4-11-08
Op Alg
1

Last lecture: amenability and inner amenability are confused throughout.

Recall Property Γ: $\exists \xi_j$ s.t. $\forall x$ $\|[x, \xi_j]\|_2 \to 0$, $\|\xi_j\| = 1$

Once you know this for a dense subset y, can get for all x by

$$\|[x, \xi_j]\|_2 \le \|[y, \xi_j]\| + 2C\|x-y\|_2 \quad \text{where} \quad \sup_j \|\xi_j\| < C.$$

If Γ is an ICC group, TFAE:

(a) Γ inner amenable

(b) $\exists \xi_j \in \ell^2(\Gamma)$ s.t. $\pi = \lambda\rho^{-1}$ satisfies

$$\|\pi(\gamma)\xi_j - \xi_j\|_2 \to 0 \quad \forall \gamma \in \Gamma$$
$$\xi_j \perp \delta_e, \quad \|\xi_j\|_2 = 1$$

(c) $\|\lambda(\gamma)\xi_j - \rho(\gamma)\xi_j\|_2 \to 0 \quad \forall \gamma \in \Gamma$

If $L(\Gamma)$ has property (Γ), then (c) holds with $\lambda(\gamma), \rho(\gamma)$ replaced by the left and right action of any $x \in L(\Gamma)$, so (a) holds too. The converse is open.

Amenability of Groups

- We defined amenability in terms of Følner sets. Let's call this condition F.
- Condition A: $\exists \mu : \ell^\infty(\Gamma) \to \mathbb{C}$, $\mu \ge 0$, $\mu(1) = 1$,

$$\mu[\varphi(g^{-1}\cdot)] = \mu(\varphi(\cdot)) \quad \forall \varphi \in \ell^\infty(\Gamma).$$

Last time: $F \Rightarrow A$.

Condition WF: $\forall \varepsilon > 0$ $\forall g_1, \dots, g_n \in G$ $\exists \varphi \in \ell^1(G), \varphi \geq 0,$

$$\sum_{i=1}^{n} \|\varphi - \lambda_{g_i}(\varphi)\|_1 \leq \varepsilon \|\varphi\|_1 .$$ "Weak Følner"

Can easily show $F \Rightarrow WF$: let $\varphi = \frac{1}{|F_m|} \chi_{F_m}$

Use $\|\chi_X - \chi_Y\|_1 = |X - Y|$ for sets X, Y.

<u>Claim $A \Rightarrow WF$.</u>

$\ell^1(\Gamma) \hookrightarrow \ell^\infty(\Gamma)^* = \ell^1(\Gamma)^{**}$ in the usual way

Let $Q = \{ (\varphi - \lambda_{g_1}(\varphi), \dots, \varphi - \lambda_{g_n}(\varphi)) : \varphi \in \ell^1(\Gamma), \varphi \geq 0, \varphi(1) = 1 \}$.

This is a convex set. Its weak closure contains 0, because $\mu - \lambda_{g_i}(\mu) = 0$ and μ is a weak limit of $\{\varphi \in \ell^1(\Gamma), \varphi \geq 0, \varphi(1) = 1\}$.

Q convex $\Rightarrow 0 \in \overline{Q}^{\|\|} \Rightarrow \forall \varepsilon > 0$ $\exists \varphi \in \ell^1(\Gamma)$, $\sum \|\varphi - \lambda_{g_i}(\varphi)\|_1 \leq \varepsilon$, $\varphi \geq 0$, $\varphi(1) = 1 \Rightarrow \|\varphi\|_1 = 1$.

Here we use the "bipolar theorem": Let V be a Banach space, $V_* \subseteq V^*$ its pre-dual and dual. Let $Q \subseteq V$ a set.

$Q^\# = \mathrm{Polar}(Q) = \{\varphi \in V^*, |\varphi(x)| \leq 1 \ \forall x \in Q\}$

$Q_\# = \text{Pre-Polar}(Q) = \{\varphi \in V_*, |\varphi(x)| \leq 1, \forall x \in Q\}$

$$(Q^\#)_\# = \overline{co}^w Q$$ co = convex hull

$$(Q_\#)^\# = \overline{co}^{w*} Q$$

This implies that $\overline{Q}^w = \overline{Q}^{\|\|}$ for Q convex.

