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Recall $M = L(\mathbb{F}_n) = W^*(\lambda(\mathbb{F}_n))$. We'll prove factoriality again.

$H = \ell^2(\mathbb{F}_n)$ is an M-M bimodule.

Def A Hilbert space K is called a *bimodule* over a VNA N if K is endowed with a pair of commuting normal actions of N and N^{op}.

$\lambda : N \to B(K)$

$\rho : N^{op} \to B(K)$

$[\lambda(x), \rho(y)] = 0 \quad \forall x \in N,\ y \in N^{op}$

λ, ρ normal, i.e. σ-WOT-cts.

λ, ρ extensions of $\lambda, \rho : \mathbb{F}_n \to B(\ell^2(\mathbb{F}_n))$

Note If M is a tracial VNA then if $L^2(M)$ (viewed as an M-M bimodule) has no central vectors perp. to 1, then M is a factor.
(Converse is true — proved later.)

Here $\xi \in K$ is called *central* if $\lambda(x)\xi = \rho(x^{op})\xi \quad \forall x \in M$

[Usual notation is $x\xi$ for $\lambda(x)\xi$ and ξx for $\rho(x)\xi$, so this is written $x\xi = \xi x$.]

Example $\mathbb{F}_2 = \langle a, b \rangle \qquad A = W^*(a) \subseteq L(\mathbb{F}_2)$

$$ {}_A \ell^2(\mathbb{F}_2)_A = \overline{\text{span}}\{\delta_{a^k} : k \in \mathbb{Z}\} \oplus \bigoplus_{\substack{g \text{ not starting} \\ \text{or ending w/ a} \\ \text{power of } a}} \underbrace{\overline{\text{span}}\{a^k g a^l : k, l \in \mathbb{Z}\}}_{H_g} $$

$\ell^2(\mathbb{F}_2)$ as an A-A bimodule

$$ = H_e \oplus \bigoplus_{g \dots} H_g $$

Clearly these spaces are all A-A invariant and comprise all of $\ell^2(\mathbb{F}_2)$.

Now $H_e \simeq L^2(A)$ as an A-A bimodule (can identify A with $L(\mathbb{Z})$ by $a^k \leftrightarrow k$)

All vectors are central here.

For nontrivial g, ${}_A {H_g}_A \simeq {}_A L^2(A) \bar{\otimes} L^2(A)_A$

$$a^k g a^l \longmapsto a^k \otimes a^l$$

or if you prefer the notation $\delta_{a^k}\, g\, \delta_{a^l} \longmapsto \delta_{a^k} \otimes \delta_{a^l}$

(It's clear that this works algebraically; because it takes an ONB to an ONB, it extends to a Hilbert space isomorphism.)

No central vectors by following fact:

<u>Fact</u> If M is a VNA acting on H, then $H \otimes H$ has nonzero central vectors iff ~~M is finite dimensional.~~ $\exists$ M-invariant subspace H_0 s.t. $M|_{H_0}$ generates a finite-dim VNA.

<u>Pf</u> ${}_M H \otimes H_M \simeq$ Hilbert-Schmidt operators on H

$$T \mapsto aTb \leftrightarrow a \cdot T \cdot b$$

$T \in HS(H)$ central $\Rightarrow T \in M' \subseteq B(H)$

$T \neq 0$, T compact $\Rightarrow M' \cap \mathbb{K} \neq 0$

$\Rightarrow M'$ has a finite rank projection p $\quad (T \in M' \Rightarrow |T| \in M')$

$\Rightarrow \exists$ M-inv. subspace $H_0 = pH$ with $M|_{H_0}$ finite dim. $\square$

In our case $M = A = L(\mathbb{Z})$, $L^2(A) \bar{\otimes} L^2(A)$ has no A-central vectors because the bilateral shift, which generates A, has no finite-dim'l invariant subspaces (its spectrum is diffuse).

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Abelian case & heuristics

$$\ell^2(\mathbb{Z}) \simeq L^2(\mathbb{T})$$
$$L(\mathbb{Z}) \longleftrightarrow L^\infty(\mathbb{T})$$
$$\lambda(1) \longleftrightarrow M_{e^{i\theta}}$$

$$L^2(\mathbb{T}) \otimes L^2(\mathbb{T}) = L^2(\mathbb{T} \times \mathbb{T})$$

ξ central in $L^2(\mathbb{T}^2) \Rightarrow \xi = \xi(\theta, \varphi)$

$$e^{i\theta}\, \xi(\theta,\varphi) = \xi(\theta,\varphi)\, e^{i\varphi}$$

$\Leftrightarrow$ (supp $\xi \cap$ diagonal) has full measure.

Impossible because the diagonal has measure zero. $\square$

Lemma If $x \in L(\mathbb{F}_2)$, $x \in A' \cap L(\mathbb{F}_2) \Rightarrow \hat{x} \in L^2(A) \leq \ell^2(\mathbb{F}_2)$

Pf Any such x would give rise to a vector $\hat{x}$ which is A-central.
We've just showed $L^2(A)$ is the only place these can live. $\square$

If $x \in L(\mathbb{F}_2)$, $x \in B' \cap L(\mathbb{F}_2)$, $B = W^*(b)$ $\Rightarrow \hat{x} \in L^2(B)$.

$x \in L(\mathbb{F}_2)' \cap L(\mathbb{F}_2) \Rightarrow \hat{x} \in L^2(A) \cap L^2(B) = \mathbb{C}\,\delta_e$.

Hence $L(\mathbb{F}_2)$ is a factor. $\square$

BTW, a similar trick works for more general free products.

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Now we'll try to show $\exists\, C$ s.t.

$$\|x - \tau(x)\|_2 \leq C \max\left(\|[x,a]\|_2 , \|[x,b]\|_2\right)$$

Q Can there be a sequence $x_n \in M = L(\mathbb{F}_2)$ s.t.

(0) $\|x_n\|_2 = 1$

(1) $\|[x_n, y]\|_2 \to 0 \quad \forall y \in A$

(2) $P_{L^2(A)}(\hat{x}_n) = 0$?

I.e. a sequence that "almost commutes with A"?

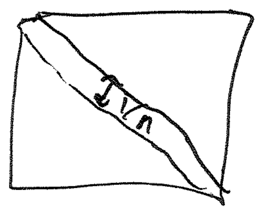
Related Q Can there be a sequence $\xi_n \in L^2(A) \otimes L^2(A)$ s.t.

(0) $\|\xi_n\| = 1$

(1) $\|a\xi_n - \xi_n a\|_2 \to 0$?

Yes. Think of the torus picture again. Can we make
$e^{i\theta}\xi_n(\theta,\varphi) \approx \xi_n(\theta,\varphi)e^{i\varphi}$?

Make ξ_n supported on a strip around the diagonal.



In $\ell^2(\mathbb{F}_n)$, given $H_b = \overline{\text{span}}\{a^k b a^l\}$,

$$\xi_N = \frac{1}{\sqrt{2N}} \sum_{t=-N}^{N-1} a^{k+t}\, b\, a^{l-t}$$

$\xi_N a$ and $a\xi_N$ differ only in a couple of terms, so the $\frac{1}{\sqrt{2N}}$ kills everything.

$\boxed{\text{Exercise 3}}$ Consider $A = W^*(a+a^{-1}+b+b^{-1}) \subseteq L(\mathbb{F}_2)$.

Show that $A' \cap L(\mathbb{F}_2)\cdot 1 \subseteq L^2(A)$.