4-14-08
Op Alg
1

End of last time

Γ gen. by $g_1, \dots, g_n$

$$L = \frac{1}{2n} \sum_{i=1}^{n} g_i + g_i^{-1} \qquad \|L\| \leq 1$$

$$\Gamma \text{ amenable} \Leftrightarrow 1 \in \sigma(L) \Leftrightarrow \rho(L) = 1 \Leftrightarrow \|L\| = 1$$

$\langle L^k \delta_e, \delta_e \rangle$ = prob. of returning to e after k steps in a random walk

So why is $\rho = \lim_{k\to\infty} \left| \langle L^{2k} \delta_e, \delta_e \rangle \right|^{1/2k}$?

$L = L^*$ $\quad L \in L(\Gamma)$ $\qquad \langle L^{2k} \delta_e, \delta_e \rangle = \tau(L^{2k})$

$W^*(L) \subseteq L(\Gamma)$ abelian, $\quad W^*(L) \sim L^\infty(X) \qquad X = \sigma(L) \subseteq \mathbb{R}$

$L \rightsquigarrow M_x$

$\tau \big|_{W^*(L)} \in L^\infty(X)_* = L^1(X) \Rightarrow \tau$ is a measure on $X = \sigma(L)$

$\|L\| = \|x\|_\infty \qquad x \in L^\infty(\sigma(L), \mu)$

$$|L^{2k}|^{1/2k} = \left(\int x^{2k} \, d\mu \right)^{1/2k} = \|x\|_{L^{2k}(\mu)}$$

Since $\lim_{p\to\infty} \|x\|_p = \|x\|_\infty$, the result follows. $\square$

Ex: $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$ amenable (semicircle law $\Rightarrow$ spectral radius 1)

$\mathbb{Z}/3\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$ not amenable (spectral radius less than 1)

4-14-08 Op Alg 2

• Affiliated operators / Murray-von Neumann dimension
• More examples of VNA's from ergodic theory

Ex $M_{k \times k} \overset{?}{=} L(\Gamma)$ for some Γ (not obvious)

Recall $\lambda = \bigoplus_{\pi \text{ irred. rep'n}} \pi^{\oplus \dim \pi}$ for finite Γ

(Note: $\pi : \Gamma \to U(H)$ irred $\Leftrightarrow \pi(\Gamma)' = \mathbb{C}1 \Leftrightarrow \pi(\Gamma)'' = B(H)$)

$$L(\Gamma) = \bigoplus_{\pi \text{ irred rep on } V_\pi} B(V_\pi) \otimes 1_{\dim \pi}$$

$L(\Gamma)$ is a matrix alg $\Leftrightarrow \Gamma$ has only 1 irr. rep $\Leftrightarrow \Gamma$ trivial
since always have the trivial rep.

Another way to get VNA's: $\Gamma \rightsquigarrow$ groupoid + crossed product constructions

A — VNA

Γ — discrete group acting on A; $\alpha : \Gamma \to Aut(A)$

Mimic

H — group

$\alpha : \Gamma \to Aut(H)$

$H \rtimes_\alpha \Gamma = (h, \gamma)$

$(h, \gamma) \cdot (h', \gamma') =$ $(h\,\alpha_\gamma(h'),\ \gamma\gamma')$

$$H \rtimes_\alpha \Gamma = H * \Gamma / \gamma h \gamma^{-1} = \alpha_\gamma(h)$$

4-14-08
Op Alg
3

Assume A has a faithful trace τ and that $\tau \circ \alpha = \tau$.
(If not, still works, but more complicated.)

Then α preserves $\|\cdot\|_2$ and so extends to $U_\alpha : L^2(A, \tau)$ $\quad U_\alpha$ unitary.

