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An equivalence relation R on (X,μ) is <u>amenable</u> if

$\exists P: L^\infty(R,\mu_R) \longrightarrow L^\infty(X,\mu)$ s.t.

$P \geq 0$, and $\forall \alpha$ R-morphism,

$$P(\xi) = P(\lambda_\alpha \cdot \xi) \quad \text{on } \operatorname{dom} \alpha.$$

Such P sometimes called "invariant mean".

<u>Følner-like conditions for equivalence relations</u>

A subset $K \subset R$ is <u>bounded</u> if

$$\sup_{x_0 \in X} \# \{(x,y) \in K : x = x_0\} < \infty$$

and $$\sup_{y_0 \in X} \# \{(x,y) \in K : y = y_0\} < \infty$$

i.e. K has ~~finitely~~ boundedly many points from each orbit.

(Corresponds to F_n finite in amenability of groups)

<u>Lemma</u>

R amenable $\Rightarrow$ $\forall \epsilon$ $\forall R$-morphisms $\alpha_1, \dots, \alpha_n$

$\exists f \in L^1(R,\mu_R)$, $f \geq 0$, $f \not\equiv 0$,

$$\sum_{j=1}^n \|\lambda_{\alpha_j} f - f\|_1 \leq \epsilon \|f\|_1.$$

See proof for groups — can copy pretty much word for word. $\square$

<u>Lemma</u>

R amenable $\Rightarrow$ $\forall \alpha_1, \dots, \alpha_n$ R-morphisms, $\forall \epsilon > 0$, $\exists f$ valued in $\{0,1\}$

s.t. $$\sum_i \|\lambda_{\alpha_i} f - f\|_1 \leq \epsilon \|f\|_1.$$

(Hence f is the characteristic function of some bounded K.)

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Pf

$E_a = \chi_{[a,\infty)}$ $\qquad t = \int_0^\infty E_a(t)\,da$ $\qquad |t-t'| = \int_0^\infty |E_a(t) - E_a(t')|\,dt$

$f_a = E_a \circ f$

$$\sum_i \iint_y |f_a(y,\alpha_i(x)) - f_a(y,x)|\,da\,d\mu_R(x,y) < \iint f_a\,da\,d\mu_R(x,y)$$

- Fubini
- "True on average $\Rightarrow$ true somewhere". $\square$

$K \subseteq R$ bdd $\Rightarrow$ $K \subseteq \bigcup$ (graphs of finitely many automorphisms)

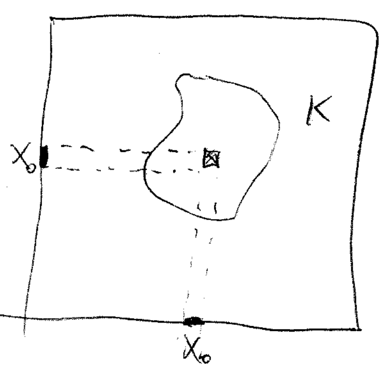
"$\forall S \subseteq R$ bdd $\exists R_0 \subseteq R$ bdd s.t. R_0 is ε-invariant under S"

Lemma R amenable $\Rightarrow \forall K \subseteq R$ bdd measurable,
$\forall \varepsilon > 0$, $\exists T \subset R$ finite equiv sub-relation s.t.

$\mu\{(x,y) \in K$ but $(x,y) \notin T$, but $x \in dom(T)$ or $y \in range(T)\}$
$\leq \varepsilon \mu(dom(T))$

Cleaner notation: T defined on $X_0 \subseteq X$

$\mu\{(x,y) \in K, (x,y) \notin T,$ but $x \in X_0$ or $y \in X_0\} \leq \varepsilon \mu(X_0)$.



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We can replace K by the union of graphs of some R-morphisms $\alpha_1, \dots, \alpha_n$

- Choose $f: R \to \{0,1\}$ as in lemma so

$$\sum_i \int_{(x,y)\in R} |f(y,\alpha_i(x)) - f(y,x)|\, d\mu_R < \varepsilon \int_{(x,y)\in R} f(y,x)\, d\mu_R(x,y)$$

$$\sum_i \int_{y\in X} \sum_{(x,y)\in R} |f(y,\alpha_i(x)) - f(y,x)|\, d\mu(y) < \varepsilon \int_y \sum_{(x,y)\in R} f(y,x)\, d\mu(y)$$

"True on avg
∴ true somewhere"

$\Rightarrow$ $\exists\, \Omega \subseteq X$ $\mu(\Omega) \neq 0$ s.t. $\forall y \in \Omega$,

$$\sum_i \sum_{(x,y)\in R} |f(y,\alpha_i(x)) - f(y,x)| < \varepsilon \sum_{(x,y)\in R} f(y,x)$$

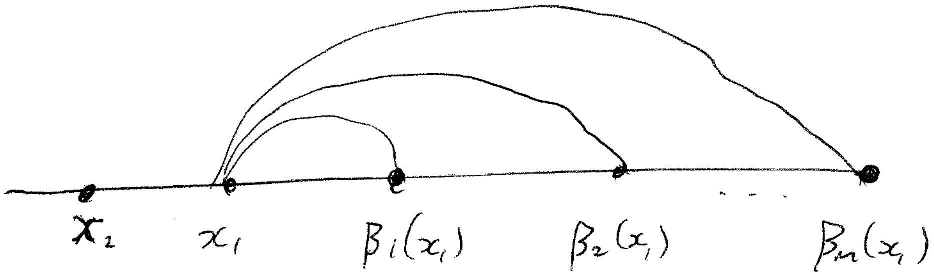
- $K' = \{(x,y) \in R : f(x,y) = 1\}$ $\quad f = \chi_{K'}$

K' bdd $\quad K_1 = (K')^{-1} \cup K'$

Choose $\Omega' \subset \Omega$, $\mu(\Omega') > 0$ s.t.

$\forall x_1, x_2 \in \Omega'$, $x_1 \neq x_2$, $\{y : (x_1,y) \in K_1\}$ and $\{y : (x_2,y) \in K_1\}$

are disjoint.



- Can make x_2 miss all these. Use Fubini and a "true on avg $\Rightarrow$ true somewhere" argument.

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$\forall y \in \Omega'$, let $F_y = \{x \in X : (y,x) \in R \text{ and } f(y,x) = 1\}$.

$= \text{orbit of } Y \cap K'$.

Choice of Ω' $\Rightarrow$ $F_{y_1} \neq F_{y_2}$ if $y_1 \neq y_2$

Now define $T \subset R$ equiv. rel. over $X_0 = \Omega'$
by "orbits of T are the sets F".

$T = \{(x,y) : x \in \Omega' \text{ and } y \in F_x\}$

$B_i(y) = \{x \in \operatorname{dom} \alpha_i \text{ s.t. } \alpha_i(x) \in F_y \text{ but } x \notin F_y\}$
$\quad \cup \{x \in \operatorname{dom} \alpha_i \text{ s.t. } \alpha_i(x) \notin F_y \text{ but } x \in F_y\}$ "error set"

$\forall y \in \Omega'$,
$$\sum_i \sum_{B_i(y)} 1 = \sum_i \sum_{x \in B_i(y)} |f(y,x) - f(y, \alpha_i(x))|$$

$$< \varepsilon \sum_{x \in R} f(x,y) = \varepsilon \sum_{x \in F_y} 1$$

Conclude that
$\forall y \in \Omega' = X_0$, $\sum_i |B_i(y)| < \varepsilon |F_y|$.

Let $B_i = \bigcup_{y \in \Omega'} B_i(y)$ "total misbehavior"

$$\Rightarrow \sum_i \mu(B_i) \leq \varepsilon \mu\Big(\bigcup_{y \in \Omega'} F_y\Big)$$

Questionable... Have
$\mu(\cup) \leq \Sigma\mu$
$\leq \Sigma\mu$
$\cancel{\leq \varepsilon}\,\mu(\cup)$
??

Resume next time ...

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From last time:

$K \subseteq R$ bdd subset $\qquad K = \bigcup$ (graphs of $\alpha_1, \ldots, \alpha_n$)

$\Omega' \subseteq X \qquad K_1 \subseteq R$ bdd subset

$\forall j \qquad \lambda_{\alpha_j} \cdot K_1 \cap K_1$ is large compared to K_1.

$\forall \varepsilon > 0$ $\exists T$ ~~finite eq. rel.~~ $\underset{\text{supp } f}{}$ s.t. $\mu_R((x,y) : (x,y) \in K$ but not $T) < \varepsilon \mu(\Omega')$

~~More precisely,~~ WTS let $\quad F_y = \{x : (x,y) \in R$ s.t. $(x,y) \in \operatorname{supp} f\}$

$B_y = \{x$ s.t. either $x \in F_y$ but $\exists_j$ with $\alpha_j(x) \notin F_y$
or $x \notin F_y$ but $\exists_j$ with $\alpha_j(x) \in F_y\}$

can split into

$B_i(y) = \{x : \quad x \in F_y,\ \alpha_i(x) \notin F_y$
or $x \notin F_y,\ \alpha_i(x) \in F_y\}$

$\forall y \in \Omega'$; $\qquad \sum_i |B_i| < \varepsilon |F_y|$

$\forall y \neq y' \in \Omega'$; $\qquad K \cdot y$ and $K \cdot y'$ disjoint

$K_0 \subseteq R \qquad K_0 \cdot y = \{x : (y,x) \in K_0\}$

Let $\quad B_i = \bigcup_{y \in \Omega'} B_i(y) \qquad \sum_i \mu(B_i) \leq \varepsilon \mu(\Omega')$

T — equiv rel: $\quad x \sim x'$ if $x, x' \in F_y$ for some y $\quad$ (finite equiv. rel.)

Suppose $(x,y) \in K$, $(x,y) \notin T$, $x \in \Omega'$, $y \in \Omega'$
$\Rightarrow \exists \alpha_j \qquad y = \alpha_j(x) \qquad \Rightarrow x \in B_i(y)$ or $y \in B_i(x)$. $\square$

Next we want to patch together the local estimates from this lemma to get some kind of global theory.

Lemma R amenable $\Rightarrow$ $\forall K \subseteq R$ bdd measurable, $\exists T$ finite equiv rel, $T \subset R$, s.t. $\mu_R(K \setminus T) < \varepsilon$.

PF We can assume $\text{proj}_x(K) \cup \text{proj}_y(K) = X$.

Consider triples (T, Y, H), $T \subset R$ finite sub-eq. rel. over X_0, $H \subseteq K$ subset

s.t. (1) $\cancel{\mu_R(H \setminus T) < \varepsilon\, \mu(Y)}$ $\mu_R\big((Y \times Y) \cap (H \setminus T)\big) < \varepsilon\, \mu(Y)$

(2) If $(x, y) \in H$ and either $x \in Y$ or $y \in Y$ then $(x, y) \in T$.

By the previous lemma, the set of such triples is nonempty. It can be partially ordered by inclusion, and unions are upper bounds, so Zorn guarantees a maximal triple (T, Y, H).

If $Y \neq X$, can enlarge Y by gluing in another finite eq. rel. obtained by approximating $K \cap (Y' \times Y')$ over Y', $Y' \subseteq X \setminus Y$.

