An equivalence relation $R$ on $(X, \mu)$ is amenable if

$$\exists P : L^\infty(R, \mu_R) \to L^\infty(X, \mu) \text{ s.t. } P \geq 0, \text{ and } \forall \alpha \text{ R-morphism, }$$

$$P(\alpha) = P(\alpha \cdot \frac{x}{x}) \text{ on some } x,$$

such $P$ sometimes called "invariant mean".

**Følner-like conditions for equivalence relations**

A subset $K \subset R$ is **bounded** if

$$\sup_{x_0 \in X} \# \{ (x, y) \in K : x = x_0 \} < \infty$$

and

$$\sup_{y_0 \in X} \# \{ (x, y) \in K : y = y_0 \} < \infty$$

i.e. $K$ has finitely many points from each orbit.

(= corresponds to $F_n$ finite in amenability of groups)

**Lemma**

$R$ amenable $\Rightarrow \forall \alpha_1, \ldots, \alpha_n$

$$\exists f \in L^1(R, \mu_R), \ f \geq 0, \ f \neq 0,$$

$$\sum_{i=1}^n \| \alpha_i f - f \|_1 \leq \varepsilon \| f \|_1.$$  

See proof for groups — can copy pretty much word for word. □

**Lemma**

$R$ amenable $\Rightarrow \forall \alpha_1, \ldots, \alpha_n$ R-morphisms, $\forall \varepsilon > 0$, $\exists f$ valued in $\mathbb{R}$ or

$$\sum_{i=1}^n \| \alpha_i f - f \|_1 \leq \varepsilon \| f \|_1.$$  

(Hence $f$ is the characteristic function of some bounded $K$.)
If
\[ E_a = \chi_{[a, \infty)} \quad t = \int_0^\infty E_a(t) \, dt \quad |t - t'| = \int_0^\infty |E_a(t) - E_{a'}(t)| \, dt \]

\[ f_a = E_a \circ f \]

\[ \sum \sum |f_a(y, x) - f_a(y, x)| \, d\mu(x,y) < \sum f_a \, d\mu(x,y) \]

Fubini

"True on average = true somewhere". \(\Box\)

\[ K \in \mathbb{R} \text{ bdd} \Rightarrow K \subseteq \bigcup \text{ (graphs of finitely many automorphisms) } \]

"\(\forall \mathcal{R} \in \mathbb{R} \text{ bdd} \exists \mathcal{R} \text{ bdd s.t. } \mathcal{R} \text{ is } \varepsilon \text{- invariant under } S"\]

**Lemma** \(\mathcal{R} \text{ amenable } \Rightarrow \forall K \subseteq \mathcal{R} \text{ bdd measurable,} \)

\[ \forall \varepsilon > 0, \exists T \subseteq \mathcal{R} \text{ finite equiv sub-relation s.t.} \]

\[ \mu \{ (x,y) \in K \setminus T, \text{ but } x \in \text{dom}(T) \text{ or } y \in \text{range}(T) \} \leq \varepsilon \mu(\text{dom}(T)) \]

"Clearer notation: \(T\) defined on \(X_0 \times K\)

\[ \mu \{ (x,y) \in K \times X_0, \text{ but } x \in X_0 \text{ or } y \not\in X_0 \} \leq \varepsilon \mu(X_0) \]
We can replace $K$ by the union of graphs of some $R$-morphism $\alpha_1, \ldots, \alpha_n$.

Choose $f : R \to \{0, 1\}$ as in lemma so

$$\sum_i \int_{(x,y) \in R} |f(y, \alpha_i(x)) - f(y, x)| \, d\mu(x) < \varepsilon \int_{(x,y) \in R} f(y, x) \, d\mu(x, y)$$

and

$$\sum_i \sum_{y \in \mathcal{X}} |f(y, \alpha_i(x)) - f(y, x)| \, d\mu(y) < \varepsilon \sum_{y \in \mathcal{X}} f(y, x) \, d\mu(y)$$

"True on avg := true somewhere"

$$\Rightarrow \exists \Omega \subset \mathcal{X} \quad \mu(\Omega) > 0 \quad s.t. \quad \forall y \in \Omega,$$

$$\sum_i \sum_{(x,y) \in R} |f(y, \alpha_i(x)) - f(y, x)| < \varepsilon \sum_{(x,y) \in R} f(y, x)$$

$$K' = \{ (x,y) \in R : f(x,y) = 1 \} \quad f = \chi_K.$$  

Choose $\Omega' \subset \Omega$, $\mu(\Omega') > 0$ s.t.

$$\forall x_1, x_2 \in \Omega', \quad x_1 \neq x_2,$$

$$\{ y : (x,y) \in K \} \quad \text{and} \quad \{ y : (x,y) \in K \}$$

are disjoint.

(can make $x_2$ miss all these. Use Fubini and a "true on avg => true somewhere" argument.)
\[
\forall y \in \Omega', \text{ let } F_y = \{ x \in X : (y, x) \in R \text{ and } f(y, x) = 1 \}.
\]

\[
= \text{ orbit of } y \text{ in } K'.
\]

\[
\text{Char of } \Omega' \Rightarrow F_y \neq F_{y_2} \text{ if } y_1 \neq y_2.
\]

Now define \( T \subset R \) equiv. rel. over \( \Omega' \) by "orbits of \( T \) are the sets \( F \)."

\[
T = \{ (x, y) : x \in \Omega' \text{ and } y \in F_x \}
\]

\[
B_i(y) = \begin{cases} 
\{ x \in \text{dom } \alpha : \text{s.t. } \alpha_i(x) \in F_y \text{ but } x \notin F_y \} & \text{error set} \\
\{ x \in \text{dom } \alpha : \text{s.t. } \alpha_i(x) \notin F_y \text{ but } x \in F_y \} & \text{set}
\end{cases}
\]

\[
\forall y \in \Omega', \quad \sum \sum 1 = \sum \sum \sum \text{ } f(y, x) - f(y, \alpha_i(x))
\]

\[
\leq \varepsilon \sum \text{ } f(x, y) = \varepsilon \sum \text{ } 1
\]

Conclude that

\[
\forall y \in \Omega' = \Omega, \quad \sum_i |B_i(y)| \leq \varepsilon |F_y|.
\]

Let \( B_i(\Theta) = \bigcup_{y \in \Omega'} B_i(y) \) "total misbehavior"

\[
\Rightarrow \sum \mu(B_i) \leq \varepsilon \mu(\bigcup_{y \in \Omega'} F_y)
\]

Resume next time...
From last time:

\[ K \subseteq R \text{ bbd subset} \quad K = \bigcup (\text{graphs of } \alpha_1, \ldots, \alpha_n) \]

\[ \Omega, \epsilon \subseteq X \quad \forall K_i \subseteq R \text{ bbd subset} \]

\[ \forall j \quad \alpha_j, \ K_i \cap K_i \text{ is large compared to } K_i. \]

\[ \supp f \cup \{ (xy) : (x, y) \in K \text{ and } t \} < \epsilon \mu (\Omega) \]

More precisely, let

\[ F_y = \{ x : (x, y) \in R \text{ s.t. } (x, y) \in \supp f \} \]

\[ B_y = \{ x \text{ s.t. either } x \notin F_y \text{ or } 3_j \text{ with } \alpha_j(x) \notin F_y \} \]

or

\[ x \notin F_y \text{ or } 3_j \text{ with } \alpha_j(x) \notin F_y \}

Can split into

\[ B_i(y) = \{ x : x \in F_y, \ \alpha_i(x) \notin F_y \} \]

or

\[ x \notin F_y, \ \alpha_i(x) \notin F_y \}

\[ \forall y \in \Omega', \quad \sum_i |B_i| < \epsilon |F| \]

\[ \forall y \neq y' \in \Omega', \quad K \cdot y \cap K \cdot y' \text{ disjoint} \]

\[ K \subseteq R \quad K \cdot y = \{ x : (y, x) \in K \} \]

Let

\[ B_i = \bigcup \{ B_i(y) \} \quad \sum_i \mu(B_i) < \epsilon \mu (\Omega') \]

**T** -- equiv rel: \( x \sim x' \) if \( x, x' \in F_y \) for some \( y \) (finite equiv. rel.)

Suppose \( (x, y) \in K, (x, y) \notin T, \ x \in \Omega', \ y \in \Omega' \)

\[ \Rightarrow \exists \alpha_j \quad y = \alpha_j(x) \quad \Rightarrow \ x \notin B_i(y) \text{ or } y \notin B_i(x). \]

Next we want to patch together the local estimates from this lemma to get some kind of global theory.
Lemma \( R \) amenable \( \Rightarrow \) \( \forall K \in R \) bold measurable, \( \exists T \) finite equiv rel, \( T \subseteq R \), s.t. \( \mu_R(K \setminus T) < \varepsilon \).

PF We can assume \( \text{proj}_x(K) \cup \text{proj}_y(K) = X \).

Consider triples \((T, Y, H)\), \( T \subseteq R \) finite sub-eq. rel. over \( X \), \( H \subseteq K \) subset s.t. (1) \( \mu_R(H \setminus T) \leq \mu_R(Y) \) \( \mu_R((Y \setminus H) \cup Y) \leq \varepsilon \mu(Y) \)
(2) If \( (x, y) \in H \) and either \( x + Y = y + Y \) then \((x, y) \in T \).

By the previous lemma, the set of such triples is nonempty. It can be partially ordered by inclusion, and unions are upper bounds, so Zorn guarantees a maximal triple \((T, Y, H)\).

If \( Y \neq X \), can enlarge \( Y \) by gluing in another finite eq. rel. obtained by approximating \( K \cap (Y' \times Y') \) over \( Y' \), \( Y' \subseteq X \setminus Y \).

Hence the maximal triple is \((T, X, H)\), \( \Box \)

Theorem \( R \) ergodic and amenable \( \Rightarrow \) \( R \) hyperfinite

PF Choose \( K_n \subseteq R \) increasing seq. of bold sets s.t. \( R = \bigcup K_n \), \( \mu_K \)
Choose \( E_n > 0 \), \( \sum E_n < \infty \), and choose \( T_n \subseteq R \) s.t. \( \mu_R(K_n \setminus T_n) < E_n \mu_R(K_k \setminus T_k) \) for \( k > n \).

Let \( T^n = \bigcap_{k \geq n} T_k \) \( T^n \subseteq T^{n+1} \)

For \( m > n \), \( \mu(K_n \setminus T^n) \leq \sum_{k \geq m} \mu(K_n \setminus T_k) \leq \sum_{k \geq m} E_k \to 0 \) as \( m \to \infty \).

This concludes the proof that hyperfiniteness is equivalent to amenability.
Theorem If $R = R_\Gamma$ for a free action of $\Gamma$ on $X$, then $R$ amenable $\iff \Gamma$ amenable.

\( \iff (\Rightarrow) \quad \text{done} \)

\( (\Rightarrow) \quad P: cL^\infty(R,\mu) \to L^\infty(X,\mu) \quad \text{invariant mean} \)

\[ L = \mu \circ P \quad L: L^\infty(R) \to C \]

\[ L^\infty(G) \xrightarrow{\mu} L^\infty(R) \]

\[ \frac{1}{n} \sum_{j=1}^{n} \alpha_j(x_j) = \frac{1}{n} \langle g \rangle \]

\[ L: L^\infty(G) \to C \quad \text{is an invariant mean on } \Gamma \]

\[ \text{Theorem } R \quad \text{amenable} \quad \Rightarrow \quad \text{if } R_0 \subseteq R \text{ sub-erg rel, then } R_0 \text{ amenable.} \]

\[ \text{PF } \quad \text{Easy. Just restrict the invariant mean.} \]

\[ \text{Thm } \quad R \text{ hyperfinite, } R_0 \subseteq R \Rightarrow R_0 \text{ hyperfinite.} \]

\[ \text{PF } \quad \text{Hard to do directly! This is why amenability is helpful.} \]

Fact $F_n$ not amenable.

