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Last Time:

Can write $M = \oint_X M_x \, d\mu(x)$ $\qquad L^\infty(X,\mu) \simeq Z(M)$

M a <u>factor</u> if $M' \cap M = \mathbb{C} 1$

Example: $M_{N\times N}$ is a factor, because $D \subseteq M$ is a MASA.

<u>Another proof that M is a factor</u>

Thm Assume M has a faithful trace, i.e. $\tau : M \to \mathbb{C}$ with $\tau(xy) = \tau(yx)$, $\tau(x^*x) \geq 0$, $\tau(x^*x) = 0 \Leftrightarrow x = 0$, and $\tau(1) = 1$.
Then M is a factor iff τ is unique.

NOTE: We can assume τ normal, i.e. WOT-cts — we won't prove this.

Pf Suppose $Z(M) = L^\infty(X,\mu)$ where X has more than 1 pt.
Let $q \in Z(M)$ be a projection, not 0 or 1.

$\varphi(x) := \frac{1}{\tau(q)} \tau(qxq)$ is another trace.

other direction: omitted for now. □

So we can show M is a factor by proving the trace is unique.

Lemma
Weyl character formula: $\frac{1}{N} Tr(x) I = \int_{u \in U(N)} u x u^* \, du$ $\qquad$ for $x \in M_N$

Pf Let $w \in U(N)$

$$w \int_{u \in U(N)} u x u^* du = \int w u x u^* du = \left(\int v x v^* dv \right) w$$

$v = wu$
$dv = du$

$$\Rightarrow \int u x u^* du \in Z(M_{N\times N}) = \mathbb{C}$$

we're cheating here and already using that M is a factor!

$$Tr\left(\int u x u^* du \right) = Tr(x)$$

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Now if φ is any normalized trace on $M_{N\omega}$,

$$\frac{1}{N}\operatorname{Tr} x = \varphi\left(\frac{1}{N}\operatorname{Tr} x \; I\right) = \varphi\left(\int u x u^* \, du\right) \overset{\text{using normality of } \varphi \text{ here}}{=} \int \varphi(u x u^*)\, du = \varphi(x).$$

Fact. If A is a C^*-algebra, and $T = \{\text{trace-states on } A\}$, for any $\tau \in T$ we have a GNS construction for A, τ

$$\mathcal{H} = L^2(A,\tau) \qquad \overline{A}^{\|\cdot\|_2} \qquad \|x\|_2 = \tau(x^*x)^{1/2}$$

$$W^*(A \text{ in the GNS rep'n for } \tau) \overset{\text{def}}{=} W^*(A,\tau)$$

(This is the non-commutative analogue of Borel measures on topological spaces:

X – top space

μ – Borel measure

$A = C(X)$

$\tau \in T(A)$

$L^\infty(X,\mu)$: X as measure space

$L^2(A,\tau) = W^*(A,\tau)$)

$\boxed{\text{Exercise 1}}$ $W^*(A,\tau)$ is a factor $\Leftrightarrow$ τ is extremal in $T(A)$.

Third Proof that M_N Is a Factor (this one also cheats and assumes factoriality!)

(M,τ) $\quad \tau = \text{trace}$

$$L^2(M,\tau) = \overline{M}^{\|\cdot\|_2} \qquad \|x\|_2 = \tau(x^*x)^{1/2} = \tau(xx^*)^{1/2}$$

Claim $\|xy\|_2 \le \|x\|_2 \|y\|_\infty$
$\qquad\quad \le \|x\|_\infty \|y\|_2$ $\qquad$ (Hölder-like inequality)

Pf $\|xy\|_2^2 = \tau(xyy^*x^*) = \tau(yy^*x^*x) = \tau\left((x^*x)^{1/2}\, yy^*\, (x^*x)^{1/2}\right)$
$\qquad = \varphi(yy^*)$ where $\varphi(z) = \tau\left((x^*x)^{1/2}\, z\, (x^*x)^{1/2}\right)$

Now $\|\varphi\| = \varphi(1)$ for any cts lin. functional $\varphi: A \to \mathbb{C}$ on a unital C^*-alg A, where $\varphi(aa^*) \ge 0$.