Hence the maximal triple is (T, X, H). $\square$

Theorem R ergodic and amenable $\Rightarrow$ R hyperfinite

Pf Choose $K_n \subseteq R$ increasing seq. of bdd sets, s.t. $R = \bigcup K_n$

Choose $\varepsilon_n > 0$, $\sum_n \varepsilon_n < \infty$, and choose $T_n \subset R$

s.t. $\mu_R(K_n \setminus T_n) < \varepsilon_n\, \mu_R(K_k \setminus T_k)$ for $k \geq n$.

Let $T^n = \bigcap_{k \geq n} T_k$ $\quad T^n \subseteq T^{n+1}$

For $m \geq n$, $\mu(K_n \setminus T^m) \leq \sum_{k \geq m} \mu(K_n \setminus T_k) \leq \sum_{k \geq m} \varepsilon_k \to 0$ as $m \to \infty$.

$\square$

This concludes the proof that hyperfiniteness is equivalent to amenability.

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Theorem If $R = R_\Gamma$ for a free action of Γ on X, then R amenable $\Leftrightarrow$ Γ amenable.

Pf
$(\Leftarrow)$ done

$(\Rightarrow)$ $P : L^\infty(R,\mu) \longrightarrow L^\infty(X,\mu)$ invariant mean

$$L = \mu \circ P \qquad L : L^\infty(R) \longrightarrow \mathbb{C}$$

$$\ell^\infty(G) \longrightarrow L^\infty(R)$$

$$\xi \qquad \hat{\xi}(x, \alpha_g(x)) = \xi(g)$$

$L : \ell^\infty(G) \longrightarrow \mathbb{C}$ is an invariant mean on Γ.

Theorem R amenable $\Rightarrow$ if $R_0 \subseteq R$ sub-eq rel, then R_0 amenable.

~~Pf~~ Easy. Just restrict the invariant mean.

Thm R hyperfinite, $R_0 \subseteq R$ $\Rightarrow$ R_0 hyperfinite.

Pf Hard to do directly! This is why amenability is helpful.

Fact $\mathbb{F}_n$ not amenable.

One way: Calculate spectral radius (use stuff from last quarter)

Another way: Write $w \in \mathbb{F}_n$ $\quad w = g_{i_1}^{\pm a_1} \cdots g_{i_k}^{\pm a_k}$ $\quad i_1, \dots, i_k \in \{1, \dots, n\}$

$a_j \in \{1, 2, \dots\}$

$k \geq 0$

$i_1 \neq i_2,\ i_2 \neq i_3, \dots$

$A_1 = \{w \text{ start with } g_1^{-1}\}$ (not g_1^{-2})

$A_2 = \{w \text{ starts with } g_2^{-1}\}$ (not g_2^{-2})

$$G = A_1 \cup A_2 \cup g_1 A_1 \cup g_2 A_2 \qquad \text{disjoint}$$

$$f_i = \chi_{A_i} \qquad 0 \leq f_1 + f_2 + f_1(g_1) + f_2(g_2) - 1$$

$$P(f_1 + f_2 + f_1(g_1) \cdots) = -1. \qquad \Rightarrow \Leftarrow$$

inv. mean

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Recall

Amenability of a group Γ: (gen. by $S = \langle g_1, \dots, g_n \rangle$)

- Følner condition: $\exists F_n$ finite, $\frac{|gF_n \triangle F_n|}{|F_n|} \to 0$
- $\exists\, \rho : \ell^\infty(\Gamma) \to \mathbb{C}$ invariant mean
- $\sum_{g \in S} (1-\lambda_g)(1-\lambda_g)^*$ has an approximate kernel
  acting on $\ell^2(\Gamma)$
- $\exists\, \xi_n \in \ell^2(\Gamma)$, $\|\xi_n\| = 1$, $\|\lambda_g \xi_n - \xi_n\| \to 0 \quad \forall g$.
- $L = \sum_{g \in S} \lambda_g + \lambda_g^{-1}$ has $\|L\| = \rho(L) = 2|S|$

We'd like to add another condition.

Full/reduced C^* algebra of a group

- Given discrete group Γ,

$$C^*_{full}(\Gamma) = \overline{\mathbb{C}\Gamma}^{\|\cdot\|_{full}}$$

where $$\left\|\sum a_i g_i\right\|_{full} = \sup_{\substack{\pi \text{ unitary rep of } \Gamma \\ \pi : \Gamma \to \mathcal{U}(\mathcal{H}_\pi)}} \left\|\sum a_i \pi(g_i)\right\|_{\mathcal{B}(\mathcal{H}_\pi)}$$

$$C^*_{red}(\Gamma) = \overline{\mathbb{C}\Gamma}^{\|\cdot\|_\lambda}$$

where $$\left\|\sum a_i g_i\right\|_\lambda = \left\|\sum a_i \lambda_{g_i}\right\|_{\mathcal{B}(\ell^2(\Gamma))}$$

- Difference is taking one rep'n vs. a sup over all rep'ns.

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Quick Observations

- $\|\cdot\|_{Full} \geq \|\cdot\|_{\lambda}$
- $\pi : \Gamma \longrightarrow U(H_\pi)$ rep induces

$$C^*(\pi(\Gamma)) \longleftarrow C^*_{full}(\Gamma) \quad \text{surjection}$$

- $\alpha : \Gamma \longrightarrow B(\mathbb{C})$ $\quad \alpha(g) = 1 \quad$ trivial rep

$\alpha : C^*_{full}(\Gamma) \longrightarrow \mathbb{C} = B(\mathbb{C})$ $\quad$ $*$-hom

Kernel has codimension 1 $\Rightarrow$ $C^*_{full}(\Gamma)$ not simple if Γ nontrivial

---

Using this, another condition for amenability is

- $C^*_{Full}(\Gamma) \longrightarrow C^*_{rel}(\Gamma)$ is injective.

OR: - $\|\cdot\|_{Full} = \|\cdot\|_{\lambda}$.

equivalent

Cor Γ amenable $\Rightarrow$ $\|\sum a_j \lambda_{g_j}\| \geq |\sum a_j| = \|\sum a_j \alpha(g_j)\|_{B(\mathbb{C})}$

Pf $\|\sum a_j \lambda_{g_j}\| \geq \langle \sum a_j \lambda_{g_j} \xi_n, \omega \xi_n \rangle \approx \langle \sum a_j \xi_n, \omega \xi_n \rangle$

$|\omega| = 1$ $\quad = \sum a_j \bar{\omega}$.

This compares the left-regular-rep norm to the trivial-rep norm.

In order to show that these are equivalent to amenability use the last condition from before (with $\|L\| = 2|S|$).

- $\|\ \|_{full} = \|\ \|_{\lambda}$ clearly implies $\|L\| = 2|S|$.
- Reverse: $\forall \pi$ unitary rep of Γ $\quad \pi \otimes \lambda = \lambda^{\oplus \infty}$

$H_\pi \otimes L^2(\Gamma) \qquad g \cdot (h \otimes \xi) = \pi(g) h \otimes \lambda_g \xi$

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<u>Claim</u> $\exists$ isometry $V: \mathcal{H}_\pi \otimes \ell^2(\Gamma) \hookrightarrow \mathcal{H}_\pi \otimes \ell^2(\Gamma)$

s.t. V intertwines $\pi \otimes \lambda$ and $id \otimes \lambda$.

<u>Ex</u> $\Gamma = \mathbb{Z}$ $\quad \mu$ = any measure on $\mathbb{T}$ $\quad \lambda$ = Haar measure

$$\mu * \lambda \ll \lambda \qquad \text{actually, } \mu * \lambda = \lambda\ ?$$

Tensor product here corresponds to convolution of measures.

$$\mathcal{H}_\pi \otimes \ell^2(\Gamma) = \ell^2(\Gamma; \mathcal{H}_\pi)$$

$$\left((\pi \otimes \lambda)_h \xi\right)(g) = \pi_h\left(\xi(h^{-1}g)\right)$$

$$\left((id \otimes \lambda)_h \xi\right)(g) = \xi(h^{-1}g)$$

$$V: \eta \otimes \delta_g \longmapsto \pi(g)\eta \otimes \delta_g \qquad \eta \in \mathcal{H}$$

$$(id \otimes \lambda) V = V(\pi \otimes \lambda) \quad \text{as claimed.}$$

<u>Cor</u> $\| \sum a_i (\pi \otimes \lambda)(g_i) \| \le \| \sum a_i \lambda(g_i) \|$

Pf $\| \sum a_i \pi(g_i) \| \approx \langle \sum a_i \pi(g_i) \xi, \xi \rangle$, $\quad \|\xi\| = 1$, $\xi \in \mathcal{H}_\pi$

$$\left\langle \sum a_i \pi(g_i) \otimes \lambda(g_i)\, \xi \otimes \xi_n,\ \xi \otimes \xi_n \right\rangle \xrightarrow{n \to \infty} \left\langle \sum a_i (\pi(g_i)\xi) \otimes \xi_n,\ \xi \otimes \xi_n \right\rangle$$

$$\approx \| \sum a_i \pi(g_i) \|$$

But $\| \sum a_i \lambda(g_i) \| \ge \left\langle \sum a_i \pi(g_i) \otimes \lambda(g_i)\, \xi \otimes \xi_n,\ \xi \otimes \xi_n \right\rangle$. $\square$

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Recall Γ amenable $\Leftrightarrow$ $\sum_i (\lambda_{g_i} - 1)(\lambda_{g_i} - 1)^*$ has approx. kernel.

So Γ not amenable $\Leftrightarrow$ " " strictly positive,

i.e. $\geq \varepsilon \cdot id$ for some $\varepsilon > 0$,

Let $Q \in \mathbb{C}\Gamma$ be the element $Q = \sum (\lambda_{g_i} - 1)(\lambda_{g_i} - 1)^*$.

Γ non-amenable $\Leftrightarrow \lambda(Q) \geq \varepsilon I \Leftrightarrow \lambda(Q)$ invertible in $B(\ell^2(\Gamma))$

~~Ex $\pi = \lambda\rho^{-1}$~~

For a rep'n π of Γ, want to ask, Does $\pi(Q)$ have non-trivial approx. kernel?

$\ker \pi(Q) = \{ h \in H \text{ s.t. } \pi(g)h = h \quad \forall g \in \Gamma \}$

Is 0 an isolated point in $\sigma(\pi(Q))$? "π has spectral gap"

Ex $\pi = \rho\lambda^{-1}$

π has spectral gap $\Leftarrow$ Γ not inner amenable

Ex $\Gamma = \mathbb{F}_2 \times S_\infty$

$\rho\lambda^{-1}$ has no spectral gap

Def
Γ has property (T) if Q has a spectral gap in any representation.

using this,

Γ has (T) $\Rightarrow L(\Gamma)$ is non-Γ.

Γ has (T) and amenable $\Rightarrow$ Γ is finite

$\sigma(\lambda(Q)) \cap (0,\varepsilon) = \{0\}$ $\quad$ $\sigma(\lambda(Q)) \cap [0,\varepsilon) \neq \emptyset$

together imply $0 \in \sigma(\lambda(Q))$

$\Rightarrow \ell^2(\Gamma)$ has inv. vector.