One way: Calculate spectral radius (use stuff from last quarter)

Another way: Write $w \in F_n$,

\[ W = g_i^{+a_i} \cdots g_k^{+a_k} \quad i, \ldots, k \in \{1, \ldots, n\} \]

\[ A_1 = \{ w \quad \text{start with } g_i^{-1} \} \quad i \in \{1, \ldots, n\} \]

\[ A_2 = \{ w \quad \text{start with } g_i^+ \} \quad k \leq 0 \]

\[ i_1 \neq i_2, \quad i_2 \neq i_3, \ldots \]

\[ G = A_1 \cup A_2 \quad \text{disjoint} \]

\[ F_i = X_A: \quad 0 \leq f_i + f_2 + f_i(g_1) + f_2(g_2) - 1 \]

\[ P(f_i + f_2 + f_i(g_1), \ldots) = -1. \quad \Rightarrow \quad \exists \]
Recall

Amenability of a group $\Gamma$:
- Følner condition: $\exists F_n$ finite, $\lim_{n \to \infty} \frac{\log F_n - \log \Gamma}{|F_n|} = 0$
- $\exists \rho : \Lambda^0(\pi) \to \mathbb{C}$, invariant mean
- $\sum_{\text{yes}} (1 - \lambda_g)(1 - \lambda_g)^* \text{ has an approximate kernel}$
- $\exists z_n \in L^2(\pi), \|z_n\| = 1, \|\lambda_g z_n - z_n\| \to 0, \forall g$
- $L = \sum_{\text{yes}} \lambda_g + \lambda_g^*$ has $\|L\| = p(L) = 2|\Sigma|$

We'd like to add another condition.

Full/reduced $C^*$ algebra of a group

- Given discrete group $\Gamma$,

$$ C^*_{\text{full}}(\Gamma) = \overline{C\Gamma}^{\|\cdot\|_{\text{full}}} $$

where $\|\sum a_i g_i\|_{\text{full}} = \sup_{\text{unitary rep } \pi} \|\sum a_i \pi(g_i)\|_{B(\ell^2(\Gamma))}$

$$ C^*_{\text{red}}(\Gamma) = \overline{C\Gamma}^{\|\cdot\|_{a}} $$

where $\|\sum a_i g_i\|_{a} = \|\sum a_i \lambda_g\|_{B(\ell^2(\Gamma))}$

Difference is taking one rep'n vs. a sup over all rep'n's.
Quick Observations

- $\| \cdot \|_{\text{full}} \geq \| \cdot \|_2$
- $\pi: \Gamma \to U(H_\pi)$ rep

\[ C^*_\text{full}(\Gamma) \twoheadrightarrow C^*(\pi(\Gamma)) \]

\[ \alpha: \Gamma \to \mathcal{B}(C) \quad \alpha(g) = 1 \quad \text{trivial rep} \]

\[ \alpha: C^*_\text{full}(\Gamma) \to C = \mathcal{B}(C) \quad + \text{- hom} \]

Kernel has co-dimension 1 $\Rightarrow C^*_\text{full}(\Gamma)$ not simple if $\Gamma$ non-trivial

Using this, another condition for amenability is

\[ \text{or: } \| \cdot \|_{\text{full}} = \| \cdot \|_{\text{rel}}. \]

Cor: $\Gamma$ amenable $\Rightarrow$ $\| \sum a_j \lambda_{\gamma_j} \| \geq |\sum a_j| = \| \sum a_j \lambda_{\gamma_j} \|$ \[ \text{for: } \| \sum a_j \lambda_{\gamma_j} \| \geq \left\langle \sum a_j \lambda_{\gamma_j} \varphi_a \omega^n, \varphi_a \omega^n \right\rangle \sim \left\langle \sum a_j \pi_n, \pi_n \right\rangle \]

This compares the left-regular-rep norm to the trivial-rep norm.

In order to show that these are equivalent to amenability use the last condition from before (with $\| L \| = 2(s/\ell)$).

- $\| \|_{\text{full}} = \| \|_2$ clearly implies $\| L \| = 2(s/\ell)$.

- Reverse: $\pi$, unitary rep of $\Gamma$ where $\pi(\mathcal{F}^\infty) = 1$.

\[ \pi \otimes L^2(\pi) \quad \pi(\cdot \otimes h) = \pi(\cdot)h \otimes \mathcal{F}^\infty \]
Claim: Isometry $V: H_{\pi} \otimes l^2(\Gamma) \hookrightarrow H_{\pi} \otimes l^2(\Gamma)$

such that $V$ intertwines $\pi \otimes \lambda$ and $\text{id} \otimes \lambda$.

• Ex: $\Gamma = \mathbb{Z}$, $\mu$ = any measure on $\Gamma$, $\lambda =$ Haar measure.

\[ \mu \ast \lambda \ll \lambda \quad \text{actually, } \mu \ast \lambda = \lambda ? \]

Tensor product here corresponds to convolution of measures.

\[ H_{\pi} \otimes l^2(\Gamma) = l^2(\Gamma; H_{\pi}) \]

\[ (\pi \otimes \lambda)(g) = \pi_h \lambda (h^{-1}g) \]

\[ (\text{id} \otimes \lambda)(g) = \lambda (h^{-1}g) \]

$V: \eta \otimes \delta_g \rightarrow \pi(g) \otimes \delta_g \quad \eta \in \mathcal{H}$

\[ (\text{id} \otimes \lambda) V = V (\pi \otimes \lambda) \quad \text{as claimed.} \]

Cor: $\| \sum a_i (\pi \otimes \lambda)(g_i) \| \leq \| \sum a_i \lambda(g_i) \|$

$\| \sum a_i \pi(g_i) \| \approx \langle \sum a_i \pi(g_i) \delta_{g_i}, \delta \rangle$

$\langle \sum a_i \pi(g_i) \otimes \lambda(g_i) \delta \otimes \delta_{g_i}, \delta \otimes \delta_{g_i} \rangle \rightarrow \langle \sum a_i (\pi \otimes \lambda)(g_i) \delta \otimes \delta_{g_i}, \delta \otimes \delta_{g_i} \rangle$

But $\| \sum a_i \lambda(g_i) \| \leq \langle \sum a_i \pi(g_i) \otimes \lambda(g_i) \delta \otimes \delta_{g_i}, \delta \otimes \delta_{g_i} \rangle$.
Recall $\Gamma$ amenable $\iff \sum_i (\lambda_i - 1) (A_i - 1)^*\ y$ has approx kernel.

so $\Gamma$ not amenable $\iff$ strictly positive.

i.e. $\geq \varepsilon d$ for some $\varepsilon > 0$.

Let $Q \in \mathcal{C}^*$ be the element $Q = \sum_i (\lambda_i - 1) (A_i - 1)^*$.

$\Gamma$ non-amenable $\iff \exists (Q) \geq \varepsilon I \iff Q^*Q$ invertible, in $\mathcal{B}(\ell^2(\mathbb{N}))$.

For a rep $\pi$ of $\Gamma$, we want to ask: Does $\pi(Q)$ have non-trivial approx kernel?

ker $\pi(Q) = \{ h \in \mathbb{H} \text{ s.t. } \pi(g) h = h \ \forall g \in \Gamma \}$

Is $0$ an isolated point in $\sigma(\pi(Q))$? "\pi has spectral gap."

Example $\pi = \rho 2^\alpha$

$\pi$ has spectral gap $\iff \Gamma$ not inner amenable.

Example $\Gamma = \mathbb{F}_2 \times S_\infty$

$\rho 2^{-1}$ has no spectral gap.

Definition $\Gamma$ has property (T) if $Q$ has a spectral gap in any representation.

Using this, $\Gamma$ has (T) $\implies \ell^2(\Gamma)$ is non-

$\Gamma$ has (T) and amenable $\implies \Gamma$ is finite.

$\sigma(\mathfrak{H}) \cap (\mathfrak{g}_e) = \{ 0 \}$

$\sigma(\mathfrak{H}) \cap \mathfrak{g}_e$ do not.

together imply $o \in o(\mathfrak{h}(\mathfrak{g}))$.

$\ell^2(\Gamma)$ has an invariant.
Examples

\( L(R) \) \hspace{1cm} R \ \text{hyperfinite}

\( L(\mathbb{F}_n) \) \hspace{1cm} \text{not hyperfinite} \ (\text{non-} R)

\( L(\mathbb{N}) \) \hspace{1cm} \mathbb{N} \ \text{has CT}

How to distinguish? Next time...
Property (T)

L(Γ) \leq L(F_m) \quad \text{if} \quad Γ \text{ has (T)}

\[ L(\Gamma) \leq L(F_m) \]

We'll prove this.

- Observe: If \( \pi \): unitary rep of \( Γ \) on \( H_\pi \) with cyclic vector \( \xi \)
  \[ \phi_\pi: \Gamma \to \mathbb{C} \quad \phi_\pi(g) = \langle \pi(g)\xi, \xi \rangle \]
  \[ \phi_\pi(g^{-1}h) = \langle \pi(g^{-1}h)\xi, \xi \rangle = \langle \pi(g)^{-1}\pi(h)\xi, \xi \rangle \]
  \[ = \langle \pi(h)\xi, \pi(g)\xi \rangle \]
  \( q_1, \ldots, q_n \in \Gamma \)

  Matrix \[ [\phi_{\pi}]_{ij} = \phi_\pi(q_i^{-1}q_j) = \langle \pi(q_i)\xi, \pi(q_j)\xi \rangle \]

  \[ \left[ \frac{\langle \eta_i, \eta_j \rangle}{\eta_i} \right]_{ij} \geq 0 \]

- \( \phi_\pi \) is the "positive definite function associated to \( \pi \)"

  \[ \mathbb{E}_\Gamma = \mathbb{Z} \quad \Gamma_\pi = \left[ \frac{2}{\pi} \left( \begin{array}{c} \pi, \nu \end{array} \right) \right] \quad \Pi = \left[ 0, 2\pi \right] \]

  \[ \phi_\pi(n) = \int_0^{2\pi} \pi(n, \theta) f(\theta) \quad f(\theta) \neq 0 \]

  This is the well-known characterization of which functions \( \phi(n) \) are Fourier transforms of positive measures: \[ [\phi_\pi(n-m)] \text{ must be positive definite.} \]

Going backwards: Given \( \phi: \Gamma \to \mathbb{C} \) s.t. \( \forall q_1, \ldots, q_n \) \[ [\phi(q_1, q_2)] \geq 0 \quad (\ast) \]

Can define a vector space \( H \) and rep \( \pi \) by

\[ H_\pi = \text{span} \{ g : g \in \Gamma \} \quad \langle g, h \rangle = \phi(g^{-1}h) \]

(\ast) \to \langle , \rangle \text{ nonnegative}
\[ \langle \sum \alpha_i g_i, \sum \alpha_j g_j \rangle = \sum \alpha_i \alpha_j \phi(g_i, g_j) \]

\[ = \left\langle \begin{bmatrix} \phi(g, g_0) \\ \vdots \\ \phi(g, g_n) \end{bmatrix}, \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} \right\rangle \]

H. has a rep of \( \Gamma \) leaving \( \langle , \rangle \) invariant

\[ \rightarrow \text{unitary rep } \pi \text{ of } \Gamma \text{ on } \mathcal{H}_0 \]

\[ \exists = \{ e \} \quad \phi(\pi \rho) = \phi \]

So pos. def. func. \( \iff \) reps.

