\quad = \{u \in A \rtimes \Gamma$ unitary, $uAu^* = A\}$

Def If (X, μ) is a measured space,

$R \subseteq X \times X$ is called a <u>Borel equivalence relation</u>

if R is an equivalence relation and a Borel subset.

R is called <u>discrete</u> if orbits are countable.

R receives a (possibly infinite) measure μ^R as follows:

For $\mathcal{O} \subseteq R$ Borel,

$$\mu^R(\mathcal{O}) = \int_X \#\{y \text{ s.t. } (x,y) \in \mathcal{O}\}\, d\mu(x)$$

Ex $\mathcal{O}_\alpha$ = graph of an automorphism

$\mu^R(\mathcal{O}_\alpha) = 1$ since integrand everywhere 1.

R <u>measurable equivalence relation</u> if it's taken up to μ^R-null sets.

When $\Gamma \curvearrowright X$, ($\Gamma$ discrete countable) Γ induces an equivalence relation of being in the same orbit: $x \sim y \iff x = \alpha_\gamma(y)$ for some $\gamma \in \Gamma$.

$$R = \coprod_{\gamma \in \Gamma} \mathcal{O}_{\alpha_\gamma}.$$