$\mathcal{H} = L^2(A) \otimes \ell^2(\Gamma)$ $\qquad \pi : \Gamma \to B(\mathcal{H})$

$\lambda : A \to B(\mathcal{H}) \quad \lambda(a) = a \otimes 1 \qquad \pi(\gamma) = U_\gamma \otimes \lambda_\gamma$

$$\pi(\gamma)\lambda(a)\pi(\gamma^{-1})(\xi \otimes \delta_g) = U_\gamma a U_{\gamma^{-1}} \xi \otimes \delta_{\gamma\gamma^{-1}g}$$
$$= U_\gamma a U_{\gamma^{-1}} \xi \otimes \delta_g$$

$$\pi(\gamma)\lambda(a)\pi(\gamma^{-1}) = U_\gamma \lambda(a)$$

$$(U_\gamma \lambda(a) U_{\gamma^{-1}}) 1 = \alpha_\gamma(a\,\alpha_{\gamma^{-1}}(1)) = \alpha_\gamma(a)\, 1$$

So $A \rtimes_\alpha^{\text{alg.}} \Gamma \hookrightarrow A \rtimes_\alpha \Gamma \overset{\text{def}}{=} W^*(\lambda(A), \pi(\Gamma)) \subseteq B(\mathcal{H})$.

algebraic, i.e. no closures $\qquad$ VNA product

$$L(\Delta \rtimes_\alpha \Gamma) = L(\Delta) \rtimes_\alpha \Gamma$$

<u>Examples</u>
- α = trivial $\qquad A \rtimes \Gamma = A \bar{\otimes} L(\Gamma)$
- $A = \ell^\infty\{1, \dots, n\}$ $\quad \Gamma = \mathbb{Z}/n\mathbb{Z}$ by cyclic permutations

$A \rtimes \Gamma = M_{n \times n}$ as follows:

$\ell^\infty\{1, \dots, n\} \hookrightarrow M_{n \times n}$ as diagonal matrices

$\mathbb{Z}/n\mathbb{Z} \ni 1 \longrightarrow \begin{bmatrix} 0 & & & 0 \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ 0 & & 1 & 0 \end{bmatrix}$

Note: $\Gamma \subseteq N(A)$ in $A \rtimes \Gamma$

--14-08 Op Alg 4

Let $\xi \in \mathcal{H} = L^2(A) \otimes \ell^2(\Gamma)$ given by $\xi = 1 \otimes \delta_e$

Then $\langle \cdot \, \xi, \xi \rangle$ is a trace on $A \rtimes \Gamma$.

$A = \mathbb{C}$, α trivial $\quad A \rtimes \Gamma = L(\Gamma)$

<u>Key example</u> Actions of Γ on measured space

(X, μ) = measure space, μ prob. measure

Γ: discrete gp acting on X, preserving μ

$\boxed{\text{Examples with } \Gamma = \mathbb{Z}}$

(a) X = unit circle $\quad \alpha_n(e^{2\pi i \theta}) = e^{2\pi i(\theta + n\theta_0)}$, θ_0 fixed (usu irrational)

(b) $X = \{0,1\}^{\mathbb{Z}}$ $\quad \Gamma = \mathbb{Z}$ acts by shifts "Bernoulli action"

(c) $X = \{0,1\}^{\mathbb{N}}$ $\quad \Gamma$ acts by an "odometer transformation"

(add and carry as appropriate)

$A = L^\infty(X, \mu)$ $\quad A \rtimes_\alpha \Gamma$

(a) $A \rtimes_\alpha \Gamma$ $\quad \alpha$ = rot. by θ_0

If θ_0 rational, action is periodic of period N

ℓ^∞(space of orbits) $\longrightarrow \ell^\infty(X/\Gamma) \otimes L(\Gamma/N\Gamma)$ $\longleftarrow$ stabilizer of a pt

If θ irrational, one gets a factor.

$A \rtimes_\alpha \Gamma$ is a "replacement" for L^∞(orbits) $\times$ stabilizer.

Examples (a) – (c) all give rise to same VNA ...

4-14-08
Op Alg
5

Another way to look at this

$$A \rtimes_{\alpha}^{alg} \Gamma = (A * \mathbb{C}\Gamma) \big/ \text{relations: } u_\gamma a u_{\gamma^{-1}} = \alpha_\gamma(a)$$

span $\{ a u_\gamma : a \in A, u_\gamma \in \mathbb{C}\Gamma \}$
with these relations.

Trace:

$$\tau \left(\sum_\gamma a_\gamma u_\gamma \right) = a_e .$$