**EXERCISE**

If \( \pi \) is a unitary rep of \( \Gamma \), \( \pi \) is a sub-rep of the left regular rep \( \Leftrightarrow \phi(\pi) \in L^2 \quad \forall \exists \in \mathcal{H}_\pi. \)

If \( \exists \) contains a \( \Gamma \)-invariant vector, then \( \exists \) cycle s.t.

\[ \phi(\pi \rho) = \langle \pi(g) \exists, \exists \rangle \text{ satisfies } \phi(\pi \rho (g)) \geq c > 0 \quad \forall g \in \Gamma. \]

\( \pi = \pi_1 \ast \pi_2 \quad \pi_1 = \text{trivial rep} \quad \pi_2 = \text{some rep} \)

\[ \exists = \exists_1 \ast \exists_2 \]

\[ \langle \pi(g) \exists, \exists \rangle = \| \exists_2 \| ^2 + \langle \pi_2(g) \exists, \exists_2 \rangle \]

Property (T): \( \exists \exists \subseteq \Gamma \) finite set, \( e > 0 \)

\( \forall \text{unitary rep } \pi : \Gamma 

\[ \rightarrow \mathcal{H}_\pi, \quad \text{if } \forall \pi(g) \exists - \exists \leq \varepsilon, \| \exists \| = 1, \]

then \( \exists \exists \), s.t. \( \pi(g) \exists_0 = \exists_0. \)
\[ \forall x \in M, \ x \ M-bdd \Rightarrow E(x) \ is \ N-bdd. \]

\[ \| E(x) \|_2 = \| E(x_n) \|_2 \leq \| x_n \|_2 \leq C \| n \|_2 \]

\[ = E \text{ takes } M \text{ to } N \text{-bdd vectors.} \]

Why is \( E \) possible?

\[ E(x \xi) \in N \text{ so } \text{tr} \langle E(x \xi), \xi \rangle = 0 \forall \xi \in L^2(N) \]

Completely positive because \( E_n = E_{M_{\mu}(M)} \)

Example: Compositions of CP maps are CP.

One common source of CP maps is to compose automorphisms with conditional expectations. In fact, it's hard to come up with an example not of this form!

Next Time

Bimodules vs CP maps
Last time we discussed completely positive maps \( \eta : A \to B \) and their C*-algebra properties.

**Fact:** There is a 1-1 correspondence between

1. C*-algebra maps \( \eta : A \to B \)
2. Pairs \( \mathcal{H}, \mathcal{A} \mathcal{T} \) where \( \mathcal{H} \) is a \( \mathcal{B} \)-Hilbert module with \( \mathcal{B} \)-linear action.

**Def.** \( \mathcal{H}, \langle \cdot, \cdot \rangle \) is called a \( \mathcal{B} \)-Hilbert module if

- \( \mathcal{H} \) is a linear space and a right \( \mathcal{B} \)-module
- \( \langle \cdot, \cdot \rangle : \mathcal{H} \otimes \mathcal{H} \to \mathcal{B} \) is a map which is linear and satisfies
  - \( \langle x, y \mathcal{B} \rangle = \langle x, y \rangle \mathcal{B} \)
  - \( \langle x \rangle = \langle y \rangle \mathcal{B} \)
  - \( \langle x, x \rangle \geq 0 \), \( \langle x, x \rangle = 0 \iff x = 0 \)
- \( \mathcal{H} \) is a Banach space under \( \| x \| = \sqrt{\langle x, x \rangle} \)

**Examples of Hilbert Modules**

- Let \( \mathcal{B} = C(X), \quad E \to X \) a vector bundle, \( g \) a metric on \( E \)
- \( \Gamma(E) \) (space of continuous sections) is a \( C(X) \)-Hilbert module

\( \langle \xi, \eta \rangle(x) = g_{\xi \eta} \langle \xi(x), \eta(x) \rangle \)

One can prove conversely that any vector bundle looks like this.

**Def.** Given a \( \mathcal{B} \)-Hilbert module \( \mathcal{H} \), an operator \( T : \mathcal{H} \to \mathcal{H} \) is called

- \( \mathcal{B} \)-linear if \( T(x \mathcal{B}) = (Tx) \mathcal{B} \)
- \( \mathcal{A} \)-adjointable if \( \exists T^* : \mathcal{H} \to \mathcal{H} \) s.t.

\( \langle \xi, y \rangle = \langle x, T^* y \rangle \quad \forall x, y \in \mathcal{H} \)

(Note that adjointable \( \implies \mathcal{B} \)-linear.)

Denote \( \mathcal{M}(\mathcal{H}) = \{ T : \mathcal{H} \to \mathcal{H} \text{ adjointable} \} \) "multiplier algebra"

**Fact.** Adjointable \( \implies \) bounded but converse is false.
So why are (1) and (2) equivalent?

(2 =⇒ 1) Given \( \bar{z} \in H \), \( \lambda : A \to M(\bar{H}) \)

define \( \eta : A \to B \)

\[
\eta(\bar{z}) = \left( \begin{array}{c}
\sum_{i,j} a_i^* b_j \bar{z}^{ij}
\end{array} \right) \in B.
\]

Claim \( \eta \) is c.p.

⇒ If \( a, z \in A \), \( a = y^* y \),

\[
\eta(\bar{z}) = \langle y^* y \bar{z}, \bar{z} \rangle = \langle y \bar{z}, y \bar{z} \rangle \geq 0
\]

More generally, let \( H_n = H^\otimes n \) viewed as an \( M_{nn}(B) \)-module. \( A \) acts on \( H_n \).

Let \( H_n = M_{nn}(H) \) viewed as an \( M_{nn}(B) \)-module.

\[
\langle \bar{z}^{ij}, \bar{s}^{ij} \rangle = \sum_{i,j} \langle \bar{z}^{ij}, \bar{s}^{ij} \rangle = \sum_{i,j} \left( \sum_k \langle \bar{z}^{kj}, \bar{s}^{kj} \rangle \right) = \sum_{i,j} \left( \sum_k \langle \bar{z}^{kj}, \bar{s}^{kj} \rangle \right)
\]

\( M_{nn}(A) \) acts by left multiplication

\[
\langle \left[ \begin{array}{c}
\bar{z}
\end{array} \right], (a_{ij}) \left[ \begin{array}{c}
\bar{s}
\end{array} \right] \rangle = \text{pos.}
\]

Now how do we go \( 1 \Rightarrow 2 ? \)

Given \( \eta : A \to B \), let \( H_0 = A \otimes B \)

\[
\langle \sum a_i \otimes b_i, \sum a_j^* \otimes b_j^* \rangle = \sum \langle b_i^* \eta(a_i a_j^*) b_j^* \rangle
\]

Let \( \bar{z} = 1 \otimes 1 \)

\[
\langle 1 \otimes 1, a(1 \otimes 1) \rangle = \eta(a)
\]

Claim \( \langle x, x \rangle \geq 0 \). Then we can take \( H_0 \) to get a Hilbert module.
Let \( \eta : M \to N \) be a normal c.p. map.

\[ \exists \]

\( H \) is a Hilbert space, normal commuting actions of \( M \) and \( N \) on \( H \)

"\( M, N \) correspondence" or "\( M, N \) bimodule"

or "\( M, N \) Hilbert bimodule"

(\( ^{\mathcal{N}} = N \& B(H) \) possible)

Why are c.p. maps analogs of pos. def. funs on \( \Gamma \)?

"\( \ldots \) \( M, N \) bimodules \( \ldots \) analogs of representations?"

**Fact** \( \pi : \Gamma \to U(\mathcal{N}) \)

\( H = K \otimes l^2(\Gamma) \)

\( \Gamma \) acts on \( H \) via \( \pi \otimes \alpha \)

\[ \text{id} \otimes \alpha \]

Both \( \pi \otimes \alpha \) and \( \text{id} \otimes \alpha \)

\[ L^\infty \]

\[ \rho^\infty \]

extend to a pair of commuting normal actions of \( L(\mathcal{N}), \ L(\mathcal{N}) \) op

\[ g \cdot (k \otimes \alpha) = \pi(g) k \otimes \alpha_g \]

\[ (k \otimes \alpha) h = k \otimes \rho_h \]

Can recover the original representation

\[ \text{span} \left\{ g (k \otimes \alpha) g^{-1} : g \in \Gamma \right\} \approx \text{span} \left\{ \pi(g) k : g \in \Gamma \right\} \]

\( \rho^{-1} \) rep of \( \Gamma \) on \( H \)

\( H = K \otimes \) (other stuff)
C.P. map associated to \( \hat{g} = k \circ l \)

\[
\langle h, \eta(g) \rangle_{L^2(L(M))} = \langle \hat{g} \circ h, g^* k \rangle = \langle k \circ l, \pi(g) k \circ g h^{-1} \rangle
\]

\[
= \sum_{g = h} \langle k, \pi(g) k \rangle = \sum_{g = h} \varphi(k)
\]

where \( \varphi \) is the positive def. fn. on \( \mathbb{R} \) you started with.

How do we translate something like property (T)?

Say \( K \) contains an invariant vector \( k \): \( \pi(g) k = k \)

\[
H = K \circ L^2(M) \quad g(k \circ \delta_e) = \pi(g) K \circ \delta_e = (k \circ \delta_e) g
\]

So fixed vectors in \( K \) translate to \( L(M) \)-central vectors.

\( H \) has a most invariant vectors if \( \{ \) sequence \( \frac{\hat{k}}{k} \), \( \| \hat{k} \| = 1 \)

s.t. \( \| x_{\hat{k}} - \hat{x} \| \to 0 \) as \( k \to \infty \). (May want to insist \( \hat{x} \) be bounded too.)

A normal C.P. map \( \eta : M \to M \) extends to a map \( L^2(M) \to L^2(M) \).

Quick way: \( \eta \) extends to \( L'(M) \to L'(M) \)

\[
\eta^* \to M_\ast \quad M_\ast \\
\leq \to \circ \eta
\]

and by "interpolation" one obtains \( L^2 \) from \( L' \).

If we also require \( \tau \circ \eta = \eta \) (note: this is equivalent to \( \eta^* \) unital) one should be able to do even more easily using Cauchy - Schwarz or something. Proof next time.
Let $\eta : \Gamma \rightarrow C$ be a positive definite function

$\eta(g) = \varphi(g)g$ associated CP map

$\eta$ extends to $\ell^2(\Gamma')$ as the operator of multiplication by $\varphi$

$\eta(\bar{z}) = \bar{\varphi} \cdot \bar{z}$

$\|\eta\|_{\ell^2(\Gamma')} = \|\varphi\|_{\ell^2(\Gamma')} \leq \varphi(e)$.

**Observation**

If $\varphi$ has the property $\varphi(g) \rightarrow 0$ as $g \rightarrow \infty$

(i.e., $\forall \epsilon > 0 \exists F \subset \Gamma$ finite with $\varphi g \leq \epsilon$ on $\Gamma \setminus F$)

then $\eta : \ell^2(\Gamma') \rightarrow \ell^2(\Gamma')$ is a compact operator.

**Def**

$M$ has the Haagerup property if for all sequence $\eta_m$ of CP maps $M \rightarrow M$ s.t.

1) $\eta_n$ unital $\Gamma'$-preserving

2) $\eta_n : \ell^2(\Gamma) \rightarrow \ell^2(\Gamma)$ compact

3) $\|\eta_n(x) - x\|_{\ell^2} \rightarrow 0$ $\forall x \in M$.

