3-31-08
Op Alg
1

VNA sort of a non-commutative analogue of a measure space.

(1) $M \subseteq B(H)$ is a VNA if it's a unital $*$-subalgebra and
$$M = \overline{M}^{SOT} = \overline{M}^{WOT}.$$

Or: (2) M is a C^*-alg which, as a Banach space, is the dual of some Banach space.

<u>Thm</u> M commutative VNA $\Leftrightarrow$ $M = L^\infty(X,\mu)$ for some (X,μ)

Since $\underbrace{L^\infty(X,\mu)}_{M} = \big(\underbrace{L^1(X,\mu)}_{M_* \text{ "pre-dual"}}\big)^*$ this is tied in with (2).

M_* gives you a topology on M, the $\sigma(M, M_*)$ topology

$$x_j \to x \iff \forall \varphi \in M_*, \quad \varphi(x_j) \to \varphi(x).$$

$\sigma(M, M_*)$ is almost the WOT - It's the σ-WOT:

$$x_j \to x \iff \langle x_j \xi, \zeta \rangle \to \langle x \xi, \zeta \rangle$$
for $\xi, \zeta \in H^{\oplus \infty}$

<u>Notation</u>
Given $x_1, \dots, x_n \in B(H)$

$W^*(x_1, \dots, x_n)$ is the WOT closure of the $*$-algebra generated by $x_1, \dots, x_n$.

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2

$\boxed{\text{Fact}}$ Given $M = L^\infty(X,\mu)$ and $M_* = L^1(X,\mu)$,

M_* is those functionals on M which are $\sigma(M,M_*)$ cts.

So the measure class is captured in the topology, so to speak.

Another way to think about VNA's is symmetry.

Observe If $S = S^* \subseteq B(H)$ is a self-adjoint subset,

$$S' = \{x \in B(H) : xy = yx \ \forall y \in S\}$$ is a VNA.

Why? • Easy to show WOT-closed.

Comes up when you have a group Γ acting on H, and decompose H into Γ-modules.
unitarily, i.e. all $g \in \Gamma$ are unitary

A subspace $K \subseteq H$ is Γ-invariant iff orthogonal projection onto K commutes with Γ.

$$\{p \in B(H) \text{ proj s.t. } [p, \pi(\Gamma)] = 0\}$$
$$= \{p \in \pi(\Gamma)' : p \text{ projection}\}$$

so set of projections $\pi(\Gamma)'$ is a VNA.

<u>Thm</u> (Bicommutant Thm)

$M \subseteq B(H)$ VNA $\Rightarrow$ $M = M''$.

(Proj(M) $\longleftrightarrow$ lattice of M'-invariant subspaces in $B(H)$.)

CLASSICAL STORY ON MEASURED SPACES

If (X, μ) is separable (meaning $L^1(X, \mu)$ a separable Banach space) then ICBST
either
- X is k atoms $k = 1, 2, \ldots, +\infty$
- X is $[0,1]$ w/ Lebesgue measure
- X is $[0,1]$ plus some atoms

- Projections in L^∞ are characteristic functions of subsets of X.
- These form a lattice: $P \leq Q$ if $Q - P$ a positive operator.
- Atoms correspond to <u>minimal projections</u>.

<u>Thm.</u> Let M be a VNA on H.

$Z(M) =$ center of $M = M \cap M'$

This is an abelian VNA, so it's $L^\infty(X, \mu)$ for some (X, μ).

One can identify $H = L^2(X, H_x, \mu)$

where there's a Hilbert space H_x corresponding to each $x \in X$

It's the set of $f : X \longrightarrow \coprod H_x$

where $f(x) \in H_x$, $\int |f(x)|^2 \, d\mu < \infty$,

and $\{\langle f(x), g(x) \rangle\}$ is μ'ble.

Basically Spectral Thm — when a VNA acts on a HS it decomposes into subspaces of common spectra. [?]

$\forall x$ $\exists$ a VNA $M_x \leq B(H_x)$ and $M = \{$m'ble sections $f : X \to M_x\}$.

<u>Example</u> $H = \mathbb{C}^3$ $\Gamma = S_3$ (permutations) $M = \pi(\Gamma)$

Invariant subspace $W = \text{span}(e_1 + e_2 + e_3)$

$W^\perp$ also invariant, has no inv. subsp. (else all of $\mathbb{C}^3$ eigenvecs $\Rightarrow S_3$ abelian)

M' contains the projections $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, no others.

- (Irreducible rep'ns correspond to trivial commutants) [?]

$$M' = \left\{ \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \beta \end{bmatrix} : \alpha, \beta \in \mathbb{C} \right\}$$

$$M = (M')' = \left\{ \begin{bmatrix} * & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{bmatrix} \right\}$$

$$Z(M) = M \wedge M' = M' \text{ since } M' \subseteq M.$$

What does this tell us?

Can write $Z(M)$ as $L^\infty(X, \mu)$ where X is 2 points $\{W, W^\perp\}$

- $H = H_W \oplus H_{W^\perp} = L^2(H; H_x)$

Have $\forall x$, $\exists$ a VNA $M_x \subseteq B(H_x)$ with $M = \{\text{m'ble } f: X \to M_x\}$ and M a <u>factor</u> meaning trivial center.

Ex $M = M_{n \times n}$ is a factor. Why?

- Consider the set $D \subseteq M$ of diagonal matrices. This is a MASA (Maximal Abelian Subalgebra). Easy to check;

$$\left[\begin{bmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{bmatrix}, [x_{ij}] \right] = [(a_i - a_j) x_{ij}]$$

- $M' \cap M \subseteq D' \cap M = D$

Can finish by finding "another D" (different basis) which also must contain the center.

Or, look at normalizer $N(D) := \{u \in M \text{ unitary} : u D u^* = D\}$.

One can find $u_1 \in N(D)$ (eg - permutation) which commutes with nothing in D except $\mathbb{C}1$.