4-23-08
Op Alg
1

Let's generalize the odometer thing.

Say we're given $\{d_j\}_{j\geq 0}$ with $d_j \in \mathbb{N}$

$$X = \prod_j \{0, \ldots, d_j - 1\} \qquad \text{(previously } d_j = 2)$$

$$T(x_1\ x_2 \ldots) = (0\ 0 \ldots 0\ (x_k+1)\ x_{k+1}\ x_{k+2} \ldots)$$

If all $x_i = d_i - 1$ then $T(x) = 0$.

where k is smallest with $x_k \neq d_k - 1$.

Also let μ_j be some measure on $\{0, 1, \ldots, d_j - 1\}$ not necessarily uniform

$$\mu = \prod \mu_j$$

"Quasi-invariant": $T_* \mu \sim \mu$

If $\mathcal{O}_n = \{\vec{x} : x_1 = x_2 = \cdots = x_n = 0\}$

then $\mathcal{O}_n, T\mathcal{O}_n, \ldots, T^{\prod_1^n d_i - 1} \mathcal{O}_n$ disjoint

$\boxed{\text{Theorem}}$ (Dye?) R ergodic and singly generated $\Rightarrow$ R isomorphic to an odometer.
(R is the orbit-equivalence relation of the odometer action)

Given R generated by $T: Y \to Y$

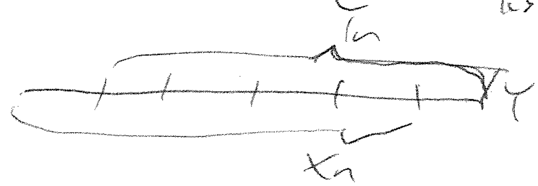
PF Suffices to construct sequence $\{Q_n, S_n\}$ s.t.

- $Q_n = \{A_{1,n} \ldots A_{d_n, n}\}$ is a partition of Y $\forall n$
- S_n map defined on a subset $X_n \subseteq Y$, $S_n : X_n \to Y_n$

s.t. $S_n(A_{k,n}) = A_{k+1,n}$ $\forall k < d_n$

$$X_n = \bigcup_{k < d_n} A_{k,n} \qquad Y_n = \bigcup_{k > 1} A_{k,n}$$

picture:



S_n: $\longrightarrow$

Also require

- Q_n nested (Q_{n+1} is a refinement of Q_n)
- S_n are measuring preserving and R-computible
- S_{n+1} is an extension of S_n, and

(✱ S = automorphism defined by the union of the S_n's
R generated by S.)

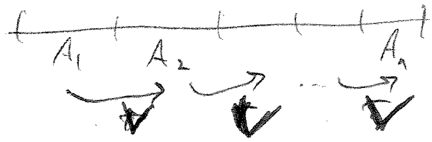
4-23-08
Op Alg
2

Can replace requirement (*) by

$\forall \varepsilon > 0 \qquad \forall N \geq 0 \qquad \exists L$ s.t.

$$\mu\left\{ x : \{T^k x\}_{k=-N}^{N} \subseteq \{S_n^k x\}_{k=-L}^{L} \right\} \geq 1-\varepsilon.$$

By Rokhlin's lemma, $\forall n \quad \exists A_1, \ldots, A_n$ s.t.



Assume V partially defined, m.p. and invertible $\qquad VA_k = A_{k+1}, \qquad k < n.$

$\forall W : A_1 \to A_1 \qquad \exists! \ \hat{V}$ extension, invertible m.p.

s.t. $\hat{V}^n\big|_{A_1} = W\big|_{A_1}$

($W = \hat{V}^n$ defines W or $\hat{V}$ as appropriate)

Rokhlin $\Rightarrow \exists F_1'$ s.t. $T^j F_1'$ disjoint for $j = 0, \ldots, n_1 - 1$

$$\mu\Big(\underbrace{X \setminus \bigcup_{j=0}^{n_1-1} T^j F_1'}_{G \text{ "garbage set"}}\Big) < \varepsilon$$

$A_{j,1} = G_j \cup T^j F_1' \qquad G = G_0 \cup \cdots \cup G_{n_1-1}$

$S_1 = \begin{cases} T & \text{on } T^j F_1' \\ S' & \text{on } G_j \end{cases} \qquad S'$ comp. with R s.t. $S' G_j = G_{j+1}$

4-23-08
Op Alg
3

We'd like to repeat the process for F_1.

<u>Fact</u> If R is singly generated by $T: X \to X$

$$F \subseteq X \qquad \mu(F) > 0$$

then $R|_F$ is also singly generated.

Define $n(y) = \inf \{ n > 0 : T^n(y) \in F \}$ "how long you have to wait to come back"

$m(y) = \sup \{ m < 0 : T^m(y) \in F \}$

$+\infty$ if you never come back

$-\infty$ if you were never there

Note that n, m are measurable and a.e. finite (by ~~ergodicity~~ meas-preserving + invariance)

Replace F by $F \cap \{ y : n(y), m(y) \text{ finite} \}$

$$S : F \longrightarrow Y \quad \text{by}$$
$$S(y) = T^{n(y)}(y)$$

$S(y)$ is the first return

<u>Claim</u> $S(y)$ generates $R_T|_F$.

Putting this into our construction allows us to iterate, etc. $\square$

<u>Note:</u> We were free to pick n_1 as we please. With a non-quasi-invariant measure you have to work harder.

~~But if it's~~

But any two singly generated ergodic <u>measure-preserving</u> equivalence relations are isomorphic. Can map both to, say, the binary odometer.