**Fact** If $\Gamma$ has the Haagerup property, so does $L(\Gamma)$.

Choose $\varphi_n : \Gamma \rightarrow C$ positive definite, $\varphi_n(e) = 1$

we can make $\varphi_n = \bar{\varphi}_n$

then $\eta_n = \bar{\eta}_n$

$\eta_n(g) - g = (\varphi_n(g) - 1)g \rightarrow 0$ as $n \rightarrow \infty$.

$\eta_n - \text{id}$ is bounded and CP dense in $\ell^2(\Gamma') \rightarrow (3)$ above.

Get $\eta_\infty \leq \eta^*$ since $\varphi_n$ real $\Rightarrow \Gamma'$-preserving $\Rightarrow (1)$.

Compactness comes from the limiting behavior of $\varphi_n$. 
Claim: $L(\mathbb{F}_n)$ has the Haagerup property.

- Given $H$ a real Hilbert space
  
  $H_c = H \otimes \mathbb{C}$ (complexification)

  $\mathcal{F}(\mathcal{H}) = \bigoplus_{n \in \mathbb{N}} H_c^n$

  "vacuum vector"

  $|0\rangle = h \in H_c$, $\mathcal{L}(h)^* \text{ bind}$

  $\langle h, g \rangle$

  $s(h) = \mathcal{L}(h) + \mathcal{L}(h)^*$ \text{ for } h \in H \text{ (not } H_c)\)

- \[ \mathcal{W}^*(s(h), h \in H) \text{ then } \langle \mathcal{F}_2, \cdot, \mathcal{F}_2 \rangle \text{ is a trace} \]

- \[ \mathcal{W}^*(s(c_1)) = \mathcal{L}(\mathbb{F}_m) \quad m = \dim_\mathbb{C} H \]

- $\mathcal{W}^*(s(h), h \in H) \text{ def}$

  The map $\mathcal{W}^*$ is a * functional * from (real $H$-valued and contraction)

  to (W*-algebra and normal cp maps).

  If $T: H_1 \rightarrow H_2$ is a contraction, then $\mathcal{W}^*(s(H_1)) \rightarrow \mathcal{W}^*(s(H_2))$ \text{ cp map s.t. } \mathcal{W}^*(s(H_1)) = s(T)$.

  \[ \mathcal{W}^*(s(h), h \in H_1) \leq \mathcal{W}^*(s(h), h \in H_2) \]

  We'll show next time that $\mathcal{W}^*(H)$ has the Haagerup property.
Recall Haagerup property for VNA's:

\[ M \text{ is a factor} \]
\[ \exists \Phi_n \text{ unital } \tau \text{-preserving CP maps} \]
\[ \| \Phi_n(x) - x \|_2 \to 0 \text{ as } n \to \infty \]
\[ \Phi_n : L^2 \to L^2 \text{ are compact.} \]

Remarks:

- For \( M = L(\Gamma) \) this is equivalent to the Haagerup property for \( \Gamma \).
- Don't actually need unital \( \tau \)-preserving, but it's convenient.
- \( L(F_n) \) has the Haagerup property

\[ L(F_n) = E(R^n) \overset{\text{def}}{=} W^* \{ p(h) : h \in R^n \} \]
\[ p(h) = 1(h) + l(h)^* l(h) \]

\[ E \text{ functor from (real Hilbert spaces, contractions)} \]
\[ \to (W^*-algebras, \text{CP maps}) \]

\[ T : R^n \to R^n \]
\[ T \frac{z}{\lambda} = (\cos \theta) \frac{z}{\lambda} \]

\[ \frac{z_1}{\lambda} \cdots \frac{z_n}{\lambda} \to (\cos \theta \frac{z_1}{\lambda}, \sin \theta \frac{z_1}{\lambda}) \cdots \to (\cos \theta \frac{z_n}{\lambda}, \sin \theta \frac{z_n}{\lambda}) \]
Remark If \( R \) is hyperfinite, it has the Haagerup property.

In fact, something stronger:

\[
R = \overline{UR_m} \quad R_m \text{ f.d. subalg}
\]

\[
R \xrightarrow{\phi_m} R \xrightarrow{\text{rad} \circ \phi} R_m
\]

\[\text{finite dim}_\mathbb{F}\]

In this case \( \phi_m \) are not just compact but actually have finite rank.

**Theorem** (Connes) \( \exists \phi_m : M \rightarrow M \) are unital CP \( \tau \)-preserving \( \Rightarrow M \) hyperfinite.

\[\text{such that} \quad \|\phi_m(x) - x\|_2 \rightarrow 0 \quad \text{and} \quad \phi_m \text{ finite rank}\]

\[\text{So why can't we embed} \quad L(P) \hookrightarrow L(F_m) \text{ if } P \text{ has } (T)?\]

**Note** If \( M \) has the Haagerup property and \( N \subseteq M \), \( N \) has the Haagerup property.

\[
\phi_m : M \rightarrow M \quad \|\phi_m(x) - x\|_2 \rightarrow 0
\]

\[
\psi_m = E_N \circ \phi_m : N \rightarrow N \quad \|\psi_m(x) - x\|_2 \rightarrow 0 \quad \forall x \in N
\]

\[
\phi_m : L^2(M) \rightarrow L^2(M) \text{ compact}
\]

then so are \( \psi_m = E_N \circ \phi_m \circ e_N \quad e_N : L^2(M) \rightarrow L^2(N) \)

If \( L(P) \hookrightarrow L(F_m) \Rightarrow L(P) \text{ has Haag.} \Rightarrow \text{P has Haag. Which contradicts (T) if} \)

\( P \) infinite (we showed this before).
Note
$M \overset{\Pi_1}{\to} \text{factor}$
$N \subseteq M$ fixed
$N_k$ s.t. $N_k \to N$
meaning $\forall x \in N, \exists x_k \in N_k$ s.t. $\|x_k - x\|_2 \to 0$

If $N_k$ has Huang.up, so does $N$. (So Huang is preserved under this kind of "limit".)

Proof using $\mathcal{P}_{\ell,k} : N_k \to N_k$ $\mathcal{P}_{\ell,k} \to \text{id}$ as $\ell \to \infty$

$\exists \mathcal{P}_{\ell,k} \text{ s.t.} \mathcal{P}_{\ell,k} \circ \mathcal{P}_{\ell,k} \to \text{Huang. for } N.$

Proposition $M$ has property $(\Pi) \iff C^*(M, JM^J) \cap K = \{0\}$.

$3x_k \in M$ s.t.
$\|x_k, x\|_2 \to 0$ as $k \to \infty$
$\|x_k\|_2 = 1$

$J : L^2(M) \to L^2(M)$
$x \Rightarrow x^* \Omega$
$R_x = JL_{x^* J}$

Remak. Given $x_0, \ldots, x_n$ consider $L = \sum |x_i - Jx_i^* J|^2$
$0 \in \sigma(L)$ since $1 \in L^2(M)$ is in kernel
$\sigma(L)$ has a gap near $0 \iff L$ on $L^2(M) \oplus C$ has full inverse
$\exists C \text{ s.t. } \|x - \mathcal{P}x\|_2 \leq C \max \|x, x_i\|_2$
$P_{\ell,0} \in C^*(L) \subseteq C^*(M, JM^J)$
Note: \( P_{c_1} = P_{c_2} = p \) rank 1 projection

\[ \sum x_i: P y_i: \text{ finite rank operator} \]

\[ (x_i: P y_i)(z) = x_i \triangleright (y_i: z) \]

So \( P_{c_1} \in C^*(M, J^*M) \Rightarrow C^*(M, J^*M) \ni k. \)

Fact: \( T: H \to H \) compact \( \iff \forall h \in H, \|h\| = 1 \)

\[ (h_n \to 0 \text{ weakly}) \Rightarrow T h_n \to 0 \text{ in norm.} \]

(Well-known)

\[ \text{(Property: r)} \iff \exists \text{ unitaries } u_k \in M, \gamma(u_k) = 0 \]

\[ \forall x \in M, \|u_k x u_k^* - x\|_2 \to 0 \]

\[ \| (x - J x J^*) u_k \| \to 0 \quad \forall x \in M. \]

Remark: Non: \( \gamma \) \( \iff \text{Inn}(M) \subset \text{Aut}(M) \) closed in the following topology:

\[ \| \gamma \circ \alpha \circ \alpha^{-1} - \alpha \|_2 \to 0 \]

[\[ \gamma \circ \alpha \underset{\text{norm}}{\longrightarrow} \text{Inn}(M) \]

Why? \( \text{Aut}(M) = \{ \alpha \text{-isomorphisms } M \to M \} \]

\( \text{Inn}(M) = \{ \alpha \in \text{Aut}(M) : \alpha(x) = u x u^*, u \in M \} \]

Corresponds to a map \( U(M) \to \text{Aut}(M) \) with range \( \text{Inn}(M) \)

These groups are all Polish (their topologies are metrizable)

\[ \Rightarrow \text{image dense only if } \mathcal{A} \text{ approximate kernel.} \]
Open Mapping Theorem for Polish groups:

\[ X: G \to H \quad G, H \text{ Polish} \]

\[ \ker(x) \text{ closed} \iff x \text{ has no approximate kernel.} \]

In our case, \( \operatorname{Inn}(M) \text{ closed} \iff X \text{ un s.t.} \]

\[ \text{Ad } u_n \to \text{id} \quad \text{but} \quad 0 \notin u_n \to \text{ in } \Pi. \]

In general, \( M \) is full if \( \operatorname{Inn}(M) \cap \operatorname{Aut}(M) \text{ closed.} \)

For type II\(_1\), \( M \) full \( \iff \text{non-}\Gamma. \)
Recall $M$ has $\Gamma \ni f$ unitary, $\| [u_n, x] \|_2 \to 0 \quad \forall x \in M$

Let $\varphi_n$ be a state on $C^* (M, JMJ)$ defined by

$$\varphi_n (T) = \langle Tu_n, u_n \rangle = \langle u_n^* Tu_n, 1 \rangle$$

Thus, $M$ non-$\mathcal{F}$ $\Leftrightarrow$ whenever $u_n$ unitaries satisfy $\| [u_n, x] \|_2 \to 0$

then $\| u_n - 2 (u_n) \|_2 \to 0$.

Let $u_n$ satisfy $\| [u_n, x] \|_2 \to 0 \quad \text{and take } \varphi_n$ as above.

$$\varphi_n (L_x R_y) = \langle u_n^* x y^* J u_n, 1 \rangle$$

$$\varphi_n (L_x R_y 1) = \langle u_n^* x u_n y^* J 1, 1 \rangle \to \langle x y^* J 1, 1 \rangle$$

Some holds true if we sum $\sum \varphi_n (L_x R_y)$, and these sums are $\| \|_2$-dense.

If $\varphi$ is any weak limit of $\{ \varphi_n \}_n$, $\varphi (T) = \langle T 1, 1 \rangle$ where

$T \in C^* (M, JMJ)$.

If $T \in K$, $\varphi_n (T) = \langle Tu_n, u_n \rangle \to \langle Tu, u \rangle$ because $Tu_n \xrightarrow{\|} Tu$ and $u_n \xrightarrow{\text{weak}} u$.

$\Gamma$ equivalent to being able to choose $s$ s.t. $Tu_n \to 0$. But then $\langle T 1, 1 \rangle = 0$

$$\Rightarrow \langle \sum \alpha_i G i, 1 \rangle = 0 \quad \forall \alpha_i \in \mathcal{M} \Rightarrow T = 0.$$

Converse: Omitted.
Recall Property (T) for $\Gamma$:

\[ \exists \varepsilon, F \subseteq \Gamma \text{ finite subset s.t. } \|x^3 - y^3\| < \varepsilon \|x\| \quad \forall x \in F \]

\[ \Rightarrow \exists \exists' \text{ near } 3 \text{ s.t. } \pi(g)\exists' = \exists' \quad \forall g, \forall \exists \text{ unitary rep on a Hilbert space.} \]

$M$ has (T) if $\exists F \subseteq M$ finite subset, $\varepsilon > 0$ s.t.