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Examples

$L(R)$ R hyperfinite

$L(\mathbb{F}_n)$ not hyperfinite (non-Γ)

$L(\Gamma)$ Γ has (T)

How to distinguish? Next time ...

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PROPERTY (T) STUFF

$L(\Gamma) \subseteq L(\mathbb{F}_m)$ if Γ has (T) — we'll prove this

- Observe: If π: unitary rep of Γ on $\mathcal{H}_\pi$ with cyclic vector ξ

$$\varphi_\pi : \Gamma \to \mathbb{C} \qquad \varphi_\pi(g) = \langle \pi(g)\xi, \xi\rangle$$

$$\varphi_\pi(g^{-1}h) = \langle \pi(g^{-1}h)\xi, \xi\rangle = \langle \pi(g)^*\pi(h)\xi, \xi\rangle$$
$$= \langle \pi(h)\xi, \pi(g)\xi\rangle$$

$g_1, \dots, g_n \in \Gamma$

Matrix $[\varphi_\pi]_{ij} = \varphi_\pi(g_i^{-1}g_j) = \langle \underbrace{\pi(g_j)\xi}_{\eta_j}, \underbrace{\pi(g_i)\xi}_{\eta_i}\rangle$ is pos. def.

$$[\langle \eta_i, \eta_j\rangle]_{ij} \geq 0$$

φ_π is the "positive definite function associated to π"

Ex
$\Gamma = \mathbb{Z}$ $\qquad \mathcal{H}_\pi = L^2(\mathbb{T}, \nu) \qquad \mathbb{T} = [0, 2\pi]$

$$(\pi(n)f)(\theta) = e^{in\theta} f(\theta)$$

$$\varphi_\pi(n) = \int_0^{2\pi} (\pi(n)\cdot \mathbb{1})(\theta)\cdot \mathbb{1}(\theta)\, d\nu(\theta) = \hat{\nu}(n)$$

This is the well-known characterization of which functions $\varphi(n)$ are Fourier transforms of positive measures: $[\varphi_\pi(n-m)]$ must be positive definite.

Going backwards: Given $\varphi : \Gamma \to \mathbb{C}$ s.t. $\forall g_1, \dots, g_n \quad [\varphi(g_i^{-1}g_j)] \geq 0 \quad (*)$

Can define a vector space H and rep π by

$$\mathcal{H}_0 = \text{span}\{g : g \in \Gamma\} \qquad \langle g, h\rangle = \varphi(g^{-1}h)$$

$(*) \Rightarrow \langle\ ,\ \rangle$ nonnegative

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$$\langle \sum \alpha_i g_i, \sum \alpha_j g_j \rangle = \sum \alpha_i \bar{\alpha}_j \varphi(g_i^{-1} g_j)$$

$$= \left\langle [\varphi(g_i^{-1}g_j)]_{ij} \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}, \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} \right\rangle$$

H_0 has a rep of Γ leaving $\langle \, , \rangle$ invariant

$\longrightarrow$ unitary rep π of Γ on $\overline{H_0 / \ker \langle \, , \rangle}^{\langle \, , \rangle}$

$\xi = [e]$ $\qquad \varphi_\pi = \varphi$.

So pos. def. fns $\longleftrightarrow$ rep'ns.

EXERCISE

If π is a unitary rep'n of Γ, π is a sub-rep of a power of the left regular rep $\Leftrightarrow$ $\varphi_\pi \in \ell^2$ $\quad \forall \xi \in H_\pi$.

- If ξ contains a Γ-invariant vector, then $\exists \xi$ cycle s.t. $\varphi_\pi = \langle \pi(g)\xi, \xi \rangle$ satisfies $\varphi_\pi(g) \geq c > 0$ $\quad \forall g \in \Gamma$.

$\pi = \pi_1 \oplus \pi_2$ $\qquad \pi_1 =$ trivial rep $\qquad \pi_2 =$ some rep

$\xi = \xi_1 \oplus \xi_2$ $\qquad \langle \pi(g)\xi, \xi \rangle = \|\xi_1\|^2 + \underbrace{\langle \pi(g)\xi_2, \xi_2 \rangle}_{|\ | \leq \|\xi_2\|}$

Property (T): $\exists F \subseteq \Gamma$ finite set, $\varepsilon > 0$ s.t. $\forall$ unitary rep $\pi : \Gamma \to H_\pi$, if $\|\pi(g)\xi - \xi\| \leq \varepsilon$ $(g \in F)$, $\|\xi\| = 1$, then $\exists \xi_0$ s.t. $\pi(g)\xi_0 = \xi_0$.

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$x \in M$, x M-bdd $\Rightarrow$ $E(x)$ is N-bdd.

$$\|E(x)\cdot n\|_2 = \|E(xn)\|_2 \le \|xn\|_2 \le C\|n\|_2$$

$\therefore$ E takes M to N-bdd vectors.

Why is E positive?

$E(x^*x) \in N$ so STP $\langle E(x^*x)\xi, \xi\rangle \ge 0 \quad \forall \xi \in L^2(N)$

$$\langle E(x^*x)\xi, \xi\rangle = \langle E(x^*x\xi), \xi\rangle = \langle x^*x\xi, \xi\rangle.$$

Completely positive because $E_n = E^{M_{n\times n}(M)}_{M_{n\times n}(N)}$ conditional expectation.

Ex Compositions of CP maps are CP.

(One common source of CP maps is to compose automorphisms with conditional expectations.)
In fact, it's hard to come up with an example not of this form!

Next Time

Bimodules vs CP maps

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Last time Completely positive maps $\eta : A \to B$ (C^* algs)

Fact There is a 1-1 correspondence between
(1) c.p. maps $\eta : A \to B$
(2) pairs $\mathcal{H}$, $\xi \in \mathcal{H}$ where $\mathcal{H}$ is a B-Hilbert module with a B-linear action.

Def $\mathcal{H}$, $\langle , \rangle$ is called a B-Hilbert module if
- $\mathcal{H}$ is a linear space and a right B-module
- $\langle , \rangle : \mathcal{H} \otimes \mathcal{H} \to B$ is a map which is linear and satisfies
  - $\langle x, yb \rangle = \langle x, y \rangle b$
  - $\langle x, y \rangle = \langle y, x \rangle^*$
  - $\langle x, x \rangle \geq 0$, $\langle x, x \rangle = 0 \Leftrightarrow x = 0$
- $\mathcal{H}$ is a Banach space under $\|x\| = \sqrt{\langle x, x \rangle}$

Examples of Hilbert Modules
- Let $B = C(X)$, $E \to X$ a vector bundle, g a metric on E
  $\Gamma(E)$ (space of continuous sections) is a $C(X)$-Hilbert module

$$\langle \xi, \eta \rangle (x) = g_x (\xi(x), \eta(x)).$$

One can prove conversely that any vector bundle looks like this.

Def Given a B-Hilbert module H, an operator $T : H \to H$ is called
- B-linear if $T(xb) = (Tx)b$
- adjointable if $\exists\, T^* : H \to H$ s.t. $\langle T^*x, y \rangle = \langle x, Ty \rangle$ $\forall x, y \in \mathcal{H}$.
  (Note that adjointable $\Rightarrow$ B-linear.)

Denote $\mathcal{M}(H) = \{T : H \to H \text{ adjointable}\}$. "multiplier algebra"

Fact Adjointable $\Rightarrow$ bounded but converse is false.

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So why are (1) and (2) equivalent?

(2 ⇒ 1) Given $\xi \in H$, $\underbrace{\lambda : A \to \mathcal{L}(H)}$

define $\eta(a) = \langle \xi, \lambda(a)\xi \rangle$ ~~$\langle \lambda(a)\xi, \xi \rangle$~~ $\in B$.

B-linear action

we'll write as $a\xi$

Claim η is c.p.

If $a \geq 0$, $a = y^* y$,

$\eta = \langle y^* y \xi, \xi \rangle = \langle y\xi, y\xi \rangle \geq 0$

~~More generally, let $H_n = H^{\oplus n}$ viewed as an $M_{n \times n}(B)$-module. A acts on H_n.~~
~~Given $A = (a_{ij})$, $A = Y^* Y$, $Y = (y_{ij})$~~

Let $H_n = M_{n \times n}(H)$ viewed as an $M_{n \times n}(B)$-module.

$$\langle \underbrace{[\xi_{ij}]}_{\Xi}, \underbrace{[\zeta_{ij}]}_{Z} \rangle = \Big(\underbrace{\sum_k \langle \xi_{ki}, \zeta_{kj} \rangle}_{\Xi^* Z} \Big)_{ij}$$

$M_{n \times n}(A)$ acts by left multiplication

$$\left\langle \begin{bmatrix} \xi & & \\ & \ddots & \\ & & \xi \end{bmatrix}, (a_{ij}) \begin{bmatrix} \xi & & \\ & \ddots & \\ & & \xi \end{bmatrix} \right\rangle \quad \text{pos.}$$

Now how do we go $\underline{1 \Rightarrow 2}$?

Given $\eta : A \to B$, let $H_0 = A \otimes B$ $\quad a\left(\sum a_i \otimes b_i\right) \cdot b = \sum a a_i \otimes b_i b$

$$\left\langle \sum a_i \otimes b_i, \sum a_j' \otimes b_j' \right\rangle = \sum_{i,j} b_i^* \, \eta(a_i^* a_j') \, b_j'$$

Let $\xi = 1 \otimes 1$

$$\langle 1 \otimes 1, a(1 \otimes 1) \rangle = \eta(a)$$

Claim $\langle x, x \rangle \geq 0$. Then we can take $\dfrac{H_0}{\{x : \langle x, x \rangle = 0\}}$

to get a Hilbert module.

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$\eta: M \to N$ normal c.p. map

$\Updownarrow$

unital
$M \hookrightarrow \tilde{N}$
$\tilde{N}$ = ampl. of N
$[\tilde{N} = N \otimes B(H)$ possible$]$

H Hilb. space, normal commuting actions of M and N on H

"M, N correspondence"
or "M,N bimodule"
or "M,N Hilbert bimodule"

Why are c.p. maps analogs of pos. def. fns on Γ?
" " M,N bimodules analogs of representations?

Fact $\pi: \Gamma \to \mathcal{U}(K)$ $\quad \mathcal{H} = K \otimes \ell^2(\Gamma)$

Γ acts on $\mathcal{H}$ via $\pi \otimes \lambda$
$\qquad\qquad\qquad\qquad \mathrm{id} \otimes \rho$

Both $\pi \otimes \lambda$ and $\mathrm{id} \otimes \rho$
$\qquad \cong \qquad\qquad\qquad \cong$
$\qquad \lambda^{\oplus \infty} \qquad\qquad \rho^{\oplus \infty}$

extend to a pair of commuting normal actions of $L(\Gamma)$, $L(\Gamma)^{op}$

$$g \cdot (k \otimes \xi) = \pi(g) k \otimes \lambda_g \xi$$
$$(k \otimes \xi) h = k \otimes \rho_h \xi$$

Can recover the original representation

$$\mathrm{span}\{ g(k \otimes 1) g^{-1} ; g \in \Gamma\} \simeq \mathrm{span}\{\pi(g)k : g \in \Gamma\}$$

$\rho^{-1}\lambda$ rep of Γ on $\mathcal{H}$ $\qquad \mathcal{H} = K \oplus$ (other stuff)

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C.P. map associated to $\xi = k \otimes 1$

$$\langle \cancel{\eta(g), h} \overset{h,\, \eta(g)}{} \rangle_{L^2(L(\Gamma))} = \langle \xi, g\xi h^* \rangle = \langle k\otimes 1, \pi(g)k \otimes gh^{-1} \rangle$$

$$= \delta_{g=h} \langle k, \pi(g)k \rangle = \delta_{g=h}\, \varphi(k)$$

where φ is the positive def. fn on Γ you started with.