Using this, $\varphi(yy^*) \le \|\varphi\|\, \|yy^*\| = \|x^*x\|_2^2\, \|y\|_\infty^2$. $\square$

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<u>Exercise 2</u>

$\|\varphi\| = \sup_{\|x\|=1} |\varphi(x)|$. Can choose $x \geq 0$.

$x \geq 0$, $\|x\| = 1 \Rightarrow x \leq 1$

so $\varphi(x) \leq \varphi(1)$.

Now $\mathcal{H} = L^2(M, \tau) \supseteq M$

$\forall x \in M$ define $L_x : M \to M$ $\quad L_x a = xa$

$R_x : M \to M$ $\quad R_x b = bx$

$\|L_x a\|_2 = \|x a\|_2 \leq \|x\|_\infty \|a\|_2$

$\|R_y b\|_2 = \|by\|_2 \leq \|y\|_\infty \|b\|_2$

Thus $\|L_x\| \leq \|x\|$, $\|R_y\| \leq \|y\|$ on $M \subseteq L^2(M)$, so both extend to bounded operators on $L^2(M)$.

Also have $L_x L_y = L_{xy}$ and $R_y R_x = R_{xy}$

so $x \mapsto L_x$ is a representation of M

$y \mapsto R_y$ is an "anti-representation" (a rep'n of the opposite algebra).

Now assume $z \in Z(M) = M' \cap M$

$az = za \quad \forall a \in M$

$L_a z = R_a z$

In other words, we have $[z, a] = 0 \iff z \in \left(\ker (L_a - R_a)\right) \cap M$.

So a strategy to prove factoriality is to analyze operators of the form $L_a - R_a$.

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Define $J : L^2(M) \to L^2(M)$ by $Ja = a^*$

This is an isometry since $\tau(a^*a)^{1/2} = \|Ja\|_2 = \|a\|_2 = \tau(aa^*)^{1/2}$

equal by trace properties

Now $R_{x^*} = J R_x J$.

So we can equivalently say $[z,a] = 0 \Leftrightarrow z \in \ker\left((L_a - J L_{a^*} J)\right) \cap M$.

usually L's omitted

$\ker(a - Ja^*J) \cap M$

<u>Matrix Case</u>

$M = M_{N \times N}$ $\quad \tau = \frac{1}{N} Tr(\cdot)$ $\quad L^2(M, \tau) = M$

L_a, R_a matrix multiplication $\quad J =$ ~~[illegible]~~ conjugate transpose

For $x_1, \ldots, x_n \in M_{N \times N}$ define

$$L_{x_1, \ldots, x_n} = \sum (L_{x_i} - R_{x_i})^* (L_{x_i} - R_{x_i})$$

$y \in \ker L_{x_1, \ldots, x_n} \Leftrightarrow [y, x_j] = 0 \quad \forall j$ (since each term in sum nonnegative)

<u>Observe</u>

1) $L_{x_1, \ldots, x_n} \in M_{N^2 \times N^2}$ so has atomic spectrum.

2) $\exists\, x_1, \ldots, x_n$ s.t. $\ker L_{x_1, \ldots, x_n} = \mathbb{C} I$. (only scalars commute w/ all basis elements)

3) $\exists\, x_1, \ldots, x_n$ s.t.

$$\|x - \tau(x)\|_2 \leq C \max_j \|[x, x_j]\|_2$$

from 1) and 2).

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Indeed, 1)+2) mean $L_{x_1,\ldots,x_n}$ is bddly invertible on $(\ker L_{x_1,\ldots,x_n})^\perp = (\mathbb{C}I)^\perp$

so bdd below

$$x \perp \mathbb{C}1 \Rightarrow \|L_{x_1,\ldots,x_n}(x)\| \geq K\|x\|_2$$

Also have $x \perp \mathbb{C}1 \Rightarrow \|x\|_2 \leq K' \max_j \|[x, x_j]\|$.

(3) results.

Enough on matrices. We'll look at some other VNA's next time.