Wherever $H$ is an $M,M$ bimodule, $\exists \in H$ satisfies

\[ \|x^3 - y^3\| < \varepsilon \|x\| \quad \forall x \in F \text{ then } \exists \exists' \text{ (near } 3) \]

s.t. $y^3 = \exists'$ $\forall y \in M$.

$\Gamma$ has (T) $\Rightarrow L(\Gamma)$ has (T).

$H$ $M,M$ bimodule $\Rightarrow$ rep of $\Gamma$ by $\pi(g)\exists = g\exists g^{-1}$

$F = \{ \lambda(g) : g \in F \text{ of the group } 3 \}.$

(Not the same $F$)

\[ \text{Equivalent: } \exists F, \varepsilon, c \text{ s.t. } \forall \text{ pos. def. map } \varphi : \Gamma \rightarrow \mathbb{C} \]

if $\forall x \in [1 - \varepsilon, 1 + \varepsilon]$ then $\varphi \geq c$ everywhere.

$M$ has (T) $\iff \exists \varepsilon \text{ s.t. if } \varphi \text{ is a CP map, } \varphi \text{ unital}$

\[ \exists F \subseteq M, = \left\{ m \in M : \|m\|_2 = 1 \right\} \]

s.t. $\|\varphi(x) - x\|_2 < \varepsilon$ $\forall x \in F$

then $\|\varphi(y) - y\|_2 < \varepsilon$ $\forall y \in M,.$
\[ \varphi \mapsto (H, \mathcal{F}) \Rightarrow \exists \text{ implements } \varphi \]

\[ \sum_{x^3 - 3x} \leq \varepsilon \text{ on } F \]

\[ \langle x^3 a, y^3 b \rangle = \gamma (a \ast \varphi (x^* y)) \]

If \( \exists \mathcal{F} \) bounded, \( \langle s_\mathcal{F}, \mathcal{F} \rangle = \gamma (\varphi) \rightarrow \varphi', \varphi'(x) = x \]

\[ \text{Consequences of T} \]

\[ \text{Fact} \quad \text{If } M \text{ has (T) and } \alpha : M \rightarrow M, \quad \beta : M \rightarrow M \]

with \( M \) a II_1 factor,

\[ \| \alpha(x) - \beta(x) \|_2 \leq \varepsilon. \]

Then

\[ \exists u \in M \text{ s.t. } \alpha(x) = u \beta(x) u^* \]

Not quite true, need to cut w/projections.

\[ \exists v \in M \text{ s.t. } v \alpha(x) v^* = \beta(x) v v^* \]

partial isom. \( \forall v \in \beta(M) \cap M \).

\[ \text{Cor} \quad \text{Let } M = M \text{ and } \beta = \text{id}, \]

\[ \alpha : M \rightarrow M \text{ s.t. } \| \alpha(x) - x \|_2 \leq \varepsilon \quad \forall x \in F \quad \Rightarrow \alpha = \text{Ad } u. \]

(Note: \( \beta(M) \cap M = M \cap M = C + \text{ maximality argument} )

\[ \text{Cor} \quad \text{Recall } T \Rightarrow \text{non} - F \Rightarrow \text{Inn}(M) \text{ closed in } \text{Aut}(M) \text{ closed}. \]

Now \( \text{Aut}(M) / \text{Inn}(M) \) is a discrete group. It's also Polish.

So it's countable.
So where do these $\alpha$ and $\beta$ maps come from?

$$\alpha L^2 M \rightarrow M, M \text{ bimodule}$$

$\alpha \cdot z = \alpha(x)z, \ x \in M, \ z \in M$

$\beta \cdot y = \beta(y), \ y \in M$, $\ z \in M$

(4) says

$$\|x \cdot \mu - \mu \cdot x\| < \epsilon \quad \forall x \in F.$$

$$\Rightarrow \exists y \in M \text{ s.t. } \alpha(x)y = y\beta(x) \quad \forall x \in M.$$

$$y = \alpha^{-1}y, \ V^*\alpha(x)V = \beta(x)V^*V$$

$$V^*V \in \beta(M) \cap M$$

Let $M_1$ be a VNA generated by $x_1, \ldots, x_n$

$$M_2$$

$$y_1, \ldots, y_n$$

$$d \left( (x_1, \ldots, x_n), (y_1, \ldots, y_n) \right) = \inf_{\alpha_1, \alpha_2} \left( \sum \|\alpha_1(x_i) - \alpha_2(y_i)\|_2^2 \right)^{1/2}$$

When $n = 1$, get $d_{x,y} = \inf \left\{ \int_{\mathbb{R} \times \mathbb{R}} |x-y|^2 \, d\pi \right\}^{1/2}$

Kantorovich-Wasserstein

Studied in problems related to "optimal transport".
Can also define

\[
\mathfrak{d}(x_1, \ldots, x_n, y_1, \ldots, y_n) = \inf \left( \mathbb{E} \left[ \sum_{i=1}^{n} (x_i - y_i)^2 \right] \right)^{1/2}
\]

if \( M_1, M_2 \)-bimodule

\[\exists \mathcal{H}\]

\[\langle \mathcal{H}, \mathcal{H} \rangle = \mathcal{C}_m, \quad \langle \mathcal{H}, \mathcal{H} \rangle = \mathcal{C}_m,\]

\(d \geq d'\), but whether they're equal in general is an open question.

(May even be open for matrix algebras... not sure...)
Recall: group measure space construction \( R \rightarrow (X, \mu) \)

\[ \text{equiv. relation } R \subseteq X \times X \]

\[ \rightarrow A \subseteq \mathcal{L}(R) \quad (\supseteq \mathcal{L}^\infty(X, \mu) \times \mathcal{M}) \]

If \( R \) is free, \( A \) is a MASA. \( A \cap M = A \)

\[ E : M \rightarrow A \text{ and } \exp \text{ s.t. } \mu \circ E \text{ is a trace on } M. \]

We can abstractly characterize the inclusion \( \kappa \kappa M \).

**Def:** \( A \subseteq M \) *is called a Cartan subalgebra* if

- \( A \) is a MASA, \( A \cap M = A \)

- \( \mathcal{N}(A) = \{ u \in M \text{ unitary s.t. } uAu^* = A \} \) generates \( M \)

\[ \mathcal{U}(A) \subseteq \mathcal{N}(A) \]

\[ (\mathcal{R}, \sigma) \iff (A \subseteq M) \]

**Theorem:** let \( A \subseteq M \) be a MASA, \( \kappa \kappa A \)

- \( A \subseteq M \) Cartan

- \( \mathcal{L}^2(M) \) viewed as \( A, A \) bimodule

\[ = \mathcal{L}^2(X \times X, \eta) \]

\[ \eta \text{ is } \mu \text{-discrete} \]

\[ \text{supp } (\eta) \text{ is a set } B \subseteq X \times X \text{ s.t. each fiber of } B \text{ is countable} \]

- \( A \) is a MASA and \( \forall x \in M \)

\[ A \times A \overset{\mu}{\longrightarrow} \mathcal{L}^2(X \times X, \bar{\eta}) \] with \( \bar{\eta} \text{ discrete} \).
Let \( R_0 = \text{supp } \eta \). By a Borel selection theorem, \( R_0 \) is a union of graphs.

Choose such a graph \( \alpha \in R_0 \). \( X_\alpha \) corresponds to some element \( \exists \delta \in L^2(M) \). Then \( \int \delta \delta = \int f \delta \).

Can do a polar decomp \( \exists \delta = U_\alpha \delta \), \( U_\alpha \in M \)

\( \delta = (\delta \delta)^{1/2} \).

In fact \( b_\delta \in L^2(A) \) and \( U_\alpha \delta \) supported on \( \alpha \).

\( \int U_\alpha U_\beta = U_{\alpha \beta} \) \( \Rightarrow \) \( R_0 \) equiv rel.

\( (A \subseteq M) \mapsto U_\alpha \quad \alpha \in R_0 \) graph

\[ U_{\alpha \beta} f U_{\alpha \beta}^* = U_\alpha U_\beta f(U_{\alpha \beta}^*) \]

\( \Rightarrow U_\alpha U_\beta \left( U_{\alpha \beta} \right)^{-1} f = f U_\alpha U_\beta \left( U_{\alpha \beta} \right)^{-1} \)

\( \Rightarrow U_{\alpha \beta} \left( U_{\alpha \beta} \right)^{-1} \in U(A \cap M) = U(A) \)

\( \Rightarrow \) we get a cocycle \( \sigma(\alpha, \beta) \) with values in \( U(A) \)

\[ \sigma(\alpha, \beta) = U_\alpha U_\beta \left( U_{\alpha \beta} \right)^{-1} \]

\( A \subseteq M \) comes from \( R \) \( \Rightarrow \) can replace \( u_\alpha \) by \( u_\alpha V_\alpha \), \( V_\alpha \in U(A) \)

\[ \sigma(\alpha, \beta) = \forall \alpha, \beta \]

"Cartan subalgebras are the same as equivalence relations"
Another way of stating this third condition (distinctness) is that $A^* \langle Me_e \rangle$ is generated by projections which are finite in $\langle Me_e \rangle$.

Question about free group: Is $\mathbb{F}_n \cong \mathbb{F}_n$? No. Then $\mathbb{F}_n$ would be $m$-generated by universality; various ways to get a contradiction from this.

Harder question: Is (a) $C^*_{\text{full}} (\mathbb{F}_n) \cong C^*_{\text{full}} (\mathbb{F}_m)$? (b) $C^*_{\text{red}} (\mathbb{F}_n) \cong C^*_{\text{red}} (\mathbb{F}_m)$? (c) $L(\mathbb{F}_n) \cong L(\mathbb{F}_m)$?

If (a) then any $C^*$-alg generated by $n$ unitaries is a quotient of $C^*_{\text{full}} (\mathbb{F}_n)$.

(?) => would imply every $m$-generated $C^*$-alg was $m$-generated.

Can show that $\mathbb{T}^n$ works (given by $n$ card projections)

Q (Prof. S would really like to know the answer!)

Give a "nice" example (simple, trivial) of a $C^*$-alg generated by $n$ but not $m$ unitaries.

For example, it's an open problem, whether $C^*_{\text{red}} (\mathbb{F}_n)$ is generated by $m$ unitaries.
\[
K_1(C^*(\mathbb{F}_n)) \sim \mathbb{Z}^n \quad \text{(has rank n)}
\]
\[
K_1(C^*(\mathbb{Z})) = K_1(\mathbb{T})
\]

where \( K_1 \) is some invariant. So (b) can be done with \( K \)-theory.

(c) is open.

A fourth question is whether \( F_n \simeq X \) free, ergodic can be \( ME \) to \( F_n \simeq X \).

We'll look at 2 different approaches.

**Cost of an equivalence relation \( R \)**

- \( R \) generated by \( \{ \alpha_1, \alpha_2, \ldots \} \)

\( \alpha_j : A_j \to B_j \) partial \( R \)-morphism

If \( (x-a.e.) \quad x \sim y \Rightarrow x = \alpha_{i_1} \alpha_{i_2} \ldots \alpha_{i_k}(y) \)

for some \( k, i_1, \ldots, i_k \).

\[
C(R) = \inf \sum_j \mu(A_j) \quad \text{\{generate } R \text{ \}}
\]

\( \mu \) \text{ support of } \alpha_j \text{ } \quad \text{(definition from Lenni in early 90's)}

\[\]

\[
X = \xymatrix@C=25pt{ \alpha_1 & \alpha_2 & \alpha_3 \ar[l] \ar[r] & \cdots \\
& n \in \mathbb{N} \\
\alpha_i & \alpha_j \ar[l] & \alpha_k \ar[l]
}
\]

\( R = \text{"all are equiv"} = \mathbb{Z}/2 \) by cyclic shift

On each orbit draw a graph

\( x \overset{\alpha_j}{\sim} y \iff \exists x' = x, y' : \text{ for some } j \).
In general, we can do homology in any category where there's a concept of "exact sequence" and where we can "add" morphisms.