How do we translate something like property (T)?

- Say K contains an invariant vector k: $\pi(g)k = k$

$\mathcal{H} = K \otimes \ell^2(\Gamma)$ $\qquad g(k\otimes\delta_e) = \pi(g)k \otimes \delta_g = (k\otimes\delta_e)g$

So fixed vectors in K translate to $L(\Gamma)$-central vectors.

$\mathcal{H}$ has almost invariant vectors if $\exists$ sequence ξ_k, $\|\xi_k\| = 1$

s.t. $\|x\xi_k - \xi_k x\| \to 0$ as $k \to \infty$. (May want to insist ξ_k bi-bounded too.)

A normal C.P. map $\eta : M \to M$ extends to a map $L^2(M) \to L^2(M)$.

Quick way: η extends to $L^1(M) \to L^1(M)$

$$\begin{array}{ccc} L^1(M) & \longrightarrow & L^1(M) \\ \| & & \| \\ M_* & & M_* \\ \varphi & \longmapsto & \varphi\circ\eta \end{array}$$

and by "interpolation" one obtains L^2 from L^1.

If we also require $\tau\circ\eta = \eta$ [note: this is equivalent to η^* unital) should be able to do even more easily using Cauchy-Schwarz or something. Proof next time.

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Let $\varphi : \Gamma \to \mathbb{C}$ pos. def. function

$\eta(g) = \varphi(g) \cdot g$ associated CP map

η extends to $\ell^2(\Gamma)$ as the operator of multiplication by φ

$\eta(\xi) = \varphi \cdot \xi \qquad \|\eta\|_{L^2 \to L^2} = \|\varphi\|_\infty \leq \varphi(e).$

__Observation__

Iff φ has the property $\varphi(g) \to 0$ as $g \to \infty$

(i.e. $\forall \varepsilon > 0 \;\; \exists F \subseteq \Gamma$ finite with $|\varphi| < \varepsilon$ on $\Gamma \setminus F$)

then $\eta_\varphi : \ell^2(\Gamma) \to \ell^2(\Gamma)$ is a compact operator.

__Def__

M has the __Haagerup property__ if $\exists$ sequence η_n of C.P. maps $M \to M$ s.t.

1) η_n unital τ-preserving
2) $\eta_n : L^2 \to L^2$ compact
3) $\|\eta_n(x) - x\|_2 \to 0 \quad \forall x \in M.$

__Fact__ If Γ has the Haagerup property, so does $L(\Gamma)$.

Choose $\varphi_n : \Gamma \to \mathbb{C}$ pos def, $\varphi_n(e) = 1$
can make $\varphi_n = \overline{\varphi_n}$

$\varphi_n(g) \to 0$ as $g \to \infty$ with n fixed

$\varphi_n(g) \to 1$ as $n \to \infty$ with g fixed

$t\ \eta_n = \eta_{\varphi_n}$

$\eta_n(g) - g = (\varphi_n(g) - 1)\, g \to 0$ as $n \to \infty.$

$\eta_n - id$ bdd and $\mathbb{C}\Gamma$ dense in $\ell^2(\Gamma) \to (3)$ above.

Get $\eta_n = \eta_n^*$ since φ_n real $\Rightarrow \tau$-preserving $\Rightarrow (1)$.

Compactness comes from the limiting behavior of φ_n.

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Claim: $L(\mathbb{F}_n)$ has the Haagerup property.

- Given H a real Hilbert space

$$H_{\mathbb{C}} = H \otimes_{\mathbb{R}} \mathbb{C} \qquad \text{"complexification"}$$

$$\mathcal{F}(H) = \underset{\substack{\nearrow \\ \text{"vacuum vector"}}}{\mathbb{C}\Omega} \oplus \bigoplus_{n \geq 1} H_{\mathbb{C}}^{\otimes n}$$

$$\ell(h)\xi = h \otimes \xi \qquad h \in H_{\mathbb{C}}, \quad \ell(h)^* \text{ bdd}$$

$$\ell(h)^* \ell(g) = \langle h, g \rangle$$

$$s(h) = \ell(h) + \ell(h)^* \qquad h \in H \qquad (\text{not } H_{\mathbb{C}})$$

- $W^*(s(h) : h \in H)$ then $\langle \Omega, \cdot\, \Omega \rangle$ is a trace

$W^*(s(e_i))$ $\quad e_i = $ ONB for H $\quad$ are free and each is $\simeq L(\mathbb{Z})$

$$\simeq \mathop{\ast}_{\{e_i\} \text{ ONB}} W^*(s(e_i)) = L(\mathbb{F}_m) \qquad m = \dim H$$

$$\Phi(H) \overset{\text{def}}{=} W^*(s(h) : h \in H)$$

The map Φ is a functor from (real H sp. and contractions) to (W^*-algebras and normal c.p. maps).

If $T : H_1 \to H_2$ is a contraction, $\exists\ \Phi_T : \Phi(H_1) \to \Phi(H_2)$ c.p. map st. $\Phi_T(s(h)) = s(Th)$.

$$\begin{tikzcd} & \tilde{H} \arrow[dr, "\text{proj}"] & \\ H_1 \arrow[ur, "\text{isom}"] \arrow[rr, dashed, "T"'] & & H_2 \end{tikzcd}$$

- $W^*(s(h) : h \in H_1) \subseteq W^*(s(h) : h \in H_2)$

We'll show next time that $\Phi(H)$ has the Haagerup property.

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Recall Haagerup property for VNA's:

M II_1 factor

$\exists\ \varphi_n$ unital τ-preserving CP maps

$\|\varphi_n(x) - x\|_2 \longrightarrow 0$ as $n \to \infty$

$\varphi_n : L^2 \to L^2$ are compact.

<u>Remarks:</u>

- For $M = L(\Gamma)$ this is equivalent to the Haagerup property for Γ.
- Don't actually need unital τ-preserving, but it's convenient.
- $L(\mathbb{F}_n)$ has the Haagerup property

$$L(\mathbb{F}_n) = \Phi(\mathbb{R}^n) \overset{\text{def}}{=} W^*\{p(h) : h \in \mathbb{R}^n\}$$

$$p(h) = \ell(h) + \ell(h)^*$$

$$\ell(h) : \underset{\substack{\| \\ \mathbb{C}\Omega \oplus \bigoplus_{k \geq 1} (\mathbb{C}^n)^{\otimes k}}}{\mathcal{F}(\mathbb{C}^n)} \to \mathcal{F}(\mathbb{C}^n)$$

Φ functor from (real Hilbert spaces, contractions)
to (W^*-algebras, CP maps)

$$\begin{tikzcd} & \tilde{H} \arrow[dr] & \\ H_1 \arrow[ur] & & H_2 \end{tikzcd}$$

$T : \mathbb{R}^n \to \mathbb{R}^n \qquad T\xi = (\cos\theta)\xi$

$$\begin{tikzcd} & \mathbb{R}^n \times \mathbb{R}^n \arrow[dr, "\text{proj. onto 1st coord}"] & \\ \mathbb{R}^n \arrow[ur] & & \mathbb{R}^n \\ H_1 \arrow[u, phantom, "\shortparallel"] \arrow[rr, dotted] & & H_2 \arrow[u, phantom, "\shortparallel"] \end{tikzcd}$$

$\xi \mapsto (\cos\theta\,\xi, \sin\theta\,\xi)$

equals the "number spectrum"
$\xi_1 \otimes \cdots \otimes \xi_\ell \mapsto (\cos\theta)^\ell \xi_1 \otimes \cdots \otimes \xi_\ell$

$$\mathcal{F}(H_1) \hookrightarrow \mathcal{F}(\tilde{H}) \xrightarrow{\text{proj}} \mathcal{F}(H_2)$$

$$\xi_1 \otimes \cdots \otimes \xi_n \longmapsto (\cos\theta\,\xi_1, \sin\theta\,\xi_1) \otimes \cdots \otimes (\cos\theta\,\xi_n, \sin\theta\,\xi_n)$$

$=$

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Remark If R is hyperfinite, it has the Haagerup property. In fact, something stronger:

- $R = \overline{\bigcup R_n}$ $\quad R_n$ f.d. subalg

$$\begin{tikzcd} R \arrow[rr, "\varphi_n"] \arrow[dr, "\text{cond. exp.}"'] & & R \\ & R_n \arrow[ur, hook] & \end{tikzcd}$$

finite dim'l

In this case φ_n are not just compact but actually have finite rank.

Theorem (Connes) If $\varphi_n : M \to M$ are unital CP τ-preserving such that $\|\varphi_n(x) - x\|_2 \to 0$ and φ_n finite rank $\Rightarrow M$ hyperfinite.

- So why can't we embed $L(\Gamma) \hookrightarrow L(\mathbb{F}_n)$ if Γ has (T)?

Note If M has the Haagerup property and $N \subseteq M$, N has the Haagerup property.

$\varphi_n : M \to M$ $\qquad \|\varphi_n(x) - x\|_2 \to 0$

$\psi_n = E_N \circ \varphi_n : N \to N$ $\qquad \|\psi_n(x) - x\|_2 \to 0 \quad \forall x \in N$

$\varphi_n : L^2(M) \to L^2(M)$ compact

then so are $\psi_n = e_N \circ \varphi_n \circ e_N$ $\qquad e_N : L^2(M) \to L^2(N)$

If $L(\Gamma) \hookrightarrow L(\mathbb{F}_n) \Rightarrow L(\Gamma)$ has Haag.

- $\Rightarrow \Gamma$ has Haag. which contradicts (T) if Γ infinite (we showed this before).

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Note

M $\quad II_1$ factor

$N \subseteq M$ $\quad$ fixed

$N_k \subseteq M$ s.t. $\quad N_k$ "$\longrightarrow$" N

meaning $\forall x \in N, \ \exists x_k \in N_k$ s.t. $\|x_k - x\|_2 \to 0$.

If N_k has Haagerup, so does N. (So Haag is preserved under this kind of "limit".)

Prove using $\varphi_{l,k} : N_k \to N_k$ $\quad \varphi_{l,k} \to id$ as $l \to \infty$

$E_N \circ \varphi_{l,k} \circ E_N \rightsquigarrow$ Haag. for N.

