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Tentative Learning Seminar: "Classical and Quantum Aspects of Entropy" (Shlyakhtenko + Effros)
Tue 1-2 ? MS 6221

$L(\mathbb{F}_2)$, $\mathbb{F}_2 = \langle a,b \rangle$ Aim: $\|x - \tau(x)\|_2 \le C \max\left(\|[x,a]\|_2 , \|[x,b]\|_2 \right)$

$\mathcal{H} = \ell^2(\mathbb{F}_2)$

${}_x\mathcal{H}_y = \operatorname{span}\{ \delta_h : h \text{ starts w/ } x^{\pm 1}, \text{ ends w/ } y^{\pm 1} \}$ $\quad x,y \in \{a,b\}$

Vectors $\perp$ to $L^2(W^*(A))$ and almost commuting w/ a must look like

$$\sum a^k (\cdots) a^{N-k} \qquad N \text{ large}$$

(Last time.)

OTOH, if $x \in {}_b\mathcal{H}_b \oplus {}_a\mathcal{H}_b \oplus {}_b\mathcal{H}_a$ then $\|\lambda(a)x - \rho(a)x\|_2 \ge C\|x\|_2$

so $\lambda(a) - \rho(a)$ is bdd from below on this subspace

Similarly, if $x \in {}_a\mathcal{H}_a \oplus {}_a\mathcal{H}_b \oplus {}_b\mathcal{H}_a$ then $\lambda(b) - \rho(b)$ is bdd below.

Conclusion: $|\lambda(a) - \rho(a)|^2 + |\lambda(b) - \rho(b)|^2$ is bounded below on

$${}_a\mathcal{H}_a \oplus {}_a\mathcal{H}_b \oplus {}_b\mathcal{H}_a \oplus {}_b\mathcal{H}_b = \ell^2(\mathbb{F}_2) / \mathbb{C}\delta_e$$

Hence if $\tau(x) = 0$ $\quad (\Rightarrow x \perp \delta_e)$ then

$$\|x\|_2 \le C \underbrace{\left\| \left(|\lambda(a) - \rho(a)|^2 + |\lambda(b) - \rho(b)|^2 \right)(x) \right\|_2}_{\|T^*Tx\| \text{ where } T(x) = \left(\lambda(a) - \rho(a), \lambda(b) - \rho(b)\right)x},$$

$\therefore$ T bdd below on $\ell^2(\mathbb{F}_2)/\mathbb{C}\delta_e$.

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So

$$\|x\|_2 \le C'\|Tx\|_2 = C'\left\|\left((\lambda(a)-\rho(a))x,\ (\lambda(b)-\rho(b))x\right)\right\|_2$$

$$= C'\left(\|[a,x]\|_2^{\,2} + \|[b,x]\|_2^{\,2}\right)^{1/2}$$

$$\le C'' \max\left(\|[a,x]\|_2,\ \|[b,x]\|_2\right). \qquad \square$$

__Cor__ $L(\mathbb{F}_2)$ does not have property Γ.

__Cor__ $L(\mathbb{F}_2) \not\cong L(S_\infty)$ (Murray–von Neumann)

Historically, this was a surprise. All measure spaces look like
atoms + $[0,1]$
but there are fundamentally different VNA's. Not analogous
to the commutative case here!

So when does $L(\Gamma)$ have property Γ? Open question.

not the same Γ!

__Def__ Γ is called __amenable__ if $\exists$ sequence of subsets $F_n \subseteq \Gamma$, $|F_n| < \infty$,
s.t. $\cancel{\bigcup_n F_n = \Gamma}$ and $\forall g \in \Gamma$,

$$\frac{|F_n - gF_n|}{|F_n|} \longrightarrow 0 \text{ as } n \to \infty. \qquad (*)$$

"weak averaging property"

"Følner condition"

F_n called Følner sets

__Example__ $\exists\ \mu : \ell^\infty(\Gamma) \to \mathbb{C}$ s.t.
$\mu(\varphi) \ge 0$ if $\varphi \ge 0$
$\mu(1) = 1$
and $\mu(\varphi \cdot g) = \mu(\varphi)$ $\forall g \in \Gamma$.

means $\varphi \circ \lambda_{g^{-1}}$

This is a typical application.

__Ex__ S_∞ amenable
Take $F_n = S_n$

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Γ is called <u>inner amenable</u> if $\ell^2(\Gamma)$ and the rep'n $\pi : \Gamma \longrightarrow \mathcal{U}(\ell^2(\Gamma))$ given by

$$(\pi(g)\varphi)(h) = \varphi(g^{-1}hg) \qquad \pi(g) = \lambda(g)\rho(g)^{-1}$$

have approximately invariant vectors ξ_n s.t.

$$\xi_n \perp_{g} \ \operatorname{span}\{\chi_C : C \text{ finite conj. class}\}.$$

these ones are actually invariant, hence not an interesting example of "approximately invariant"

$$\|\xi_n\| = 1, \quad \|\pi(g)\xi_n - \xi_n\| \to 0 \quad \text{as } n \to \infty.$$

If Γ is ICC then this just becomes $\xi_n \perp \delta_e$.

$\boxed{\text{This may not work for } \Gamma \text{ abelian...}}$

<u>Amenable</u> (ICC+) $\Rightarrow$ <u>inner amenable</u>

$$\pi(g) \cdot \frac{1}{|F_n|^{1/2}} \chi_{F_n} = \left(\chi_{g \cdot F_n} - \chi_{F_n \cdot g^{-1}}\right) |F_n|^{-1/2}$$

$$\|\chi_{gF_n} - \chi_{F_n}\|_2 \leq |F_n - gF_n|^{1/2}, \text{ etc.}$$

Converse turns out to be false.

$\mathbb{F}_2 \times S_\infty$ is inner amenable because you can take F_n for S_∞ and they'll be $\mathbb{F}_2$-invariant.

$$\xi_n = \frac{1}{|F_n|^{1/2}} \chi_{\{e\} \times F_n}$$

But $\mathbb{F}_2 \times S_\infty$ isn't amenable.