Examples
- Modules over a group $G$
  \[ V, \ G\text{-action on vector space } V, \ \text{G-equiv. bilinear map} \]
- Bimodules over an algebra, bimodule homomorphisms

Suppose $F$ is a functor between two such categories.

\[
\mathcal{F}_V(V) = \frac{V \otimes V}{\text{fixed}}
\]

\[
\mathcal{F}_X(Y) = \frac{Y \otimes_{\text{A,A}} X}{\text{X,Y - A,A-bimodules}}
\]

$\mathcal{F}_X: A, A$ bimodules $\to$ vector spaces

F is exact if it takes exact sequences to exact sequences.

Homology "measures the extent to which $F$ fails to be exact."

Example Let $X$ be a CW-complex and $\Gamma$ a group acting on $X$
(cells map to cells).

\[
\mathcal{F}_X = \text{binary tree} \quad \Gamma = \text{graph of } \Gamma
\]

Easier Ex $X = \mathbb{R}$
\[
\Gamma = \mathbb{Z} \text{ acting by translations}
\]
\[
\begin{align*}
C_0(X) &= \text{span} (0 - \text{cells}) \\
C_1(X) &= \text{span} (1 - \text{cells}) \\
&\vdots \\
C^2 \xleftarrow{\partial} C_0(X) &\xleftarrow{\partial} C_1(X) &\xleftarrow{\partial} C_2(X) &\xleftarrow{\partial} &\cdots
\end{align*}
\]
boundary maps

\(H_k(X) = \frac{\ker \partial_k}{\text{im} \partial_{k+1}}\)

In our examples above, \(X\) is contractible and hence has trivial homology, i.e. \((\star)\) is exact.

Consider the functor \(F(V) = V \otimes \mathbb{C}\)
\[
\begin{align*}
C_0(X) \otimes \mathbb{C} &\cong \mathbb{C} \\
C_1(X) \otimes \mathbb{C} &\cong \mathbb{C} \\
\text{from our second example, with } \partial \text{ acting trivially}
\end{align*}
\]

\(0 \xleftarrow{\partial} C \xleftarrow{\partial} C_0(X) \xleftarrow{\partial} C_1(X) \xleftarrow{\partial} 0\) exact

\(0 \xleftarrow{\partial} \mathbb{C} \xleftarrow{\partial} \mathbb{C} \xleftarrow{\partial} \mathbb{C} \xleftarrow{\partial} 0\) no longer exact

**Note:** \(C_k(X) \otimes \mathbb{C} \cong C_k(X^{simplicial}) = C_k(\mathbb{S}^1)\)

so we recover the homology of the circle.

On the ternary tree we get the homology of the figure 8 since now our Fundamental domain is a corner \(\mathbb{L}\)

(i.e. everything is a translate of this corner).
Theorem Assume $X$ is a functor as before and

$$X_n \to X_{n-1} \to X_{n-2} \to \cdots$$

is an exact sequence with $X_0, X_1, \ldots$ free modules.

Then in $\mathcal{F}(X_1) \leftarrow \mathcal{F}(X_0) \leftarrow \cdots$,

$$D_j \left( \mathcal{F}, X_{-1} \right) = \frac{\ker \alpha_j}{\text{im} \alpha_{j+1}}$$

depend only on $\mathcal{F}$ and $X_{-1}$.

**Def** An exact sequence $X_{-1} \leftarrow X_0 \leftarrow X_1 \leftarrow \cdots$ s.t. $X_j$, $j \geq 0$ are free is called a free resolution of $X_{-1}$.

**Def** $\mathcal{H}_k(G; V) = D_k \left( \bullet \otimes_n V, C \right)$

**Ex** $H_* \left( \mathbb{Z}; C \right) = H_\text{red} (\mathbb{P})$

$H_* \left( \mathbb{F}_2, C \right) = H_* (\infty)$

**Ex** What's $H_* (G; C, G)$?

- $X_k \otimes (C \otimes X_k)$ so tensoring does nothing

$\therefore$ all homology groups trivial.

**Observation** If $G$ acts freely on a contractible CW complex $X$, then $\{C_k(X), \partial_k\}$ is a free resolution of $C$.

But note that $C_k(X)$ may not be finitely generated.
A universal example

Given $G$, construct a CW complex as follows:

$k$-cells indexed by $(k+1)$-tuples of group elements

$\langle g_0, \ldots, g_k \rangle \sim C_{g_0, \ldots, g_k}$

$\partial_C \langle g_0, \ldots, g_k \rangle = \sum (-1)^i C_{g_0, \ldots, \hat{g}_i, \ldots, g_k}$

Basically, you keep throwing in discs to fill all possible holes.

The resulting topological space is called $EG$

and carries a natural $G$-action

$g \cdot C_{(g_0, \ldots, g_n)} = C_{(gg_0, \ldots, gg_n)}$

Note $C_k(EG)$ will never be finitely generated unless $G$ is finite.

$C_k(EG) \sim \bigoplus_{k \text{ times}} C \otimes C \otimes \cdots \otimes C$

$C_{g_0, \ldots, g_k} \rightarrow g_0 \otimes g_0^{-1} \otimes \cdots \otimes g_k^{-1}$

$\partial_n (h_0 \otimes \cdots \otimes h_n) = \sum (-1)^k h_0 \otimes \cdots \otimes h_k h_{k+1} \otimes \cdots \otimes h_n$,

Observation: If $G$ acts freely on a CW complex which is contractible and has a compact fundamental domain, then $\{C_k(X)\}$ is a free resolution of $\mathbb{Z}$ by finitely generated modules.
Last Time Defined $H_k (G; V)$

1st way to compute

Find some free resolution of $C$

$$0 \leftarrow C \leftarrow X_0 \leftarrow X_1 \leftarrow \ldots$$

Make a new sequence

$$0 \leftarrow C \otimes_V X_0 \otimes_V X_1 \otimes_V \ldots$$

$$\ker \alpha_k \otimes_V \alpha_{k+1} = H_k (G; V)$$

2nd way Write down the bar complex

$$E \leftarrow CG \leftarrow CG \otimes CG \leftarrow CG \otimes CG \otimes CG \leftarrow \cdots$$

$$g_0 g_1 \otimes \ldots \otimes g_n \leftarrow g_0 g_1 \otimes \ldots \otimes g_n \leftarrow g_0 \otimes g_1 \otimes \ldots \otimes g_n$$

$$V/G \leftarrow V \leftarrow V \otimes CG \leftarrow V \otimes CG \otimes CG \otimes CG \otimes \cdots$$

$$\mathcal{I} (V \otimes g_1 \otimes \ldots \otimes g_n) = V g_1 \otimes g_2 \otimes \ldots \otimes g_n - V g_1 \otimes g_2 \otimes \ldots \otimes g_n$$

$$+ \ldots \pm V \otimes g_1 \otimes \ldots \otimes g_{n-1}$$

Example

$G =$ amenable group

$$V = (\mathbb{Z}/p \mathbb{Z})^G$$

$\mathbb{Z}/p \mathbb{Z}$ vector space

$G$ acts by Bernoulli shifts

$H (G; V)$ is $\infty$-dim

$\mathbb{Z}/p \mathbb{Z}$ space

It's a left $G$-module, but is also a compact topological space.

Its entropy becomes a nice invariant.
Let $\pi : G \to \mathcal{U}(H)$ be a unitary rep. of $G$ on a Hilbert space $H$.

Then $\pi \in L^2(\Gamma') ^{\otimes \infty}$ $\Rightarrow$ the action of $C^*G$ on $H$
extends to a normal action of $L^G$.

**Fact.** If $H$ is a Hilbert space and $M$ a II$_1$ factor acting
normally and unitarily on $H$, one can define the dimension
$\dim_M H$

Such $H \leq L^2(M) ^{\otimes \infty}$

$$P_H : L^2(M) ^{\otimes \infty} \rightarrow H \leq M' \cap B(L^2(M) ^{\otimes \infty})$$

$$= M^{op} \otimes B(L^2(M))$$

Trace $\text{tr} = \gamma \otimes \text{tr}$

Define $\dim_M H = \text{tr} \gamma (P_H)$.

Can show that this is well-defined, i.e. independent of the
choice of embedding.

Called the Murray-von Neumann dimension of $H$.

**Fact.** Let $z \in H$ be any nonzero vector.

Consider $P_{\frac{1}{z^2}} \in M'$

Coupling constant $\gamma_M \left( \frac{P_{\frac{1}{z^2}}} {z^2} \right)$

For any $\gamma' \leq \gamma_M \left( \frac{P_{\frac{1}{z^2}}} {z^2} \right)$

is independent of $z$ and equals $\dim_M H$. 
Examples

\[ \dim_M L^2(M) = 1 \]
\[ \dim_M L^2(M)^{\otimes N} = N \]
\[ p \in M^1 \quad \dim_M pL^2(M) = \tau_M(p). \]

Fact

For any abelian group \( \Gamma \), any repn of \( \Gamma \) is determined by a measure class on \( \hat{\Gamma} \) plus a multiplicity function \( n: \hat{\Gamma} \to \mathbb{N} \). (Discrete)

\( \Rightarrow \) left regular repn. \( \Leftarrow \) measure class is absolutely continuous wrt Haar measure.

(Don’t actually need \( \Gamma \) discrete.)

Given any unitary repn \( \Pi \) of \( \Gamma \), get an action of \( \Gamma \) on \( \mathsf{CAR}(H) \), whatever that is (some non-commutative topological space).

Entropy = \[ \dim \left( \text{max. part of } \Pi \text{ in left regular repn} \right) \]

(result of Voiculescu - Størmer)

Consider \( H_k(G, L^2(G)) \). Homology "eats up the right module structure," but this is still a left \( G \)-module.

Can we view \( \ker(L) \) as a Hilbert space?

If \( G \) has a cocompact (or just domain) contraction

\[ C \leftarrow (C^*G)^\pi \leftarrow (C^*G)^\pi \]

\[ C \leftarrow (C^*G)^\pi \leftarrow (C^*G)^\pi \]
We tensor the whole thing \( l^2(G) \otimes \cdot \)

and get

\[
\left( l^2(G) \right)^{n_0} \xrightarrow{\alpha_1} \left( l^2(G) \right)^{n_1} \xrightarrow{\alpha_2} \left( l^2(G) \right)^{n_2} \cdots
\]

\( \alpha_j \) are still \( L(G) \)-linear maps.

\( \alpha_j \in M_{n_j \times n_j}(C(G)) \)

\[ \ker \alpha_j / \operatorname{im} \alpha_{j+1} \text{ is not closed in general} \]

\[ H^1_k (G; l^2(G)) = \frac{\ker \alpha_j}{\operatorname{im} \alpha_{j+1}} \]

In co-compact case, the alg \( pF \) still "works" to show

this is independent of the choice of embedding

(need to assume finitely generated, though).