Proposition M has property $(\Gamma) \Leftrightarrow C^*(M, JMJ) \cap K = \{0\}$.

compact operators

$\exists x_k \in M$ s.t.
$\|[x_k, x]\|_2 \to 0$ as $k \to \infty$
$\|x_k\|_2 = 1$

$J : L^2(M) \to L^2(M)$
$x\Omega \longrightarrow x^*\Omega$
$R_x = J L_{x^*} J$

Remark Given $x_1, \dots, x_n$ consider $\mathcal{L} = \sum |x_i - Jx_i^*J|^2$

$0 \in \sigma(\mathcal{L})$ since $1 \in L^2(M)$ is in kernel

$\sigma(\mathcal{L})$ has a gap near $0 \Leftrightarrow \mathcal{L}$ on $L^2(M) \ominus \mathbb{C}1$ has bdd inverse

$\Leftrightarrow \exists C$ s.t. $\|x - \tau(x)\|_2 \leq C \max \|[x, x_i]\|_2$

$\Updownarrow$

$P_{\{0\}} \in C^*(\mathcal{L}) \subseteq C^*(M, JMJ)$

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Note $P_{\{0\}} = P_{\mathbb{C}1} = p$ rank 1 projection

$\sum_{\text{finite}} x_i P y_i$ finite rank operator

$$\xi \mapsto \sum x_i \langle \xi, y_i^* \rangle$$

$$(x_i P y_i)(\xi) = x_i \tau(y_i \xi)$$

so $P_{\mathbb{C}} \in C^*(M, JMJ) \Rightarrow C^*(M, JMJ) \supseteq K.$

Fact $T: H \to H$ compact $\Leftrightarrow \forall h_n, \|h_n\| = 1$
$(h_n \to 0$ weakly$) \Rightarrow Th_n \to 0$ in norm. (well-known.)

(Property Γ) $\Leftrightarrow \exists$ unitaries $u_k \in M$, $\tau(u_k) = 0$

s.t. $\|u_k x u_k^* - x\|_2 \to 0 \quad \forall x \in M$

$\|(x - Jx^*J)u_k\| \to 0 \quad \forall x \in M.$

Remark Non-Γ $\Leftrightarrow$ $\mathrm{Inn}(M) \subseteq \mathrm{Aut}(M)$ closed in the following topology:
$\alpha_j \to \alpha$ if $\|\alpha_j(x) - \alpha(x)\|_2 \to 0$

$$\left[\varphi \circ \alpha_j \xrightarrow{\text{norm}} \varphi \circ \alpha \quad \forall \varphi \in M_* \right]$$

Why? $\mathrm{Aut}(M) = \{*\text{-isomorphisms } M \to M\}$
$\mathrm{Inn}(M) = \{\alpha \in \mathrm{Aut}(M) : \alpha(x) = uxu^*, u \in M\}$

Corresponds to a map $U(M) \to \mathrm{Aut}(M)$ with range $\mathrm{Inn}(M)$ and kernel $\mathbb{T}$.

These groups are all Polish (their topologies are metrizable)
$\Rightarrow$ image dense only if $\nexists$ approximate kernel.

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Open Mapping Theorem for Polish groups:

$\alpha : G \to H$ $\qquad$ G, H Polish

$\alpha(G)$ closed $\Leftrightarrow$ α has no approximate kernel.

In our case, $\mathrm{Inn}(M)$ closed $\Leftrightarrow$ $\nexists\, u_n$ s.t.
$\mathrm{Ad}\, u_n \to \mathrm{id}$ but $u_n \not\to$ in $\mathbb{T}$.

In general, M is <u>full</u> if $\mathrm{Inn}(M) \subseteq \mathrm{Aut}(M)$ closed.

For type II_1, full $\Leftrightarrow$ non-Γ.

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Last Time

Connes thm:

(M,τ) non-Γ $\Leftrightarrow$ $K \wedge C^*(M, JMJ) \neq 0$
II_1 factor

Recall M has Γ $\Leftrightarrow$ $\exists$ u_n unitary, $\|[u_n, x]\|_2 \to 0$ $\forall x \in M$
$u_n \not\to \mathbb{C}1$ in $\| \|_2$

Let φ_n be a state on $C^*(M, JMJ)$ defined by
$\varphi_n(T) = \langle T u_n, u_n \rangle = \langle u_n^* T u_n \cdot 1, 1 \rangle$

Thus, M non-Γ $\Leftrightarrow$ whenever u_n unitaries satisfy $\|[u_n, x]\|_2 \to 0$,
then $\|u_n - \tau(u_n)\|_2 \to 0$.

Let u_n satisfy $\|[u_n, x]\|_2 \to 0$ and take φ_n as above.

$$\varphi_n(L_x R_y) = \langle u_n^* x J y^* J u_n 1, 1 \rangle$$
$$= \langle u_n^* x u_n J y^* J 1, 1 \rangle \to \langle x J y^* J 1, 1 \rangle,$$

$\downarrow$
$\langle L_x R_y 1, 1 \rangle$

Same holds true if we sum $\sum \varphi_n(L_{x_i} R_{y_i})$, and these sums are $\| \|_2$-dense.

If φ is any weak limit of $\{\varphi_n\}$, $\varphi(T) = \langle T1, 1 \rangle$ where $T \in C^*(M, JMJ)$.

If $T \in K$, $\varphi_n(T) = \langle T u_n, u_n \rangle \to \langle Tu, u \rangle$ because $T u_n \xrightarrow{\| \|} Tu$
and $u_n \xrightarrow{\text{weak}} u$.

Γ equivalent to being able to choose s.t. $T u_n \to 0$. But then $\langle T1, 1 \rangle = 0$
$\Rightarrow \langle \sum a_i T b_i 1, 1 \rangle = 0$ $\forall a_i, b_i \in M$ $\Rightarrow T = 0$.

Converse: Omitted.

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Recall Property (T) for Γ:

$\exists \varepsilon$, $F \subseteq \Gamma$ finite ~~sub~~ subset s.t.

$\|\pi(g)\xi - \xi\| < \varepsilon\|\xi\|$ $\quad \forall g \in F$

$\Rightarrow \exists \xi'$ near ξ s.t. $\pi(g)\xi' = \xi'$ $\quad \forall g$, $\forall \pi$ unitary rep on a Hilbert space.

M has (T) if $\exists F \subseteq M$ finite subset, $\varepsilon > 0$ s.t. whenever H is an M,M bimodule, $\xi \in H$ satisfies $\|x\xi - \xi x\| < \varepsilon\|\xi\|$ $\quad \forall x \in F$ then $\exists \xi'$ (near ξ) s.t. $y\xi' = \xi' y$ $\quad \forall y \in M$.

Γ has (T) $\Rightarrow L(\Gamma)$ has (T).

H M,M bimodule $\Rightarrow$ rep of Γ by $\pi(g)\xi = g\xi g^{-1}$

$F = \{\lambda(g) : g \in F$ of the group $\}$.

(not the same F)

• Equivalent: $\exists F, \varepsilon, c$ s.t. $\forall$ pos-def. map $\varphi : \Gamma \to \mathbb{C}$ if $\varphi|_F \in [1-\varepsilon, 1+\varepsilon]$ then $\varphi \geq c$ every where.

M has (T) $\Leftrightarrow \exists \varepsilon$ s.t. if φ is a CP map, φ unital, $\tau \circ \varphi \leq \tau$

$\exists F \subseteq M_1 = \{m \in M : \|m\|_2 = 1\}$

s.t. $\|\varphi(x) - x\|_2 < \varepsilon$ $\quad \forall x \in F$ $\quad (*)$

then $\|\varphi(y) - y\|_2 < \varepsilon$ $\quad \forall y \in M_1$.

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$\varphi \rightsquigarrow (H, \xi)$ $\quad \xi$ implements φ

$(*) \Rightarrow \|x\xi - \xi x\|_2 < \varepsilon$ on F

$$\langle x\xi a, y\xi b\rangle = \tau(a^*\varphi(x^*y)b)$$

If $\exists \xi'$ bibounded, $\langle \xi' x, \xi'\rangle = \tau(x) \longrightarrow \varphi'$; $\quad \varphi'(x) = x$

Consequences of T

<u>Fact</u> If M has (T) and $\quad \alpha: M \to \mathcal{M}$
$\beta: M \to \mathcal{M}$ $\quad$ homomorphisms

with $\mathcal{M}$ a II_1 factor,
assume $\|\alpha(x) - \beta(x)\|_2 < \varepsilon$. Then
"$\exists u \in \mathcal{M}$ s.t. $\alpha(x) = u\beta(x)u^*$" $\quad$ Not quite true, need to cut w/projections.

$\exists v \in \mathcal{M}$ partial isom. s.t. $\quad v\alpha(x)v^* = \beta(x)vv^*$
$vv^* \in \beta(M)' \cap \mathcal{M}$.

<u>Cor</u> Let $M = \mathcal{M}$ and $\beta = id$.
$\alpha: M \to M$ s.t. $\|\alpha(x) - x\|_2 < \varepsilon \quad \forall x \in F \quad \Rightarrow \alpha = Ad\, u$.
(Note: $\beta(M)' \cap \mathcal{M} = M' \cap M = \mathbb{C}$ + maximality argument)

<u>Cor</u> Recall $T \Rightarrow$ non-$\Gamma \Leftrightarrow Inn(M)$ closed in $Aut(M)$ closed.
Now $Aut(M)/Inn(M)$ is a discrete group. It's also Polish.
So it's countable.

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So where do these α and β maps come from?

${}_{\alpha}L^2M_{\beta}$ M, M bimodule

$$x \cdot \xi = \alpha(x)\xi \qquad x \in M \quad \xi \in M$$
$$\xi \cdot y = \xi \beta(y) \qquad y \in M \quad \xi \in M$$

(*) says $\|x \cdot 1_M - 1_M x\| < \varepsilon \quad \forall x \in F.$

$\Rightarrow \exists\, y \in M$ s.t. $\alpha(x) y = y \beta(x) \quad \forall x \in M.$

$y = v|y|$

$$v^* \alpha(x) v = \beta(x) v^* v$$
$$v^* v \in \beta(M)' \cap M$$

Let M_1 be a VNA generated by $x_1, \ldots, x_n$
M_2 " " $y_1, \ldots, y_n$

$$d_{(x_1,\ldots,x_n),(y_1,\ldots,y_n)} = \inf_{\substack{\alpha_1 : *\text{-hom } M_1 \hookrightarrow M \\ \text{trace-pres.} \\ \alpha_2 : M_2 \hookrightarrow M}} \left(\sum \| \alpha_1(X_i) - \alpha_2(Y_i) \|_2^2 \right)^{1/2}$$

~~M has (T) $\Rightarrow \exists\, F \in M$~~

When $n=1$, get $d_{x,y} = \inf_{\substack{\pi \text{ measures on } \mathbb{R}\times\mathbb{R} \\ (\mathrm{proj}_x)_* \pi = \mu \\ (\mathrm{proj}_y)_* \pi = \nu}} \left[\int |x-y|^2 \, d\pi \right]^{1/2}$

Kantorovich–Wasserstein dist (μ, ν)

Studied in problems related to "optimal transport".

