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$$A \rtimes_\alpha \Gamma = \langle A, u_g \rangle$$

$$u_g a u_g^* = \alpha_g(a) \qquad u_g \text{ unitary}$$

A abelian, $\quad \sigma_g u_g \quad \sigma_g \in U(A) \qquad u_g u_h = u_{gh}$

$$\sigma_g u_g \sigma_h u_h = \sigma_{gh} u_{gh}$$

"perturb by a cocycle"

$$\sigma_g \alpha_g(\sigma_h) = \sigma_{gh}$$

"σ_g is a 1-cycle from $\Gamma \to U(A)$ for the action α"

⋮

SINGLY GENERATED EQUIVALENCE RELATIONS, HYPERFINITE EQUIVALENCE RELATIONS, AND AMENABILITY

Def A II_1 factor M is hyperfinite if $M = \bigcup M_k$ with M_k an increasing family of unital finite-dim algebras.

Example: $S_\infty = \bigcup_n S_n$. $\quad L(S_\infty) = \bigcup L(S_n)$ hyperfinite.

Example $R^{(p)} = \bigotimes_{k=1}^{\infty} M_{p \times p}$ How does this work?

$$R_n^{(p)} = \bigotimes_{k=1}^{n} M_{p\times p} \qquad R_n^{(p)} \hookrightarrow R_{n+1}^{(p)} \qquad x \longmapsto x \otimes 1$$

$$R^{(p)} = \bigcup R_n$$

Turns out these are isomorphic for different p (nontrivial!)

Def An equivalence relation R is called hyperfinite if

$$R \underset{a.e.}{=} \bigcup R_n \quad \text{with } R_n \text{ finite (meaning finite orbits)}$$

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Ex

$X = \{0,1\}^{\mathbb{N}}$ $\quad A = L^\infty(X) = \bigcup A_k$

A_k = fns constant on intervals $\left[\frac{i}{2^k}, \frac{i+1}{2^k}\right]$

R gen. by $\sigma_1, \sigma_2, \sigma_3, \ldots$ $\quad$ R_k gen. by $\sigma_1, \ldots, \sigma_k$
orbit size $= 2^k$

$\sigma_j(a_1\ a_2\ \ldots) = (a_1\ a_2\ \ldots\ (\text{not } a_j)\ a_{j+1}\ \ldots)$

Meaning, complement of a_j (ie. $1-a_j$)

$W^*(R) \simeq R^{(?)}$

equiv. rel

hyperfinite II_1 factor

Theorem A (Connes) TFAE:
(i) R II_1 factor is hyperfinite
(ii) R II_1 factor is amenable
(iii) $R \simeq L(S_\infty) \simeq R^{(p)}$

Theorem B (Connes - Feldman - Weiss)

Let R be an equivalence relation with an invariant measure (meaning, $\forall \alpha$ s.t. graph $\alpha \subseteq R$ μ_R a.e., $\alpha_* \mu = \mu$)
TFAE:
(1) R is hyperfinite
(2) R is amenable
(3) $R \simeq$ the one we just described with $\{0,1\}^{\mathbb{N}}$ and σ_k's
(4) R is singly generated (meaning, $R = R_{\mathbb{Z}}$ for some free action of $\mathbb{Z}$)

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Def

Let $\Gamma_1 \overset{\alpha^{(1)}}{\curvearrowright} X$ and $\Gamma_2 \overset{\alpha^{(2)}}{\curvearrowright} X$ be free actions. Then $\alpha^{(1)}$ is <u>orbit equivalent</u> to $\alpha^{(2)}$ if $R_{\alpha^{(1)}} = R_{\alpha^{(2)}}$, i.e.,

$$x = \alpha^{(1)}_g(y) \quad \text{for some } g \in \Gamma_1$$
$$\Leftrightarrow \quad x = \alpha^{(2)}_h(y) \quad \text{for some } h \in \Gamma_2 \qquad x\text{-a.e.}$$

Why are any two free actions of $\mathbb{Z}$ orbit equivalent?

<u>Rokhlin's lemma</u>

Let α free ergodic action on X $\qquad \alpha = \alpha_1$ (action of $1 \in \mathbb{Z}$)

$\forall \varepsilon > 0 \quad \forall n \quad \exists F \subseteq X$ meas. subset s.t.

$F, \alpha(F), \ldots, \alpha^{n-1}(F)$ disjoint

and $\mu(F \cup \cdots \cup \alpha^{n-1}(F)) \geq 1 - \varepsilon$.

Pf Let $B \subseteq X$ with $\mu(B) < \frac{\varepsilon}{n}$. Let $x \in X$ and

$n_B(x) = \inf\{k : \alpha^k(x) \in B\}$ taken to be ∞ if set is empty

$n_B(x)$ is the inf of measurable fns, hence m'ble

$n_B(x) = 0 \Leftrightarrow x \in B$

Then $\{x : n_B(x) < \infty\}$ is α-invariant, and contains B, which has nonzero measure $\Rightarrow$ it's almost everything.

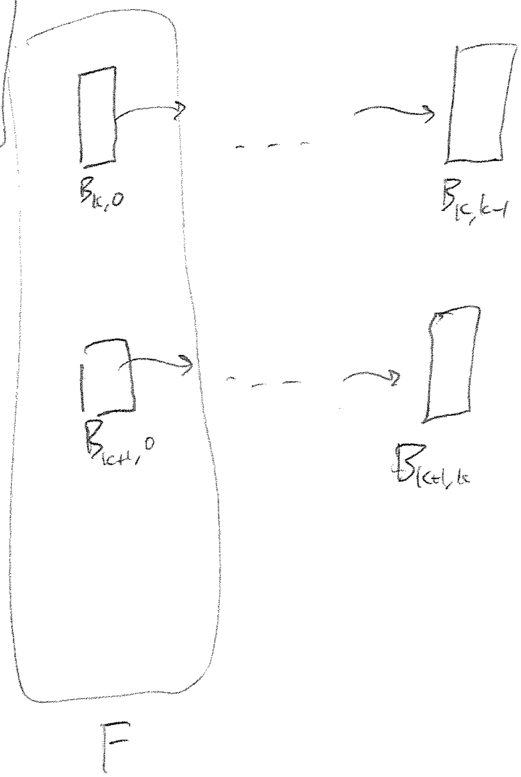
Let $B_k = \{x : n_B(x) = k\}$ Disjoint

Let $B_{k,j} = \alpha^{j}(B_k)$ $\qquad X = \bigcup_{k,j} B_{k,j}$ a.e.

Note $B_{k,k} \subseteq B$ and $\alpha(B_{k,j}) = B_{k,j+1}$

$$F = \bigcup_{k \geq n} \bigcup_{0 \leq j \leq \frac{k-n-1}{n}} B_{k,j}$$

$F, \alpha(F), \ldots, \alpha^{n-1}(F)$ disjoint $\qquad \mu(F \cup \cdots \cup \alpha^{n-1}(F)) \leq n \sum \mu(B_k) \leq n \sum \mu(B) = \frac{n\varepsilon}{n}$



- - - QED □

We'll use this next time...