**Examples**

- \( \mathbb{Z} \) has a nice resolution coming from \( \mathbb{R} \)

\[
\begin{array}{c}
\mathbb{C} \xleftarrow{\alpha_1} \mathbb{C} \mathbb{Z} \xleftarrow{\alpha_2} \mathbb{C} \mathbb{Z} \xleftarrow{\alpha_3} \cdots \\
\mathbb{C} \mathbb{Z} \xleftarrow{\alpha_1} \mathbb{C} \mathbb{Z} \xleftarrow{\alpha_2} \mathbb{C} \mathbb{Z} \xleftarrow{\alpha_3} \cdots
\end{array}
\]

\[ \operatorname{im} \alpha_1 = \operatorname{im} (I - U) = \text{dense} \]

\[ \ker (\alpha_0) = \text{everything} \]

\[ H^1_k (\mathbb{Z}; l^2(\mathbb{Z})) = 0 \]
Example

\[ \Gamma^2 = \Gamma_2 \]

\[ \begin{align*}
\mathbb{C} \leftarrow \text{span(verties)} & \quad \leftarrow \text{span(edges)} \quad \leftarrow \mathbb{C}^2
\end{align*} \]

Tensor everything and get

\[ \ell^2(\text{vert}), \quad \ell^2(\text{edges}) \]

Say \( f \in \ker(\alpha_1) \)

\( f : \text{edges} \rightarrow \mathbb{C} \)

\( \partial f = 0 \) is a sort of "conservation law"

\( f \) represents the flow along each edge

\( \partial f = 0 \) means no leakage at any juncture

Can make such an \( f \) which is in \( \ell^2 \), e.g., start with some values on

\[ \text{and then go outward, with each vertex dividing the outflow in equal parts} \]

Thus \( \ker \alpha_1 \neq 0 \)

\[ H_0^\text{rel} = 0 \]

\[ H_1^\text{rel} = \vdots \approx \ell^2(G) \]
L²-homology has to do with

\[(\mathbb{C} \mathfrak{M})^n \rightarrow (\mathbb{C} \mathfrak{M})^n\]

\[\mathfrak{M} - \text{module map}\]

\[F = (b_{i,j}) \in \mathfrak{M}_{mn} (\mathbb{C} \mathfrak{M})\]

\[(l^2 \mathfrak{M})^n \xrightarrow{f^{(2)}} (l^2 \mathfrak{M})^n\]

\[\ker f^{(2)} \subset \ker f\]

\[\text{Theorem} \quad \mathfrak{M} \text{ amenable} \Rightarrow \overline{\ker f} = \ker f^{(2)}\]

\[K = \ker f \quad \overline{K} = \overline{\ker f} \subset \ker f^{(2)}\]

\[\text{Idea} \quad \text{Given an element in } \ker f^{(2)}, \text{ need to approximate by elements in } \ker f.\]

\[\text{Projection} \quad \text{pr}: (l^2 \mathfrak{M})^n \rightarrow \overline{K} \cap \ker f^{(2)}\]

\[\text{Goal: Argue that } \gamma (\text{pr}) = 0\]

\[u \in \mathbb{C} \mathfrak{M}, \quad u = \sum \lambda_i g_i, \quad \lambda_i \in \mathbb{C}\]

\[\text{supp } u = \{g_i : \lambda_i \neq 0\} \subset \mathfrak{M}, \quad S = \bigcup_{i,j} \text{supp } (b_{i,j})\]
\[ \Delta = \mathcal{Z}_A \cup \bigcup_{t \in S} \mathcal{Z}_A \cdot t \]

\[ p_{\mathcal{A}_t} : l^2 \Gamma^t \rightarrow l^2 \Gamma \quad p_{\mathcal{A}_t}(\Sigma_{g \in A} g) \rightarrow \sum_{g \in A} g \]

Similarly,

\[ u \in l^2 \Gamma \quad p_{\mathcal{A}_t} : u = u \cdot p_{\mathcal{A}_t} \quad \text{if} \quad p_{\mathcal{A}_t} (u) = 0 \]

Doing this entry wise,

\[ \left[ (\hat{\Theta} p_{\mathcal{A}_t}) \circ f^{(2)} \right] (u) = \left[ f^{(2)} \circ \hat{\Theta} p_{\mathcal{A}_t} \right] (u) \quad \text{if} \quad p_{\mathcal{A}_t} (u) = 0 \]

\[ \Rightarrow p_{\mathcal{A}_t} (u) \leq \ker f \quad \text{if} \quad u \in \ker f^{(2)} \quad \text{and} \quad p_{\mathcal{A}_t} (u) = 0 \]

\[ \Rightarrow p \circ (\hat{\Theta} p_{\mathcal{A}_t}) = 0 \quad \text{on} \quad p_{\mathcal{A}_t}^\perp. \]

Now

\[ \dim_c \left( (p \circ \hat{\Theta} p_{\mathcal{A}_t}) (\ker f^{(2)}) \right) \leq \text{rank} (p_{\mathcal{A}_t}) \leq |\mathcal{Z}_A| (1 + l_1 + 1) \cdot m \]

\[ \zeta (p_{\mathcal{A}_t}) = \frac{\text{Tr} (p \circ \hat{\Theta} p_{\mathcal{A}_t})}{|\mathcal{A}_t|} \]
\( A = \{ g_i : i \in I \} \)

\[
pr_A = \sum_{g \in A} g \, P_e \, g^{-1}
\]

\[
\text{Tr} \left( pr \cdot \Theta \, pr_A \right) = \text{Tr} \left( pr \cdot \Theta \left( \sum_{g \in A} g \, P_e \, g^{-1} \right) \right)
\]

\[
= \sum_{g \in A} \text{Tr} \left( pr \cdot \Theta \, P_e \right) = |A| \, \gamma(pr)
\]

This gives us \( \gamma(pr) \leq \frac{m(I|A| \, \epsilon)}{|A|} < \epsilon \).

True for all \( \epsilon \Rightarrow \gamma(pr) = 0. \)

Cor \quad \Gamma \text{ amenable} \Rightarrow B_j^{(2)}(\Gamma^2) = 0 \text{ except possibly } B_0^{(2)} = 1.

This is an example of "flatness" — the homology is trivial, if you take the extra step of passing to the closure of the kernel.
One can try to define $L^2$ homology for objects other than groups.

Hochschild homology basically involves passing from the category of $\Gamma^\ast$-modules to the category of $A,A$-bimodules,

$$C \Gamma \quad \text{analogous to} \quad M \otimes M$$

$$\ell^2 \Gamma \quad \text{"} \quad \ell^2(M \otimes M) \otimes \ell^2(M \otimes M)$$

Look at dimensions over $M \otimes M^\ast$.

- $H_n(A, V)$, $A,A$-bimodule
- Resolution of $A \leftarrow X_0 \leftarrow X_1 \leftarrow \cdots$
  - exact sequence of $A,A$-bimodules $X_i \cong (A \otimes A)^n$
- $A \otimes V \leftarrow X_0 \otimes V \leftarrow \cdots$

$$H_k(A, V) \overset{\text{def}}{=} \ker \frac{J_k}{\operatorname{im} J_{k-1}}$$

$\exists$ universal resolution (bar resolution)

$A \leftarrow A \otimes A^{\ast \ast} \leftarrow A \otimes A \otimes A^{\ast \ast} \leftarrow A \otimes A^{\ast \ast \ast} \otimes A^{\ast \ast} \leftarrow \cdots$

$$\partial(a \otimes a_1 \otimes \cdots \otimes a_{n+1}) = a \otimes a_1 \otimes a_2 \otimes \cdots \otimes a_{n+1} - a \otimes a_{a_2} \otimes \cdots a_{a_{n+1}} + \cdots + (-1)^n a \otimes \cdots \otimes a_{a_{a_{a_{a_{\cdots}}}} a_{a_{a_{a_{a_{a_{\cdots}}}}}}}$$
The kinds of computations you want to do are

\[ (A \otimes A^\tau)^n \xrightarrow{f} (A \otimes A^\tau)^n \]

\[ A \quad \text{*-algebra with } A \quad \text{N} \quad \text{positive "bounded" trace} \]

\[ M = W^*(A) \]

\[ L^2(M \otimes M^\tau) \xrightarrow{f^{\otimes^2}} L^2(M \otimes M^\tau)^n \]

\[ F \text{ is an } A, A \text{ - bimodule map so } f = (b_{ij}) \text{ with } b_{ij} \in (A \otimes A^\tau) \]

\[ G \quad \text{ has 2 commuting } G \text{ - actions to } 3 \in G \]

\[ g \circ 3 \text{ or } 3 \circ g \]

\[ A \otimes A^\tau \quad \text{ has 2 commuting } A \otimes A^\tau \text{ actions:} \]

\[ 3 \otimes \eta \in A \otimes A^\tau \]

\[ a \otimes b \in A \otimes A^\tau \]

\[ (a \otimes b) \# \eta \in (3 \otimes \eta) = 3 a \otimes b \eta \]

\[ (a \otimes b) \#_{\text{out}} (3 \otimes \eta) = a (3 \otimes \eta) b \]
\[ \text{Ex } A = \mathcal{C}(\Gamma), \quad \chi = \chi_r, \quad M = L(\Gamma) \]

\[ f : (\mathcal{C}(\Gamma))^n \rightarrow (\mathcal{C}(\Gamma))^n \quad \mathcal{C}(\Gamma) \circ \mathcal{C}(\Gamma) \]

\[ g \mapsto g \circ g^{-1} \]

\[ (\mathcal{C}(\Gamma) \circ \mathcal{C}(\Gamma))^n \xrightarrow{\tilde{f}} (\mathcal{C}(\Gamma) \circ \mathcal{C}(\Gamma))^n \]

\[ \text{Th } A(\ker f)A \leq \ker \tilde{f} \leq A(\ker f)A \]

\[ f^{(i)} : (\ell^1(\Gamma))^n \xrightarrow{\delta} (\ell^1(\Gamma))^n \]

\[ \ker f^{(i)} = \ker f \]
\[ B_k^{(2)}(A, \tau) = \dim_{\text{Mom}} H_k(A, L^2(M) \otimes L^2(M)) \]

\[ B_k^{(2)}(\mathbb{C}^n, \tau) = B_k^{(2)}(\mathbb{C}) \]

Now say \( \Gamma \) acts freely and m.p. on \((X, \mu) \) → equiv. \( \mathbb{R} \)

\[ L^\infty(X) \leq L^\infty(X) \times_{\text{alg}} \Gamma \leq A \leq L^\infty(X) \times \Gamma \]

only finite sums allowed

where \( A = \{ \sum_{\text{finite}} f_\alpha \nu_\alpha \} \) with \( f_\alpha \in L^\infty(X) \)

and \( \nu_\alpha \) a partial \( R \)-morphism.

The thing to do is some sort of relative homology over \( L^\infty(X) \).

\[ H_k^{L^\infty(X)}(A) \] is obtained from a sort of bar resolution

\[ L^\infty(X) \hookrightarrow A \hookrightarrow A \otimes A \otimes A \cdots \]

with the obvious boundary maps.

Tensor the whole thing over \( A \) with \( L^\infty(X) \times \Gamma \).