Can also define

$$d'_{(X_1,\dots,X_n),(Y_1,\dots,Y_n)} = \inf_{\substack{H\ M_1,M_2\text{-bimodule} \\ \xi\in H \\ \langle \cdot\xi,\xi\rangle = \tau_{M_1} \\ \langle \xi\cdot,\xi\rangle = \tau_{M_2}}} \left(\sum \| X_i\xi - \xi Y_i\|^2\right)^{1/2}$$

$d \geq d'$, but whether they're equal in general is an open question.

(May even be open for matrix algebras ...)
not sure...

3-19-08
Op Alg
1

Recall: group-measure space construction $\Gamma \curvearrowright (X,\mu)$

$\to$ equiv. relation $R \subseteq X \times X$

If $\Gamma \curvearrowright X$ free, R is what you're really interested in.

$\to A \subseteq \underset{M}{\underset{\shortparallel}{L(R)}}$ $\left(\simeq L^\infty(X,\mu) \rtimes \Gamma \right)$

$A \simeq L^\infty(X)$

If the action is free, A is a MASA. $A' \cap M = A$

$E : M \to A$ cond. exp. s.t. $\mu \circ E$ is a trace on M.

We can abstractly characterize the inclusion $A \subset M$.

Def $A \subseteq M$ (II_1 factor) is called a <u>Cartan subalgebra</u> if

- A is a MASA, $A' \cap M = A$
- $\mathcal{N}(A) = \{ u \in M \text{ unitary s.t. } uAu^* = A \}$ generates M

$\left(\mathcal{U}(A) \subseteq \mathcal{N}(A) \right)$

$(R, \sigma) \iff (A \subseteq M)$

Theorem. ~~Let $A \subseteq M$ be a MASA.~~ TFAE:

- $A \subseteq M$ Cartan
- $L^2(M)$ viewed as A,A bimodule $A = L^\infty(X,\mu)$

$= L^2(X \times X, \eta)$ η is "discrete"

supp $(\eta) \subseteq$ a set $B \subseteq X \times X$ s.t. each fiber of B is countable

- A is a MASA and $\forall x \in M$,

${}_A\overline{AxA}^{L^2(M)}{}_A \simeq {}_A L^2(X \times X, \tilde{\eta})_A$ with $\tilde{\eta}$ discrete.

Pf

Let $R_0 = \operatorname{supp} \eta$. By a Borel selection theorem,

R_0 is a union of graphs.

Choose such a graph $\alpha \subseteq R_0$. χ_α corresponds to some element $\xi_\alpha \in L^2(M)$. Then $f\xi_\alpha = \xi f \circ \alpha$.

Can do a polar decomp $\xi_\alpha = U_\alpha b_\alpha$, $u_\alpha \in M$,

$b_\alpha = (\xi_\alpha^* \xi_\alpha)^{1/2}$,

In fact $b_\alpha \in L^2(A)$ and $U_\alpha \cdot 1$ supported on α.

$f U_\alpha U_\beta = U_{\alpha\circ\beta} f \circ (\alpha\beta) \Rightarrow R_0$ equiv. rel.

$(A \subseteq M) \rightsquigarrow U_\alpha$ $\quad \alpha \subseteq R_0$ graph

$$U_{\alpha\beta} f U_{\alpha\beta}^* = U_\alpha U_\beta f (U_\alpha U_\beta)^*$$

$$\Rightarrow U_\alpha U_\beta (U_{\alpha\beta})^{-1} f = f U_\alpha U_\beta (U_{\alpha\beta}^{-1})$$

$$\Rightarrow U_\alpha U_\beta (U_{\alpha\beta})^{-1} \in \mathcal{U}(A' \cap M) = \mathcal{U}(A)$$

$\rightarrow$ we get a cocycle $\sigma(\alpha,\beta)$ with values in $\mathcal{U}(A)$

$$\sigma(\alpha,\beta) = U_\alpha U_\beta (U_{\alpha\beta})^{-1}.$$

$A \subseteq M$ comes from R $\Leftrightarrow$ can replace u_α by $U_\alpha \cdot V_\alpha$, $V_\alpha \in \mathcal{U}(A)$ s.t. $\sigma(\alpha,\beta) = 1$ $\forall \alpha, \beta$.

"Cartan subalgebras are the same as equivalence relations"

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$(2 \Rightarrow 3)$ easy. How about $(3 \Rightarrow 2)$?
Do the polar decomposition thing, etc, and it all works out (exercise). $\square$

Another way of stating this third condition (discreteness) is that $A' \cap \langle M, e_A \rangle$ is generated by projections which are finite in $\langle M, e_A \rangle$.

Question about free groups: Is $\mathbb{F}_n \simeq \mathbb{F}_m$? No.
Then $\mathbb{Z}^n$ would be m-generated by universality; various ways to get a contradiction from this.

Harder question: Is (a). $C^*_{full}(\mathbb{F}_n) \simeq C^*_{full}(\mathbb{F}_m)$?
(b). $C^*_{red}(\mathbb{F}_n) \simeq C^*_{red}(\mathbb{F}_m)$?
(c). $L(\mathbb{F}_n) \simeq L(\mathbb{F}_m)$?

If (a) then any C^* alg generated by n unitaries is a quotient of $C^*_{Full}(\mathbb{F}_n)$
$\Rightarrow$ would imply every n-generated C^* alg was m-generated.
Can show that $\mathbb{T}^n$ works (given by n coord. projections)

Q (Prof. S would really like to know the answer!)
Give a "nice" example (simple, tracial) of a C^*-alg generated by n but not m unitaries.
For example, it's an open problem whether $C^*_{red}(\mathbb{F}_n)$ is generated by m unitaries.

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Pimsner - Voiculescu '80s computed

$$K_1(C^*_{full}(\mathbb{F}_n)) \simeq \mathbb{Z}^n \quad \text{(has rank } n\text{)}$$

$$K_1(C^*(\mathbb{Z})) = K_1(\mathbb{T})$$

where K_1 is some invariant. So (b) can be done with K-theory.
(c) is open.

A fourth question is whether $\mathbb{F}_n \curvearrowright X$ free, ergodic can be ME to $\mathbb{F}_m \curvearrowright X$.
We'll look at 2 different approaches.

Cost of an equivalence relation R

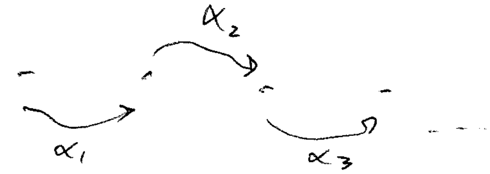
R generated by $\{\alpha_1, \alpha_2, \ldots\}$ $\quad \alpha_j : A_j \to B_j$ partial R-morphism

If (x-a.e.) $x \overset{R}{\sim} y \iff x = \alpha_{i_1} \alpha_{i_2} \cdots \alpha_{i_k}(y)$
for some $k, i_1, \ldots, i_k$.

$$C(R) = \inf_{\substack{\{\alpha_1, \alpha_2, \ldots\} \\ \text{generate } R}} \sum_j \mu(A_j)$$

"support of α_j"

definition from Levitt in early 90's

$X =$ 

n pts

$R =$ "all are equiv" $= R_{\mathbb{Z}/n\mathbb{Z}}$ by cyclic shift

On each orbit draw a graph
$x \xrightarrow{\text{edge}} y$ if $x = \alpha_j^{\pm 1} y$ for some j.

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Group Homology

In general, we can do homology in any category where there's a concept of "exact sequence" and where we can "add" morphisms.

Examples
- Modules over a group G
  V, G-action on vector space V, G-equiv. bilinear map
- Bimodules over an algebra, bimodule homomorphisms

Suppose $\mathcal{F}$ is a functor between two such categories.

Ex $\mathcal{F}_{V_0}(V) = V \otimes_\Gamma V_0 \leftarrow V \otimes V_0 / vg \otimes w = v \otimes gw$

fixed

$\mathcal{F}_{V_0}$: group modules $\longrightarrow$ vector spaces

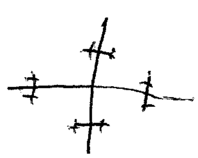
Ex $\mathcal{F}_X(Y) = Y \otimes_{A \otimes A^{op}} X$

X, Y A,A-bimodules ($\Leftrightarrow A \otimes A^{op}$-modules)

$\mathcal{F}_X$: A,A bimodules $\longrightarrow$ vector spaces

functor $\mathcal{F}$ is exact if it takes exact sequences to exact sequences.
Homology "measures the extent to which $\mathcal{F}$ fails to be exact".

Example Let X be a CW-complex and Γ a group acting on X (cells map to cells).

Ex X = ternary tree  $\Gamma = \mathbb{F}_2$ (X = Cayley graph of Γ)

Easier Ex $X = \mathbb{R}$
$\Gamma = \mathbb{Z}$ acting by translations

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$C_0(X) = \text{span}(0\text{-cells})$

$C_1(X) = \text{span}(1\text{-cells})$

$\vdots$

$$\mathbb{C} \xleftarrow{\partial} C_0(X) \xleftarrow{\partial} C_1(X) \xleftarrow{\partial} C_2(X) \xleftarrow{\partial} \cdots \qquad \text{boundary maps} \quad (*)$$

$$H_k(X) = \frac{\ker \partial_k}{\operatorname{im} \partial_{k+1}}$$

In our examples above, X is contractible and hence has trivial homology, i.e. $(*)$ is exact.

Consider the functor $\mathcal{F}(V) = V \otimes_\Gamma \mathbb{C}$

$\mathbb{C} \otimes_\Gamma \mathbb{C} \simeq \mathbb{C}$

$C_0(X) \otimes_\Gamma \mathbb{C} \simeq \mathbb{C}$ (from our second example, with Γ acting trivially)

$C_1(X) \otimes_\Gamma \mathbb{C} \simeq \mathbb{C}$

$$0 \leftarrow \mathbb{C} \xleftarrow{\partial} C_0(X) \xleftarrow{\partial} C_1(X) \leftarrow 0 \qquad \text{exact}$$

$$0 \leftarrow \mathbb{C} \xleftarrow{id} \mathbb{C} \xleftarrow{0} \mathbb{C} \xleftarrow{0} 0 \qquad \text{no longer exact}$$

Note: $C_k(X) \otimes_\Gamma \mathbb{C} \simeq C_k(X/\mathbb{Z}) = C_k(\mathbb{T})$

so we recover the reduced homology of the circle.

On the ternary tree we get the homology of the figure 8  since now our fundamental domain is a corner $\llcorner$ (i.e. everything is a translate of this corner).

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Theorem Assume X is a functor as before and

$$\cdots X_2 \to X_1 \to X_0 \to X_{-1}$$

is an exact sequence with $X_0, X_1, \ldots$ free modules.

Then in $\mathcal{F}(X_{-1}) \xleftarrow{\alpha_0} \mathcal{F}(X_0) \xleftarrow{\alpha_1} \cdots$

$$D_j(\mathcal{F}, X_{-1}) = \frac{\ker \alpha_j}{\operatorname{im} \alpha_{j+1}}$$ depend only on $\mathcal{F}$ and X_{-1}.

Def An exact sequence $X_{-1} \leftarrow X_0 \leftarrow X_1 \leftarrow \cdots$

s.t. X_j, $j \geq 0$ are free is called a

free resolution of X_{-1}.

