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UNFINISHED BUSINESS

Claim: $N \in M \subset N', \quad \overline{M}^{L^2} = \overline{N'}^{L^2} \Rightarrow M = N.$

PF Say $x \in N' \subset M'$

$x(\overline{M}) = x(\overline{N'}) \Rightarrow \overline{M \cdot x} = \overline{N' \cdot x}$

$\overline{M}^{SOT} = N$

Use a matrix trick (as in bicommutant theorem)

$\forall x_1, \dots, x_d \quad \forall \epsilon \quad \forall n \quad \exists m \text{ s.t.}$

$\|mx_j - nx_j\| < \epsilon \quad j=1, \dots, d. \quad \square$

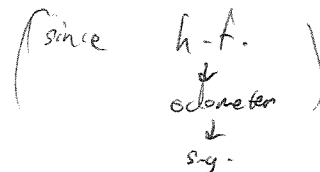
LAST TIME

$R_T \xrightarrow{\text{singly generated}} R_S \text{ (odometer)}$

Suppose now we're given a hyperfinite $R = UR_n$.

Could follow same thing: define S_j 's to move your equivalence classes around as needed.

Conclusion: hyperfinite \Rightarrow singly generated



\therefore HF and SG equivalence relations are the same.

Explicitly:

Thm Let R be measure-preserving and ergodic. TFAE:

(1) R hyperfinite

(2) $R = R_T$ for some T m.p. erg. trans.

(3) $R = R_S$, $S \Rightarrow$ odometer (m.p.)

Later we'll add (4) R amenable

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Recall

- R, μ_R ∞ measure on $R \subseteq X \times X$
- $L^2(R, \mu_R)$ Hilbert space
- V_α α -partial R -compatible morphism

$$[(V_\alpha)f](x, y) = f(x, \alpha^{-1}(y))$$

operator on L^2

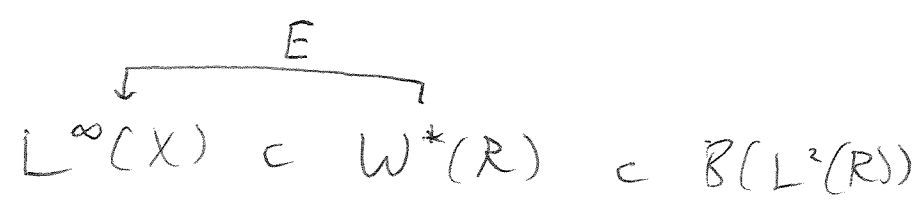
$$f \in L^\infty(X, \mu) \quad (f \circ \xi)(x, y) = f(x) \xi(x, y)$$

$$W^*(R) \stackrel{\text{def}}{=} W^*(L^\infty(X), V_\alpha : \alpha \text{ partial } R\text{-morph})$$

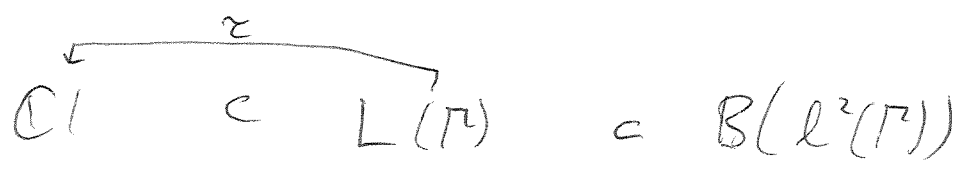
$$E : W^*(R) \rightarrow L^\infty(X) \quad \text{restr. } E : L^2(R) \rightarrow L^\infty(X)$$

$$\xi \longmapsto (x \mapsto \xi(x, x))$$

where $E(fxg) = fE(x)g \quad \forall f, g \in L^\infty(X), x \in W^*(R)$
 $E(xc^*x) \geq 0 \quad E(xc^*x) = 0 \Leftrightarrow x = 0$



"noncommutative fibration"



Classical fibered spaces: $\begin{array}{c} Y \\ \downarrow \pi \\ X \end{array}$

Measures on each fiber are " μ_x -measurable"

Probabilists write

$$E(\varphi)(x) = \int \varphi(x, y) d\mu_x(y)$$

$$E(\varphi(x, y) | \pi(y) = x)$$

conditional expectation

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Recall

Γ amenable $\Leftrightarrow \exists P: L^\infty(\Gamma) \rightarrow \mathbb{C}$ s.t.
 P positive, $P(1) = 1$, and $P(\lambda_g f) = P(f) \quad \forall g \in \Gamma$.

Def $P: L^\infty(X) \rightarrow L^\infty(X)$ is called a (left) invariant mean if

1. $P(\xi) \geq 0$ if $\xi \geq 0$
 2. ~~$P(1) = 1$~~ $P|_{\text{diagonal } x \in \mathbb{R}} = \text{identity}$
 3. $P(\lambda_g(\xi)) = P(\xi) \quad \forall g \in \Gamma$ \mathbb{R} -compat. morphisms.
- ↑
technically $P(\lambda_{\text{diag } g} \xi)$ or something

Examples singly generated

• \mathbb{R} hyperfinite $\Rightarrow \mathbb{R}$ amenable

Why? $\mathbb{R} = \varinjlim \mathbb{R}_T$ $(P_n \xi)(x) = \frac{1}{2N+1} \sum_{j=-N}^N \xi(x, T^j(x))$
 $P = \lim P_n$

α - any \mathbb{R} -morphism we arranged that $\forall \epsilon \forall M \exists N$

s.t. $\mu \left\{ x : \{x, \alpha^{\pm 1}x, \dots, \alpha^{\pm M}x\} \subset \{x, T^{\pm 1}x, \dots, T^{\pm N}x\} \right\} \geq 1 - \epsilon$

• You can do the same sort of thing for any group action.

$\mathbb{R} = \mathbb{R}_\Gamma$ amenable

$P_\Gamma: L^\infty(\Gamma) \rightarrow \mathbb{C}$ invariant mean
 $\xi \in \mathbb{R} \quad \forall x \in X, \{(\alpha_\gamma x) : \gamma \in \Gamma\} \subseteq \mathbb{R}$
 $P(\xi)(x) = P(\gamma \mapsto \xi(\alpha_\gamma x))$

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Why is this invariant?

 α morph compact. with R_Γ

\exists partition $\{X_\gamma : \gamma \in \Gamma\}$ of X s.t. $\alpha|_{X_\gamma} = \text{action of } \gamma$.

• Can check that $P(\bar{z}) = P(\text{action of } \gamma \text{ on } \bar{z})$

• If $\text{supp } \bar{z}(x, y)$ lies above $X_0 \subseteq X$

~~then~~

$\left(\begin{array}{l} \bar{z}(x, y) = 0 \text{ if } x \neq X_0 \\ \text{implies } P(\bar{z})(x) \text{ if } x \in X_0 \end{array} \right)$

$$P(\alpha \bar{z})(x) = P(\alpha \bar{z}_\gamma(x)) = P(\bar{z}_\gamma(x))$$

if $x \in X_\gamma$ (P is γ -inv.)

Summary: Amenable group action \longrightarrow amenable equivalence relation.