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Group von Neumann Algebras

Let Γ be a discrete group.

$$\ell^2(\Gamma) = \mathcal{H} = \overline{\text{span}}\ \{\delta_g : g \in \Gamma\} \qquad \langle \delta_g, \delta_h \rangle = \delta_{g=h}$$

$$\lambda : \Gamma \to B(\mathcal{H}) \qquad \rho : \Gamma \to B(\mathcal{H})$$

$$\lambda(g)\cdot\delta_h = \delta_{gh} \qquad \rho(g)\cdot\delta_h = \delta_{hg}$$

Then $\big(\lambda(g)\varphi\big)(h) = \varphi(g^{-1}h)$
$\big(\rho(g)\varphi\big)(h) = \ldots$

$\forall g \in \Gamma$, $\left.\begin{matrix}\lambda(g)\\ \rho(g)\end{matrix}\right\}$ unitary

$g \mapsto \lambda(g)$ group hom $\Gamma \to \mathcal{U}(\mathcal{H})$

$g \mapsto \rho$ is an anti-homomorphism.

Def $L(\Gamma) = W^*\big(\lambda(g) : g \in \Gamma\big)$
$R(\Gamma) = W^*\big(\rho(g) : g \in \Gamma\big) \quad \subseteq B(\ell^2(\Gamma))$

Define τ on either of them by $\tau(x) = \langle x \cdot \delta_e, \delta_e \rangle$

Lemma τ is a faithful trace on $L(\Gamma)$ and on $R(\Gamma)$. I.e.
$\tau(1) = 1$, $\tau(x^*x) \geq 0$, $\tau(x^*x) = 0 \Leftrightarrow x = 0$, and $\tau(xy) = \tau(yx)$.
Also note it's WOT-cts.

Pf Because of continuity, suffices to check $\tau(xy) = \tau(yx)$ on a dense subset, say $x, y \in \mathbb{C}\lambda(\Gamma)$.

Note $\tau(\lambda(k)) = \begin{cases} 0 & k \neq e \\ 1 & k = e \end{cases}$

Using this, $gh = 1 \Leftrightarrow hg = 1$ becomes $\tau(\lambda(g)\lambda(h)) = \tau(\lambda(h)\lambda(g))$.

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Also easy to check $\tau(x^*x) \geq 0$.

Def Given $M \subseteq \mathcal{B}(\mathcal{H})$ VNA, $\xi \in \mathcal{H}$ is cyclic for M if $\overline{M\xi} = \mathcal{H}$ and separating for M if $x \cdot \xi = 0 \Rightarrow x = 0$ for $x \in M$.

Remark: ξ sep $\Leftrightarrow \varphi(x) = \langle x \cdot \xi, \xi \rangle$ faithful. (Easy.)

Thm ξ cyclic for $M \Leftrightarrow \xi$ separating for M'.

Using this, our trace τ is faithful iff δ_e is separating for $L(\Gamma) \Leftrightarrow \delta_e$ cyclic for $L(\Gamma)'$.

Now clearly $R(\Gamma) \subseteq L(\Gamma)'$.

Now δ_e is cyclic for $R(\Gamma)$ because $R(\Gamma)\delta_e$ contains all δ_g.

Being cyclic for a sub algebra $\Rightarrow$ cyclic for the whole thing

so δ_e cyclic for $L(\Gamma)' \Rightarrow \tau$ faithful. $\square$

Proof of Thm Assume ξ sep. for M'.

$\overline{M\xi} \subseteq \mathcal{H}$ is M-invariant $\Rightarrow P_{\overline{M\xi}} \in M'$.

WTS $P_{\overline{M\xi}} = 1$.

$$\langle (1 - P_{\overline{M\xi}})\xi, \xi \rangle = 0 \Rightarrow 1 - P_{\overline{M\xi}} = 0 \Rightarrow \xi \text{ cyclic for } M.$$

Now assume ξ cyclic for M'.