Note: $Q^\# = Q^\perp$ for Q a subspace

4-11-08
Op Alg
3

So far $F \Rightarrow A = WF$. WTS $WF \Rightarrow F$.

Have $\forall \varepsilon$ $\exists f \geq 0$ $\sum \| f - \lambda_{g_i} f \|_1 < \varepsilon \| f \|_1$

Denote by E_R the char. fn of $[R, +\infty)$

$\forall R \geq 0$ $t, t' \geq 0$, $t = \int_0^\infty E_R(t)\, dR$

$$|t - t'| = \int_0^\infty | E_R(t) - E_R(t') |\, dR$$

Now set $f_R = E_R \circ f$ i.e. $f_R(x) = \begin{cases} 1 & f(x) \geq R \\ 0 & \text{else} \end{cases}$

$$\sum_{h \in G} \sum_{i=1}^{n} | f_R(h) - \lambda_{g_i}(f_R)(h) |$$

$$= \underbrace{\sum_{h \in G} \sum_{i=1}^{n} \int | f_R(h) - \lambda_{g_i}(f_R)(h) |\, dR}_{\sum_{g_i} \| f(h) - \lambda_{g_i} f(h) \|_1} \leq \varepsilon \sum_{h \in G} | f(h) |$$

$$= \varepsilon \int \sum_h | f_R(h) |\, dR$$

"It holds on average, $\therefore$ at some R":

$\exists R$ s.t. $\sum_{h \in G} \sum_i | f_R(h) - \lambda_{g_i}(f_R)(h) | \leq \varepsilon \sum_{h \in G} | f_R(h) |$.

$\Rightarrow \| f_R - \lambda_{g_i} f_R \|_1 \leq \varepsilon \| f_R \|_1$. f_R "almost invariant"

Take $F_n = \{ x : f \geq R_\varepsilon \}$. $\square$

4-11-08
Op Alg
4

A similar proof shows that

Γ amenable $\iff \forall \varepsilon \quad \forall g_1, \dots, g_n \quad \exists f \in \ell^2(\Gamma), \ \|f\|_2 = 1,$

(*) $\quad \|f - \lambda_{g_i}(f)\|_2 \leq \varepsilon.$

"The left regular rep'n contains almost invariant vectors."

Prop $\quad \Gamma = \langle g_1, \dots, g_n \rangle$

Γ amenable $\iff 0 \in \sigma\Big(\underbrace{\sum_{i=1}^{n} |\lambda_{g_i} - 1|^2}_{T}\Big)$

Pf $\quad 0 \in \sigma(T) \iff T$ has approx kernel. Use (*).

$$|\lambda_g - 1|^2 = (\lambda_g - 1)^*(\lambda_g - 1) = (\lambda_{g^{-1}} - 1)(\lambda_g - 1)$$

$$= 2 - \lambda_{g^{-1}} - \lambda_g$$

$$T = 2n - \sum_i (\lambda_{g_i} + \lambda_{g_i^{-1}})$$

$$\frac{T}{2n} = 1 - \underbrace{\frac{1}{2n}\sum_i \lambda_{g_i} + \lambda_{g_i^{-1}}}_{L}$$ has a nice interpretation

$$L \cdot \delta_h = \frac{1}{2n} \sum_i \delta_{g_i^{\pm 1} h}$$ Related to random walk

If μ is a prob. measure on Γ, $L \cdot \mu$ is a random walk with a step uniformly chosen from $g_1^{\pm 1}, \dots, g_n^{\pm 1}$

Γ amenable $\iff 0 \in \sigma(T) \iff 1 \in \sigma(L) \Leftarrow$ spec. radius of $L = 1$

$\iff \lim_{n \to \infty} \langle \underbrace{L^n \delta_e, \delta_e}_{\text{prob. of returning to } e \text{ after } n \text{ steps}} \rangle^{1/n} = 1$

Exercise 4
Verify this for $\mathbb{Z}$.

Thus, $\boxed{\Gamma \text{ amenable} \iff \text{"return probability approaches 1"}}$