We can write \( M = \sum M_x \delta y(x) \). If \( M_y M = C \), then \( M \) is a factor, because \( D_y M \) is a module.

An alternate proof that \( M \) is a factor.

Let \( \tau: M \to X \) be a faithful trace, i.e., \( \tau: M \to X \) with \( \tau(x y) = \tau(y x) \).

Then \( M \) is a factor if \( \tau \) is unique.

If unique, \( \exists(M) = L^\infty(X,\mu) \). Let \( x \) be one of \( M \).

Let \( \mathbb{E} \) be a projection, not \( 0 \) or \( 1 \).

\[ \mu(x) = \frac{1}{\mathbb{E}(y) \mathbb{E}(xy)} \] is another trace.

So we can show \( M \) is a factor by proving the trace is unique.

Lemma

Weyl character formula: \( \frac{1}{N} \text{Tr}(\chi I) = \int_{\mathfrak{g}} \chi(x^* \gamma) \chi(x) \frac{dx}{\text{vol}(\mathfrak{g})} \)

Let \( \gamma \in \text{Aut}(\mathfrak{g}) \).

\[ \int \mu(x^* \gamma u) du = \sum_{\gamma \in \text{Aut}(\mathfrak{g})} \mu(x^* \gamma u) \int \gamma(u) du = \sum_{\gamma \in \text{Aut}(\mathfrak{g})} \mu(x^* \gamma u) \]

Therefore, \( \text{Tr} \left( \sum_{\gamma \in \text{Aut}(\mathfrak{g})} \mu(x^* \gamma u) \right) = \text{Tr} \left( \mu(x) \right) \).
Let $A$ be a $C^*$-algebra, and $T = \{\text{trace states on } A\}$.

For any $\tau \in T$, we have a GNS construction for $A, \tau$.

$$
H = L^2(A, d\tau) \quad \quad A \ni a \rightarrow \tau(a) \quad \quad (d\tau) = \tau(a*d^a)
$$

$W^*(A, \tau)$ is the GNS algebra for $\tau$.

This is the non-commutative analogue of Borel measures on topological spaces:

- $\mu$: Borel measure
- $A = C(\mathcal{X})$

$$
L^\mu(\mathcal{X}) = \{\text{functions with } L^\mu(\mathcal{X}) = \mu(A, \tau)\}
$$

$W^*(A, \tau)$ is a factor if $\tau$ is extremal in $T(A)$.

Proof: $M_\tau$ is a Factor

(We also check and assume self-adjointness)

$M_\tau = \tau$ trace

$L^\tau(M_\tau, \tau) = M_\tau$ weak-

$\|a\|_\tau = \tau(a^*a) = \tau(a^*a)^{1/2} = \tau(a^*a)^{1/2}$

$\|x\|_{\tau, y} \leq 2\|x\|_{\tau} \leq 2\|y\|_{\tau, x}$

$(\|x\|_{\tau, y} - \|y\|_{\tau, x})^2 \leq 2\|x\|_{\tau} \|y\|_{\tau, x}$

$\|xy\| = \tau((xy)^*xy) = \tau(yy^*xx^*) = \tau((x^*x)yy^*x^*) = \tau((x^*x)(yy^*)x^*) = \tau((x^*x)yy^*)x^*)$ where $\tau(x) = \tau((x^*x)(yy^*)x^*)$

Now $\|x\| = \|x\|_{\tau, 0}$ for any $x \in A$ and $\tau$-self-adjoint.

Using this, $\|xy\| \leq \|x\|_{\tau, y} \|y\|_{\tau, x}$.
Exercise 2

\(\|q\| = \sup_{q \in Q} |q(x_a)|

Conclude \(x \sqcup 0 \Rightarrow x \sqcup 1\)

Now \(W = L^1(M; \mathbb{R}) \geq M \)

\(\forall a \in M \) define \( L_a : M \to M \quad L_a a = xa \)

\(R_a : M \to M \quad R_a b = ab \)

\[\|L_a a\|_2 = \|xa\|_2 = \|a\|_2 \quad \|R_a b\|_2 = \|ab\|_2 \leq \|a\|_2 \|b\|_2\]

Thus \(\|L\| \leq \|x\|\) \(\|R\| \leq \|y\|\) on \(M_{L^1(M)}\) \(a\) both extend to bounded operators on \(L^1(A)\).

Also have \( L_a L_b = L_{a \circ b} \) and \( R_a R_b = R_{a \otimes b} \)

So \( \ast \) \( L_a \) is a representation of \(M\)

\(y \mapsto R_y\) is an anti-representation (a rep\'s of the opposite algebra).

Now choose \( z \in Z(M) = M^{\ast\ast} \)

\(za = za \quad \forall a \in M \)

\(L_z = R_{za}\)

In other words, we have \( \{za\} = 0 \iff \mathbb{E} \in \ker (L_z - R_a) \in M\).

So a strategy to prove factoriality is to analyze operators of the form \(L_z - R_a\).
Define $J : L^2(A) \rightarrow L^2(A)$ by $J_a = a^*$. This is an isometry since $\|a^*\| = \|a\|$.

Now $R_\alpha = J R_\alpha J$. So $a$ and $a^*$ are equivalent in $L^2(\mathcal{M})$, so $a^*$ is in the closure of $\mathcal{M}_d$.

Matrix Form:

$$M = M_{\text{sym}} \quad \mathcal{C} = \frac{1}{N} \text{Tr}(\cdot) \quad \mathcal{L}(\mathcal{M}, \mathcal{C}) \subseteq \mathcal{M}$$

$L_a$ and $J_a$ matrix multiplication. $J_a$ is a conjugate transpose.

For $x, x_0 \in \mathcal{M}_{\text{sym}}$, define

$$L_{x, x_0}(x) = \sum \left( (L_{x_0} - L_a) \right)^* (L_{x_0} - L_a)$$

$y \in \ker L_{x, x_0} \iff \langle y, x_0 \rangle = 0 \quad \forall y$ (prove this claim).

Choose $0 \leq L_{x, x_0} \leq M_{\text{sym}}$ so that $a$ is atomic specimen.

1) $\|x - x_0\|_{\mathcal{M}} \leq \mathcal{C} \max \|x_0\|_{\mathcal{M}}$.

2) $\|x^* - x_0^*\|_{\mathcal{M}} \leq \mathcal{C} \max \|x_0\|_{\mathcal{M}}$.

3) $\|x - x_0\|_{\mathcal{M}} \leq \mathcal{C} \max \|x_0\|_{\mathcal{M}}$.
Indeed, \((f+g)\) maps \(L_{x_1, \ldots, x_n}\) to \(L_{x_1, \ldots, x_n}\) and is locally invertible on 
\[
\text{ker } L_{x_1, \ldots, x_n}^* = (g, x) = 0
\]
so that below
\[
x \leq C \Rightarrow \|L_{x_1, \ldots, x_n}^* \| x \geq K \| x \|
\]
also have \(x \leq C \Rightarrow \|x\| \leq K \|x\| \|Ex_i x_j\|.\)

(3) results

Enough for notation. We’ll look at some other VNA’s next time.