Then you compute the homology and look at dimension over \( L^\infty(X) \times \Gamma \).
\[ A \otimes \cdots \otimes A = \sum_{\alpha_1, \ldots, \alpha_k} f_{\alpha_1, \ldots, \alpha_k} V_{\alpha_1} \otimes \cdots \otimes V_{\alpha_k} \]

\[ \mathfrak{b}(f_{\alpha_1, \ldots, \alpha_n}) = f_{\alpha_1, \alpha_2, \ldots, \alpha_n} V_{\alpha_1} \otimes V_{\alpha_2} \otimes \cdots \otimes V_{\alpha_n} \]

This gives you \( \mathbb{B}_k^{(2)}(R) \), the \( L^2 \) Betti numbers of the equivalence relation \( R \).

\[ \mathbb{B}_k^{(2)}(R) = \mathbb{B}_k^{(2)}(L^\infty(X) \times A) \]

Turns out these equal \( \mathbb{B}_k^{(2)}(L^\infty(X) \times L^\infty(X) \times \text{alg } \Gamma) \)

even though \( A \) is bigger than \( L^\infty(X) \times \text{alg } \Gamma \),

\( (A \) has infinite sums \( \sum_{x_k} X_{F_k} \) provided \( F_k \) disjoint. \)

Key observation is that any \( x \in A \) is a finite sum of elements from \( L^\infty(X) \times \Gamma \) except at a small set:

\[ \forall x \in A \, \forall \epsilon > 0 \, \exists X_\epsilon \subset X \text{ with } \mu(X_\epsilon) < \epsilon \]

and \( x \cdot X_{\chi \chi^*} \in L^\infty(X) \times \Gamma \).

Details of proof omitted.
Now \( \beta_k^{(2)}(L^\infty(X) \otimes L^\infty(X) \times_{\text{diag}} \Gamma) \) is the same as \( -\beta_k^{(2)}(\Gamma) \), which leads to:

\[ \text{Theorem (Gaboriau 2000)} \]

**If** \( R \) **is induced by a free action** \( \Gamma \), **i.e.** \( R = R_\Gamma \), **then** \( \beta_k^{(2)}(R) = \beta_k^{(2)}(\Gamma) \).

\[ \text{Corollary} \] For \( n \neq m \), no actions of \( \Gamma_n \) and \( \Gamma_m \) can be orbit-equivalent.

**Note**: If \( \alpha \in \text{Aut}(R) \) then \( \alpha \) induces a map on cohomology

\[ H_k^{(2)}(R) \xrightarrow{\alpha^*} H_k^{(2)}(R) \]

\[ \text{modules over } L^\infty(X) \times \Gamma \text{ (often finite)} \]

This corresponds to a map \( \text{Aut}(R) \rightarrow \text{Aut} L^\infty(X) \otimes_c \Gamma \).
$L^\infty(X) \rightarrow C_0 \rightarrow C_1 \rightarrow \cdots$

$C_k = \text{"span of } k\text{-chains"}$

Example $\mathcal{F} \leq M$ a foliation of a manifold

Given a transversal $T$, define two points of $T$ to be equivalent if they intersect the same leaf.

This equivalence relation is discrete.

Assume $R$ on $T$ is measure-preserving.

Each orbit of $R$ is a discrete subset of a manifold (the corresponding leaf).

With an appropriate triangulation of the leaves, you get a "simplicial complex over $R$"

$C_k = \text{span of } k\text{-chains in the leaves}$

$C_k$ is acted on by $A$ (holonomies)

These $C_k$'s roughly correspond to these ones
Example

\( C^0 \in L(\Gamma) \)

\( L^\infty(X) \times_{alg} \Gamma \subseteq L^\infty(X) \times \Gamma \)

\( A \subseteq L^\infty(X) \times \Gamma \)

\( M \otimes M^\circ \subseteq M \otimes M^\circ \)

Q. Is \( \dim_{M \otimes M^\circ_{k}} H^0_k(M \otimes M^\circ, L^2(M \otimes M^\circ)) \) something reasonable? (Open)

Do know \( \dim_{M \otimes M^\circ_{k}} H^0_k(M \otimes M^\circ, M \otimes M^\circ) = 0 \) unless \( M \)'s summands are matrix algebras.

(A. Thom)

\( \beta_{i}^{(2)} = \dim_{M \otimes M^\circ} (\text{derivations } M \rightarrow \text{Aff}(M \otimes M^\circ)) \)

Unfortunately nontrivial examples are lacking...
1. If \( X_1, \ldots, X_n \) generate \( L(\mathbb{F}_n) \) are group generators

\[
\exists \delta_j : \mathbb{A}^{X_i \rightarrow X_n} \rightarrow M(L(\mathbb{F}_n))
\]

\[
\delta_j(X_i) = \delta_{i,j}
\]

(extend from group)

\[
\dim_{\mathbb{M}_{n \times n}} \text{Der}(\mathbb{A}^{X_1, \ldots, X_n} / \mathbb{A}^{X_1, \ldots, X_n}) = n.
\]

2. \( Y_1, \ldots, Y_m \in \mathbb{W}^+(X_1, \ldots, X_n) \)

\[
\dim_{\mathbb{M}_{n \times n}} \text{Der}(\mathbb{A}^{Y_1, \ldots, Y_n} / \mathbb{A}^{Y_1, \ldots, Y_n}) = m.
\]
To answer "yes" want to do:

\[ V(X_1, \ldots, X_n, Y_1, \ldots, Y_m) = \left\{ (S(X_1), \ldots, S(X_n)) \in L^2(M) \otimes L^2(M) : S \in \text{Der}(M \otimes \mathbb{C}, L^2(M)), X_1, \ldots, X_n, Y_1, \ldots, Y_m \in M \right\} \]

\[ \Delta(X_1, \ldots, X_n, Y_1, \ldots, Y_m) = \dim_{M \otimes \mathbb{C}} V(X_1, \ldots, X_n, Y_1, \ldots, Y_m) \]

\[ \dim_{M \otimes \mathbb{C}} (\text{Der}(M, \text{Aff}(M \otimes \mathbb{C}))) = \sup_{n, X_1, \ldots, X_n} \inf_{n, Y_1, \ldots, Y_m} \Delta(X_1, \ldots, X_n, Y_1, \ldots, Y_m) \]

Want lower bound on \( \Delta \).

**Theorem**

\[ L(F_n) = W^*(X_1, \ldots, X_n) \]

\text{free semicircular family}

\[ \Rightarrow \Delta(X_1, \ldots, X_n, Y_1, \ldots, Y_m) \geq n \quad \forall Y_1, \ldots, Y_m. \]

\textbf{Pf Sketch}

1. \( \Delta(X_1, \ldots, X_n) \geq S^*(X_1, \ldots, X_n) \)
2. \( S^*(X_1, \ldots, X_n) \geq S(X_1, \ldots, X_n) \)
3. \( S(X_1, \ldots, X_n) \geq S_0(X_1, \ldots, X_n) \)
4. \( S_0(X_1, \ldots, X_n, Y_1, \ldots, Y_m) \geq S_0(X_1, \ldots, X_n) \)
   if \( Y_1, \ldots, Y_m \in W^*(X_1, \ldots, X_n) \).
So what are these things?

\[ \mathcal{S}_0(Z_1, \ldots, Z_n) = \text{"Free entropy dimension"} \]

\[ \Gamma(Z_1, \ldots, Z_n, k, l, \varepsilon) = \begin{cases} \{ (z_1, \ldots, z_n) \in (M_{k,k}^{st})^n \} & \text{s.t.} \\ \left| \frac{1}{k} \text{Tr}(\rho(z_1, \ldots, z_n)) - \mathcal{C}(\rho(z_1, \ldots, z_n)) \right| < \varepsilon \\ \forall p \text{ monomial of degree } \leq l \end{cases} \]

\[ \mathcal{S}_0(Z_1, \ldots, Z_n) = \limsup_{s \to 0} \inf_{l, \varepsilon} \limsup_{k \to \infty} \frac{1}{k} \log K_s(\Gamma(Z_1, \ldots, Z_n), k, l, \varepsilon) \]

Let \( K_s(X) = \min \# s \)-balls required to cover \( X \)

\[ \limsup_{s \to 0} \frac{\log K_s(X)}{\log s} = \text{"covering dimension" of } X \]

Theorem 1.25: (e.g. use HW from 2594) that

\[ \mathcal{S}_0(X_1, \ldots, X_n) = n \text{ if } X_1, \ldots, X_n \text{ semicircular.} \]

Now \( \Gamma(x, \Gamma(X_1, \ldots, X_n, Y_1, \ldots, Y_n)) \subseteq \Gamma(X_1, \ldots, X_n) \)

In fact they're equal (use the fact that \( Y_i \in \mathbf{W}^2(X_1, \ldots, X_n) \)

so they can be approximated by polynomials in \( X_1, \ldots, X_n \)).

Then you just use the fact that projections cannot increase covering numbers.
Free entropy is
\[
\chi(A_1, \ldots, A_n) = \inf_{k \to \infty} \limsup_{k \to \infty} \frac{1}{k^2} \log \text{Vol} \left( (A_1, \ldots, A_n), k, k^2 + \frac{n}{2} \log k \right)
\]

\[
\delta(Z_1, \ldots, Z_n) = n - \limsup_{t \to 0} \frac{\chi(Z_1 + tS_1, \ldots, Z_n + tS_n)}{t^2 \log t + \frac{n}{2}}
\]

\(S_1, \ldots, S_n\) free semicircular, free from \(Z_1, \ldots, Z_n\)

\(\delta \geq \delta_0\)

Use L'Hopital to get
\[
n - \lim_{s \to 0} \frac{d}{ds} \chi(Z_1 + sS_1, \ldots, Z_n + sS_n)
\]

\(\frac{1}{4E}\)

How to compute this derivative?

\(\Omega = \text{Alg } (A_1, \ldots, A_n)\)

\(\partial_j : L^2(\Omega) \to L^2(\Omega) \otimes L^2(\Omega)\) densely defined on \(\Omega\)

\(\partial_j(A_i) = S_{ij} |1|\)

If \(|1|\) in domain \(\partial_j^+\)

\(\Theta^+(A_1, \ldots, A_n) \overset{\text{def}}{=} \sum_{j=1}^{n} \| \partial_j^+ (|1|) \|_2^2\) "free Fisher information"

Th (Biane/Captain/Guionnet)

\[
\frac{d}{ds} \chi(Z_1 + sS_1, \ldots, Z_n + sS_n) \geq \Theta^+(Z_1 + tS_1, \ldots, Z_n + tS_n)
\]

This is a hard theorem.
Define
\[ S^*(Z_i, \ldots, Z_n) = \limsup_{t \to 0} \Phi^*(Z_i + tS_i, \ldots, Z_n + tS_n) \]

Then (2) in the proof works.

\[ S^* (X_1, \ldots, X_n) = n \]

free semicircular.

\[ X_j = l_j + l_j^* \]

\[ l_j = \text{left creation ops on } \bigoplus \left( \mathbb{C}^n \right)_{\text{DO}} \]

\[ r_j = \text{right creation operator} \]

\[ [r_j, l_i] = 0 \quad [r_j, l_i^*] = -S_{ij} P \]

This is Hilbert-Schmidt (it's in \( L^2(M) \otimes L^2(M) \))

where \( M = W^*(X_1, \ldots, X_n) \)

\[ \partial_j (x) = [x, r_j] : L^2(M) \to L^2(M) \]

If \( \partial_j = [x_0, B_j] \) then \( \partial_j^d (1 \otimes x) = (B_j - iB_j^* i) \otimes \]

\[ W^*(Z_i + tS_i, \ldots, Z_n + tS_n) \leq W^*(Z_i, \ldots, Z_n, S_i, \ldots, S_n) \]

\[ \leq B \left( L^2\left( W^*(Z_i, \ldots, Z_n) \right) + L^2\left( W^*(S_i, \ldots, S_n) \right) \right) \]

\[ r_j \in B \left( L^2\left( W^*(S_i, \ldots, S_n) \right) \right) \]

\[ [r_j, x] = 0 \quad \forall x \in W^*(Z_i, \ldots, Z_n) \]

\[ [S_i, r_j] = S_{ij} P \]

\[ B_j^{\text{f}} = \frac{1}{t} E_{t} r_j E_{t} \]

\( Z \)'s satisfy algebraic relations \( \Rightarrow Z_j + tS_i \) almost satisfy them \( \Rightarrow \text{yay} \).