Def $H_k(G; V) = D_k(\cdot \otimes_\Gamma V, \mathbb{C})$

Ex $H_*(\mathbb{Z}; \mathbb{C}) = H_*^{red}(\mathbb{T})$

$H_*(\mathbb{F}_2, \mathbb{C}) = H_*(\infty)$

Ex
What's $H_*(G; \mathbb{C}G)$?

- $X_k \otimes_\Gamma \mathbb{C}\Gamma \simeq X_k$ so tensoring does nothing

$\therefore$ all homology groups trivial.

Observation If G acts freely on a contractible CW complex X

then $\{C_k(X), \partial_k\}$ is a free resolution of $\mathbb{C}$.

But note that $C_k(X)$ may not be finitely generated.

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A universal example

Given G, construct a CW complex as follows:

k-cells indexed by $(k+1)$-tuples of group elements

$$(g_0, \dots, g_k) \rightsquigarrow C_{g_0, \dots, g_k}$$

$$\partial C_{g_0 \cdots g_k} = \sum (-1)^j C_{g_0 \cdots \hat{g}_j \cdots g_k}$$

Basically, you keep throwing in discs to fill all possible holes.

The resulting topological space is called EG and carries a natural G-action

$$g\, C_{(g_0 \cdots g_n)} = C_{(gg_0 \cdots gg_n)}$$

Note $C_k(EG)$ will never be finitely generated unless G is finite.

$$C_k(EG) \cong \underset{\text{left mult.}}{\mathbb{C}G} \otimes \overbrace{\mathbb{C}G \otimes \cdots \otimes \mathbb{C}G}^{k \text{ times}}_{\text{trivial}}$$

$$C_{g_0 \cdots g_k} \longrightarrow g_0 \otimes g_0^{-1} g_1 \otimes \cdots \otimes g_{k-1}^{-1} g_k$$

$$\partial_n (h_0 \otimes h_1 \otimes \cdots \otimes h_n) = \sum_{k<n} (-1)^k h_0 \otimes \cdots \otimes h_k h_{k+1} \otimes h_{k+2} \cdots \otimes h_n$$

<u>Observation</u> If G acts freely on a CW complex which is contractible and has a compact fundamental domain, then $\{C_k(X)\}$ is a free resolution of $\mathbb{C}$ by finitely generated modules.

<u>Def</u> "Fundamental domain" $X_0 \subseteq X$ means $X = \bigcup_{g \in G} gX_0$ and $gX_0 \cap X_0 \subseteq \partial X_0$.

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Last Time
Defined $H_k(G;V)$

1st way to compute
Find some free resolution of $\mathbb{C}$

$$0 \leftarrow \mathbb{C} \leftarrow X_0 \leftarrow X_1$$

X_j free G-modules
exact sequence

Make a new sequence

$$0 \leftarrow \mathbb{C} \otimes_G V \leftarrow X_0 \otimes_G V \leftarrow \cdots$$

$$\ker \alpha_k / \operatorname{im} \alpha_{k+1} = H_k(G;V)$$

2nd way Write down the bar complex

$$\mathbb{C} \xleftarrow{1 \mapsto g} \mathbb{C}G \leftarrow \mathbb{C}G \otimes \mathbb{C}G \leftarrow \mathbb{C}G \otimes \mathbb{C}G \otimes \mathbb{C}G \leftarrow \cdots$$

$$g_0 g_1 \otimes \cdots \otimes g_n - g_0 \otimes g_1 g_2 \otimes \cdots \otimes g_n \cdots \pm g_0 \otimes \cdots \otimes g_{n-1} \leftarrow g_0 \otimes g_1 \otimes \cdots \otimes g_n$$

$$V/G \leftarrow V \leftarrow V \otimes \mathbb{C}G \leftarrow V \otimes \mathbb{C}G \otimes \mathbb{C}G \leftarrow \cdots$$

$$\partial(v \otimes g_1 \otimes \cdots \otimes g_n) = vg_1 \otimes g_2 \otimes \cdots \otimes g_n - v \otimes g_1 g_2 \otimes \cdots \otimes g_n + \cdots \pm v \otimes g_1 \otimes \cdots \otimes g_{n-1}$$

Example
G = amenable group
$V = (\mathbb{Z}/p\mathbb{Z})^G$
∞-dim $\mathbb{Z}/p\mathbb{Z}$ vector space
G acts by Bernoulli shifts

$H(G,V)$ ∞-dim
$\mathbb{Z}/p\mathbb{Z}$ space
It's a left G-module, but is also a compact topological space.
Its entropy becomes a nice invariant.

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Let $\pi : G \to \mathcal{U}(H)$ be a unitary rep. of G on a Hilbert space H.

Then $\pi \subseteq \ell^2(\Gamma)^{\oplus\infty}$ $\Leftrightarrow$ the action of $\mathbb{C}G$ on H extends to a normal action of LG.

<u>Fact</u> If H is a Hilbert space and M a II_1 factor acting normally and unitally on H, one can define the dimension
$\dim_M H$

Such $H \subseteq L^2(M)^{\oplus\infty}$

$$P_H : L^2(M)^{\oplus\infty} \longrightarrow H \in M' \cap B(L^2(M)^{\oplus\infty})$$
$$= M^{op} \otimes B(\ell^2(\mathbb{N}))$$

Trace $Tr_{M'} = \tau \otimes Tr$

Define $\dim_M H = Tr_{M'}(P_H)$.

Can show that this is well-defined, i.e. independent of the choice of embedding.

Called the Murray-von Neumann dimension of H.

<u>Fact</u> Let $\xi \in H$ be any nonzero vector

Consider $P_{\overline{M\xi}} \in M'$

$P_{\overline{M'\xi}} \in M$

Coupling constant $\dfrac{\tau_M(P_{\overline{M'\xi}})}{\tau_{M'}(P_{\overline{M\xi}})}$

is independent of ξ and equals $\dim_M H$.

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Examples

$$\dim_M L^2(M) = 1$$

$$\dim_M L^2(M)^{\oplus N} = N$$

$p \in M'$ $\quad \dim_M pL^2(M) = \tau_{M'}(p).$

Fact

For any (discrete) abelian group Γ, π rep'n of Γ is determined by a measure class on $\hat{\Gamma}$ plus a multiplicity function $n : \hat{\Gamma} \to \mathbb{N} \cup \{+\infty\}$ by the spectral theorem.

$\pi \subseteq$ left regular rep. $\iff$ measure class is absolutely continuous wrt Haar measure.

(Don't actually need Γ discrete.)

Given any unitary rep'n π of Γ, get an action of Γ on $CAR(H)$, whatever that is (some non-commutative topological space).

Entropy $= \dim_{L(\Gamma)}$ (max. part of π in left regular rep'n)

(result of Voiculescu – Stormer)

Consider $H_k(G, \ell^2(G))$. Homology "eats up the right module structure," but this is still a left G-module.

Can we view $\ker(\ )/\mathrm{im}(\ )$ as a Hilbert space?

If G has a cocompact (cpct fund. domain) contraction

$$\mathbb{C} \leftarrow \underset{C_0(X)}{(\mathbb{C}G)^{n_0}} \leftarrow \underset{C_1(X)}{(\mathbb{C}G)^{n_1}} \leftarrow \cdots$$

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We tensor the whole thing $\ell^2(G) \otimes_G (\qquad)$ and get

$$(\ell^2(G))^{n_0} \xleftarrow{\alpha_1} (\ell^2(G))^{n_1} \xleftarrow{\alpha_2} (\ell^2(G))^{n_2} \leftarrow \cdots$$

α_j are still $L(G)$-linear maps

$\alpha_j \in M_{n_{j+1} \times n_j}(\mathbb{C}(G))$

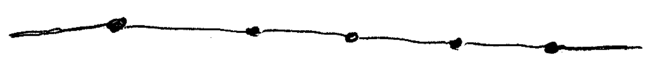
$\ker \alpha_j / \mathrm{im}\, \alpha_{j+1}$ ← not closed in general

$$H_k^{red}(G; \ell^2(G)) = \ker \alpha_j \Big/ \overline{\mathrm{im}\, \alpha_{j+1}}$$

In co-compact case, the alg pf still "works" to show this is independent of the choice of embedding (need to assume finitely generated, though).

$\boxed{\text{Examples}}$

- $\mathbb{Z}$ has a nice resolution coming from $\mathbb{R}$



$$\mathbb{C} \xleftarrow{\alpha_0} \mathbb{C}\mathbb{Z} \xleftarrow{\alpha_1} \mathbb{C}\mathbb{Z} \leftarrow 0$$

$k\,1 \leftarrow \delta_m$ $\qquad$ $\delta_m - \delta_{m-1} \longleftarrow \delta_m$

$1-u$ $\qquad$ u = bilateral shift

$\mathrm{im}(\alpha_1) = \mathrm{im}(1-u) = $ dense

$\ker(\alpha_0) = $ everything

$\therefore H_1^{red}(\mathbb{Z}; \ell^2\mathbb{Z}) = 0$

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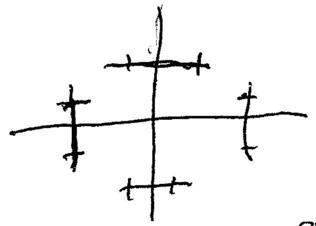
$\boxed{\text{Example}}$

$\Gamma = \mathbb{F}_2$



$$\mathbb{C} \leftarrow \underset{\mathbb{C}\mathbb{F}_2}{\text{span(vertices)}} \leftarrow \underset{\mathbb{C}\mathbb{F}_2^{\oplus 2}}{\text{span (edges)}} \leftarrow 0$$

Tensor everything and get $\quad \ell^2(\text{vert}), \xleftarrow{\alpha_1} \ell^2(\text{edges})$

Say $f \in \ker(\alpha_1)$

f: edges $\to \mathbb{C}$

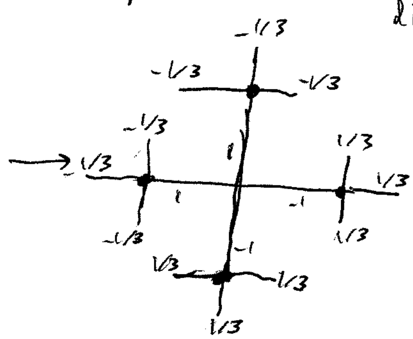
$\partial f = 0$ is a sort of "conservation law"

f represents the flow along each edge
$\partial f = 0$ means no leakage at any juncture

Can make such an f which is in ℓ^2, e.g. start with some values on

and then go outward, with each vertex dividing the outflow in equal thirds

e.g.



$\longrightarrow$ etc ...

Thus $\ker \alpha_1 \neq 0$

$H_0^{\text{red}} = 0$

$H_1^{\text{rel}} = \ldots \simeq \ell^2(G)$?

