Tentative Learning Seminar:
"Classical and Quantum Aspects of Entropy" (Shlyakhhtenko + Effros)

Tue 1-2 ? MS 6221

\[ L^2(\mathcal{F}_2) \]

Aim: \[ \| x - \tau(x) \|_2 \leq C \max \left( \| [x,a] \|_2, \| [x,b] \|_2 \right) \]

\[ H = \mathcal{L}^2(\mathcal{F}_2) \]

\[ x^k \mathcal{H}_y = \text{span} \left\{ S_h : h \text{ starts } \sqrt{x^+}, \text{ ends } \sqrt{y^+} \right\} \quad x, y \in \{a, b\} \]

Vectors \perp \mathcal{L}^2(W^w(\mathcal{A})) \text{ and almost commuting \( \lambda \alpha \) must look like: } \sum a^k (\ldots) a^{N-k} \quad N \text{ large} \]

(\text{Last time.})

\[ \Theta \mathcal{H}, \text{ if } x \in \mathcal{H}_a \oplus \mathcal{H}_b \oplus \mathcal{H}_a \oplus \mathcal{H}_b \text{ then } \| \lambda(a)x - p(a)x \|_2 \geq C \| x \|_2 \]

so \[ \lambda(a) - p(a) \text{ is bdd from below on this subspace} \]

Similarly, if \[ x \in \mathcal{H}_a \oplus \mathcal{H}_b \oplus \mathcal{H}_a \oplus \mathcal{H}_b \text{ then } \lambda(b) - p(b) \text{ is bdd below} \]

Conclusion: \[ |\lambda(a) - p(a)|^2 + |\lambda(b) - p(b)|^2 \text{ is bounded below on } \mathcal{H}_a \oplus \mathcal{H}_b \oplus \mathcal{H}_a \oplus \mathcal{H}_b = \mathcal{L}^2(\mathcal{F}_2) / C \mathcal{S}_e \]

Hence if \[ \tau(x) = 0 \quad (\Rightarrow x \perp \mathcal{S}_e) \text{ then} \]

\[ \| x \|_2 \leq C \sqrt{\left( \| (\lambda(a) - p(a))^2 + (\lambda(b) - p(b))^2)(x) \|_2 \right.} \]

\[ \left. \| \Gamma^* \tau x \| \text{ where } \Gamma(x) = (\lambda(a) - p(a), \lambda(b) - p(b))_x \right. \]

\[ \therefore \Gamma \text{ bdd below on } \mathcal{L}^2(\mathcal{F}_2) / C \mathcal{S}_e. \]
\[ \|x\|_2 \leq C' \|Tx\|_2 = C' \left\| \left( (x_1 - p(a))x_1, (x_2 - p(b))x_2 \right) \right\|_2 \]
\[ = C' \left( \|a_n x\|_2^2 + \|b_n x\|_2^2 \right)^{1/2} \]
\[ \leq C'' \max \left( \|a_n x\|_2, \|b_n x\|_2 \right). \]

Cor: \( L(F_2) \) does not have property \( \Gamma \).

Cor: \( L(F_2) \not\cong L(S_\infty) \) (Murray-von Neumann).

Historically, this was a surprise. All measure spaces look like atoms + [0,1]
but there are fundamentally different VNA's. Not analogous to the commutative case here!

So when does \( L(\Gamma) \) have property \( \Gamma \)? Open question, not the same \( \Gamma \)!

Def \( \Gamma \) is called **amenable** if \( \exists \) sequence of subsets \( F_n \subset \Gamma \), \( |F_n| < \infty \), s.t. \( \forall g \in \Gamma \),
\[
\frac{|F_n - gF_n|}{|F_n|} \to 0 \text{ as } n \to \infty. \]

"weak averaging property"

Example \( \exists \mu: \ell^\infty(\Gamma) \to C \) s.t.
\[
\mu(e) = 0 \text{ if } e \neq 0 \]
\[
\mu(1) = 1 \]
and \( \mu(ef) = \mu(e) \forall g \in \Gamma. \)

This is a typical application.

Ex: So \( S_\infty \) amenable
Take \( F_n = S_n \)
\( \Gamma \) is called **inner amenable** if \( L^2(\Gamma) \) and the rep'n \( \pi: \Gamma \to U(L^2(\Gamma)) \) given by

\[
(\pi(g)v)(h) = v(g^{-1}hg) \quad \pi(g) = \mathcal{A}(g)\mathcal{D}(g)^{-1}
\]

have approximately invariant vectors \( z_n \) s.t.

\[
\|z_n\| = 1, \quad \|\pi(g)z_n - z_n\| \to 0 \quad \text{as} \quad n \to \infty.
\]

If \( \Gamma \) is ICC then this just becomes \( z_n \perp z_n \).

\[ \text{Amenable} \implies \text{inner amenable} \]

\[
\pi(g) \cdot \frac{1}{|F_n|^{1/2}} \chi_{F_n} = (\mathcal{X}_{gF_n} - \mathcal{X}_{F_n}) |F_n|^{-1/2}
\]

\[
\|\mathcal{X}_{gF_n} - \mathcal{X}_{F_n}\|_2 \leq \left( |F_n - gF_n| \right)^{1/2}, \text{ etc.}
\]

Converse turns out to be false.

\( F_2 \times S_{\infty} \) is inner amenable because you can take \( F_n \) for \( S_{\infty} \) and they'll be \( F_2 \)-invariant.

\[
\exists \quad z_n = \frac{1}{|F_n|^{1/2}} \chi_{\mathfrak{g}e^2 \times F_n}
\]

But \( F_2 \times S_{\infty} \) isn't amenable.