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UNFINISHED BUSINESS

Claim: $1_N \in M \subset N$, $\overline{M1}^{L^2} = \overline{N1}^{L^2} \Rightarrow M = N$.

Pf Say $x \in N' \subset M'$

$x(\overline{M1}) = x(\overline{N1}) \Rightarrow \overline{M x} = \overline{N x}$

$\overline{M}^{SOT} = N$ Use a matrix trick (as in bicommutant theorem)

$\forall x_1, \ldots, x_d \quad \forall \varepsilon \quad \forall n \quad \exists m$ s.t.

$\| m x_j - n x_j \| < \varepsilon \quad j = 1, \ldots, d.$ $\square$

LAST TIME $R_T \overset{\text{singly generated}}{\rightsquigarrow} R_S$ (odometer)

Suppose now we're given a hyperfinite $R = \bigcup R_n$.

Could follow same thing: define S_j's to move your equivalence classes around as needed.

Conclusion: hyperfinite $\Rightarrow$ singly generated (since h.f. $\downarrow$ odometer $\downarrow$ s.g.)

$\therefore$ HF and SG equivalence relations are the same.

Explicitly:

Thm Let R be measure-preserving and ergodic. TFAE:

(1) R hyperfinite

(2) $R = R_T$ for some T m.p. erg. trans.

(3) $R = R_S$, S = odometer (m.p.)

Later we'll add (4) R amenable

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Recall

- R, μ_R ∞ measure on $R \subseteq X \times X$
- $L^2(R, \mu_R)$ Hilbert space
- V_α α-partial R-compatible morphism

$$[(V_\alpha) f](x,y) = f(x, \alpha^{-1}(y))$$ operator on L^2

$$f \in L^\infty(X, \mu) \qquad (f\xi)(x,y) = f(x)\xi(x,y)$$

$$W^*(R) \overset{def}{=} W^*(L^\infty(X), V_\alpha : \alpha \text{ partial } R\text{-morph})$$

$$E : W^*(R) \to L^\infty(X) \qquad \text{restr.} \quad E : L^2(R) \to L^\infty(X)$$
$$\xi \longmapsto (x \mapsto \xi(x,x))$$

where $E(fxg) = f\,E(x)\,g \quad \forall f, g \in L^\infty(X),\ x \in W^*(R)$
$E(x^*x) \geq 0 \qquad E(x^*x) = 0 \Leftrightarrow x = 0$.

$$\begin{tikzcd} L^\infty(X) & \subset & W^*(R) \arrow[ll, bend right, "E"'] & \subset & B(L^2(R)) \end{tikzcd}$$

"noncommutative fibration"

$$\begin{tikzcd} \mathbb{C} & \subset & L(\Gamma) \arrow[ll, bend right, "\tau"'] & \subset & B(\ell^2(\Gamma)) \end{tikzcd}$$

Classical fibered spaces: $\begin{matrix} Y \\ \downarrow \pi \\ X \end{matrix}$ Measures on each fiber are "μ_x - measurable"

$$E(\varphi)(x) = \int \varphi(x,y)\, d\nu_x(y)$$

Probabilists write $E(\varphi(x,y) \mid \pi(y) = x)$ conditional expectation

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Recall

Γ amenable $\Leftrightarrow$ $\exists\ P : \ell^\infty(\Gamma) \longrightarrow \mathbb{C}$ s.t.

P positive, $P(1) = 1$, and $P(\lambda_g f) = P(f) \quad \forall g \in \Gamma$.

Def $P : L^\infty(R) \longrightarrow L^\infty(X)$ is called a (left) invariant mean if

1. $P(\xi) \geq 0$ if $\xi \geq 0$
2. ~~$P(1) = 1$~~ $P|_{\text{diagonal } x \in R}$ = identity
3. $P(\lambda_g \xi) = P(\xi) \quad \forall g$ R-compat. morphisms.

technically $P(\chi_{\text{dom } g}\ \xi)$ or something

Examples

- R hyperfinite ($=$ singly generated) $\Rightarrow$ R amenable

Why? $R = R_T \qquad (P_n \xi)(x) = \frac{1}{2N+1} \sum_{j=-N}^{N} \xi(x, T^j(x))$

$P = \lim P_n$

α — any R-morphism we arranged that $\forall \varepsilon\ \forall M\ \exists N$

s.t. $\mu\left\{x : \{x, \alpha^{\pm 1}x, \dots, \alpha^{\pm M}x\} \subset \{x, T^{\pm 1}x, \dots, T^{\pm N}x\}\right\} \geq 1 - \varepsilon$.

- You can do the same sort of thing for any group action.

$R = R_\Gamma$ amenable

$P_\Gamma : \ell^\infty(\Gamma) \rightarrow \mathbb{C}$ invariant mean

$\xi \in R \qquad \forall x \in X, \quad \{(x, \gamma x) : \gamma \in \Gamma\} \subseteq R$

$P(\xi)(x) = P(\gamma \longmapsto \xi(x, \gamma x))$

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Why is this invariant?

α morph compat. with R_Γ

$\exists$ partition $\{X_\gamma : \gamma \in \Gamma\}$ of X s.t. $\alpha|_{X_\gamma}$ = action of γ.

- Can check that $P(\xi) = P(\text{action of } \gamma \text{ on } \xi)$
- If $\operatorname{supp} \xi(x,y)$ lies above $X_0 \subseteq X$ ($\xi(x,y) = 0$ if $x \notin X_0$ implies $P(\xi)(x)$ if $x \notin X_0$)

~~then $P(\xi)(x)$~~

$$P(\alpha\xi)(x) = \underset{\text{if } x \in X_\gamma}{P(\alpha\xi_\gamma(x))} = \underset{(P \text{ is } \gamma\text{-inv}).}{P(\xi_\gamma(x))}$$

Summary: Amenable group action $\longrightarrow$ amenable equivalence relation.