Let $x \in M$, $x\xi = 0$. For any $y \in M'$,

$$0 = y(x\xi) = x(y\xi) \Rightarrow x(M'\xi) = \{0\}$$
$$\Rightarrow x(\overline{M'\xi}) = \{0\}$$
$$\Rightarrow x = 0.$$

$\square$

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Theorem If Γ is ICC (Infinite Conjugacy Classes)
i.e. all nontrivial conjugacy classes infinite
then $L(\Gamma)' \cap L(\Gamma) = \mathbb{C}$.

Pf

- Because δ_e is cyclic for $L(\Gamma)$, $\ell^2(\Gamma) = \underbrace{L^2(L(\Gamma), \tau)}_{\text{GNS space}}$.
- Because δ_e is separating for $L(\Gamma)$, we have an injection

$$\wedge : L(\Gamma) \to L^2(L(\Gamma), \tau) = \ell^2(\Gamma)$$
$$x \mapsto \hat{x} = x \cdot \delta_e$$

Now as before, if $x \in L(\Gamma)' \cap L(\Gamma)$

then $\hat{x} \in \ker\{\lambda(g) - \rho(g)\}$ $\forall g \in \Gamma$. [see last lecture]

Say $\xi \in \ell^2(\Gamma)$ s.t. $\xi \in \bigcap_{g \in \Gamma} \ker[\lambda(g) - \rho(g)]$

Translates to $\xi(h) = \xi(ghg^{-1})$ $\forall g, h \in \Gamma$

i.e. ξ constant on conjugacy classes.

Conclusion If C_h is an infinite conjugacy class, then any $\xi \in \ell^2$ which is constant on C_h is 0 on C_h (else it couldn't be in ℓ^2). Thus, if Γ is ICC then $L(\Gamma)$ is a factor. $\square$

Conversely, note that if C_h is a finite conjugacy class,

$$\frac{1}{|C_h|^2} \sum \chi_{C_h}(g) \lambda(g)$$ is in the center.

(In fact, these generate the center.)

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Examples of ICC Groups

- $\mathbb{F}_n$ is ICC. Easy to check.

  Given $h = g_{i_1}^{\pm 1} \cdots g_{i_p}^{\pm 1}$ can construct g s.t. $g^n h g^{-n}$ all distinct.

- S_∞ is ICC. Just use that two permutations are conjugate iff their cycles are the same size.

So this is one way to prove factoriality.

Another approach: Recall that we showed $\exists$ matrices $x_1, \dots, x_n$ and constant C s.t. $\forall y$,

$$\|y - \tau(y)1\|_2 \le C \max_j \|[y, x_j]\|_2 \qquad (*)$$

Can we get something like this for a VNA? Not always.

Def If (M, τ) is a tracial factor, $\{y_k\}$ is an asymptotically central sequence if $\|y_k\|_2 = 1$ and $\forall x \in M$ fixed,

$$\|[y_k, x]\|_2 \to 0.$$

$\{y_k\}$ is nontrivial if $\|y_k - \tau(y_k)1\|_2 \not\to 0$

M has property Γ if it has a nontrivial asymptotically central sequence.

Note if M has property Γ it cannot have a $(*)$-like inequality.

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<u>Example</u> S_∞ has an ACS

$$S_\infty = \bigcup_n S_n \qquad L(S_\infty) = \overline{\bigcup (L(S_n))}^{\text{wot}}$$

Take $y_k = (k\ \ k+1)$.

Note $y_k \in L(S_j)' \cap L(S_\infty) \quad \forall j < k$.

Observe $\ell^2(S_\infty) = \overline{\bigcup \ell^2(S_n)}$

Then given $x \in \ell^2(S_\infty)$, have $x_j \in \ell^2(S_j)$ with $x_j \to x$.

Then $\|[y_k, x]\| \to 0$ because $\|[y_k, x_j]\| = 0$ for large k

and $\|[y_k, x]\| \le \|[y_k, x_j]\| + \|[\underbrace{y_k}_{\text{norm } 1}, \underbrace{x - x_j}_{\text{becomes small}}]\|$ □

Nontrivial because $\tau(y_k) = 0$.