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Luck's book p. 259 Th 6.37

L^2 - homology has to do with

$$(\mathbb{C}\Gamma)^m \xrightarrow{f} (\mathbb{C}\Gamma)^n \qquad \mathbb{C}\Gamma\text{-module map}$$

$$f = (b_{ij}) \in M_{m \times n}(\mathbb{C}\Gamma)$$

$$(\ell^2\Gamma)^m \xrightarrow{f^{(2)}} (\ell^2(\Gamma))^n$$

$$\ker f^{(2)} \quad \text{vs} \quad \overline{\ker f}$$

<u>Theorem</u> Γ amenable $\Rightarrow \overline{\ker f} = \ker f^{(2)}$

$K = \ker f \qquad \bar{K} = \overline{\ker f} \subseteq \ker f^{(2)}$

<u>Idea</u> Given an element in $\ker f^{(2)}$, need to approximate by elements in $\ker f$.

Projection $\quad pr : (\ell^2\Gamma)^m \longrightarrow \bar{K}^{\perp} \cap \ker f^{(2)}$

<u>Goal:</u> Argue that $\tau(pr) = 0$
τ in $L(\Gamma)$

$u \in \mathbb{C}\Gamma, \quad u = \sum \lambda_g g, \quad \lambda_g \in \mathbb{C}$

$\operatorname{Supp} u = \{g : \lambda_g \neq 0\} \subseteq \Gamma \qquad S = \bigcup_{i,j} \operatorname{supp}(b_{ij})$

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Γ amenable $\Rightarrow$ Følner condition $\Rightarrow$ $\forall \varepsilon > 0$ $\exists A \subseteq \Gamma$ nonempty, finite with $\dfrac{m\cdot(|S|+1)\,|\partial_S A|}{|A|} < \varepsilon$ where $\partial_S A = \{a \in A : \exists g \in S \text{ s.t. } ga \notin A\}$

$$\Delta = \partial_S A \cup \bigcup_{t\in S} \partial_S A \cdot t$$

$pr_A : \ell^2\Gamma \longrightarrow \ell^2\Gamma$ $\quad pr_A(\sum \lambda_g g) \longrightarrow \sum_{g\in A} \lambda_g g$

pr_Δ similarly.

$u \in \ell^2\Gamma$ $\quad pr_A \cdot u = u \cdot pr_A$ if $pr_\Delta(u) = 0$

Doing this entrywise,

$$\left[\left(\bigoplus^{n} pr_A\right) \circ f^{(2)}\right](u) = \left[f^{(2)} \circ \bigoplus^{m} pr_A\right](u) \quad \text{if } pr_\Delta(u) = 0$$

$\Rightarrow$ $pr_A(u) \leq \ker f$ if $u \in \ker f^{(2)}$ and $p_\Delta(u) = 0$

$$\boxed{\Rightarrow\ pr \circ \left(\bigoplus^{m} pr_A\right) = 0 \text{ on } pr_\Delta^{\perp}.}$$

Now

$$\dim_{\mathbb{C}}\left(\left(pr \circ \bigoplus^{m} pr_A\right)(\ker f^{(2)})\right) \leq \operatorname{rank}(p_\Delta) \leq |\partial A|(|S|+1)\cdot m$$

$$\tau(pr) = \frac{\operatorname{Tr}\left(pr \circ \bigoplus^{m} pr_A\right)}{|A|}$$

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$A = \{g_i : i \in I\}$

$$pr_A = \sum_{g \in A} g P_e g^{-1}$$

$$Tr\left(pr \circ \overset{m}{\oplus} pr_A\right) = Tr\left(pr \circ \overset{m}{\oplus}\left(\sum_{g \in A} g J P_e J g^{-1} J\right)\right)$$
$$= \sum_{g \in A} Tr\left(pr \circ \oplus P_e\right) = |A| \, \tau(pr)$$

This gives us $\tau(pr) \leq \dfrac{m |\partial A| \, C(|S|+1)}{|A|} < \varepsilon$.

True for all $\varepsilon \Rightarrow \tau(pr) = 0$. $\square$

<u>Cor</u> Γ amenable $\Rightarrow \beta_j^{(2)}(\Gamma) = 0$ except possibly $\beta_0^{(2)} = 1/|\Gamma|$.

This is an example of "flatness" — the homology is trivial, if you take the extra step of passing to the <u>closure</u> of the kernel.

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One can try to define L^2 homology for objects other than groups.

Algebras

Hochschild homology basically involves passing from the category of Γ-modules to the category of A, A - bimodules,

$\mathbb{C}\Gamma$ analogous to $M \otimes M$

$\ell^2\Gamma$ " " $L^2(M,\tau) \bar{\otimes} L^2(M,\tau)$

Look at dimensions over $M \bar{\otimes} M^{op}$.

- $H_n(A, V)$ A,A - bimodule
- Resolution of $A \leftarrow X_0 \leftarrow X_1 \leftarrow \cdots$

exact sequence of A, A - bimodules, $X_j \simeq (A \otimes A)^{n_j}$

- $A \underset{A\otimes A^{op}}{\otimes} V \leftarrow X_0 \underset{A\otimes A^{op}}{\otimes} V \leftarrow \cdots$

$$H_k(A, V) \stackrel{def}{=} \frac{\ker \partial_{k-1}}{\operatorname{im} \partial_k}$$

$\exists$ universal resolution (bar resolution)

$$A \leftarrow A \otimes A^{op} \leftarrow A \otimes A \otimes A^{op} \leftarrow A \otimes A^{\otimes 2} \otimes A^{op} \leftarrow \cdots$$

$$\partial(a_0 \otimes a_1 \otimes \cdots \otimes a_{n+1}) = a_0 a_1 \otimes a_2 \otimes \cdots \otimes a_{n+1} - a_0 \otimes a_1 a_2 \otimes \cdots \otimes a_{n+1}$$
$$+ \cdots + (-1)^{n+1} a_0 \otimes \cdots \otimes a_n a_{n+1}$$

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The kinds of computations you want to do are

$$(A\otimes A^{op})^m \xrightarrow{f} (A\otimes A^{op})^n$$

$$\cap \qquad\qquad \cap$$

$$L^2(M\bar{\otimes}M^{op}) \xrightarrow{f^{(2)}} L^2(M\otimes M^{op})^m$$

A $*$-algebra with a positive "bounded" trace

$\cap$

(M,τ) $M = W^*(A)$

f is an A,A-bimodule map so $f = (b_{ij})$ with $b_{ij} \in (A\otimes A^{op})$,

$\mathbb{C}\Gamma$ has 2 commuting Γ-actions for $\xi \in \mathbb{C}\Gamma$

$g\xi$ or ξg

$A\otimes A^{op}$ has 2 commuting $A\otimes A^{op}$ actions:

$\xi\otimes\eta \in A\otimes A^{op}$

$a\otimes b \in A\otimes A^{op}$

$$(a\otimes b)\#_{in}(\xi\otimes\eta) = \xi a\otimes b\eta$$

$$(a\otimes b)\#_{out}(\xi\otimes\eta) = a(\xi\otimes\eta)b$$

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<u>Ex</u> $A = \mathbb{C}\Gamma$, $\tau = \tau_\Gamma$. $M = L(\Gamma)$

$f : (\mathbb{C}\Gamma)^m \to (\mathbb{C}\Gamma)^n$

$\mathbb{C}\Gamma \subseteq \mathbb{C}\Gamma \otimes \mathbb{C}\Gamma$

$g \longmapsto g \otimes g^{-1}$

$$\rightsquigarrow$$

$$(\mathbb{C}\Gamma \otimes \mathbb{C}\Gamma)^m \xrightarrow{\tilde{f}} (\mathbb{C}\Gamma \otimes \mathbb{C}\Gamma)^n$$

<u>Th</u> $A(\ker f)A \subseteq \ker \tilde{f} \subseteq A(\ker f^{(1)})A$

$f^{(1)} : (\ell^1\Gamma)^m \longrightarrow (\ell^1\Gamma)^n$

$\ker f^{(1)} = \ker$ of

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Last Time

(A,τ) $M = W^*(A)$ in GNS rep

$$\beta_k^{(2)}(A,\tau) = \dim_{M \otimes M^{op}} H_k\left(A, L^2(M) \bar{\otimes} L^2(M)\right)$$

$$\beta_k^{(2)}(\mathbb{C}\Gamma, \tau_\Gamma) = \beta_k^{(2)}(\Gamma)$$

Now say Γ acts freely and m.p. on (X,μ) $\rightarrow$ equiv. rel. R

$$L^\infty(X) \subseteq L^\infty(X) \rtimes_{alg} \Gamma \subseteq A \subseteq L^\infty(X) \rtimes \Gamma$$

only finite sums allowed

where $A = \left\{ \sum_{\text{finite}} f_\alpha v_\alpha \right\}$ with $f_\alpha \in L^\infty(X)$ and v_α a partial R-morphism.

The thing to do is some sort of relative homology over $L^\infty(X)$.

$H_k^{L^\infty(X)}(A)$ is obtained from a sort of bar resolution

$$L^\infty(X) \leftarrow A \leftarrow A \underset{L^\infty(X)}{\otimes} A \leftarrow \cdots$$

with the obvious boundary maps.

Tensor the whole thing over A with $L^\infty(X) \rtimes \Gamma$.

Then you compute the homology and look at dimension over $L^\infty(X) \rtimes \Gamma$.

$$\underbrace{A \otimes \cdots \otimes A}_{k} \simeq \sum f_{\alpha_1 \cdots \alpha_k} V_{\alpha_1} \otimes \cdots \otimes V_{\alpha_k}$$

$$\partial(f V_{\alpha_1} \otimes \cdots \otimes V_{\alpha_n}) = f \cdot V_{\alpha_1 \alpha_2} \otimes V_{\alpha_3} \otimes \cdots \otimes V_{\alpha_n}$$
$$- f \cdot V_{\alpha_1} \otimes V_{\alpha_2 \alpha_3} \otimes \cdots \otimes V_{\alpha_n} \cdots \pm f V_{\alpha_1} \otimes \cdots \otimes V_{\alpha_{n-1}}$$

This gives you $\beta_k^{(2)}(R)$, the L^2-Betti numbers of the equivalence relation R.

$$\beta_k^{(2)}(R) = \beta_k^{(2)}(L^\infty(X) \subseteq A)$$

Turns out these equal $\beta_k^{(2)}(L^\infty(X) \subseteq L^\infty(X) \rtimes_{alg} \Gamma)$ even though A is bigger than $L^\infty(X) \rtimes_{alg} \Gamma$.

(A has infinite sums $\sum x_k \cdot \chi_{F_k}$ provided F_k disjoint.)

Key observation is that any $x \in A$ is a finite sum of elements from $L^\infty(X) \rtimes \Gamma$ except at a small set:

$\forall x \in A \quad \forall \varepsilon > 0 \quad \exists X_\varepsilon \subseteq X$ with $\mu(X_\varepsilon) < \varepsilon$

and $x \cdot \chi_{X \setminus X_\varepsilon} \in L^\infty(X) \rtimes \Gamma$.

Details of proof omitted.

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Now $\beta_k^{(2)}(L^\infty(X) \subseteq L^\infty(X) \rtimes_{alg} \Gamma)$

is the same as $\beta_k^{(2)}(\Gamma)$,

which leads to

<u>Theorem</u> (Gaboriau 2000)

If R is induced by a free action Γ, i.e. $R = R_\Gamma$,

then $\beta_k^{(2)}(R) = \beta_k^{(2)}(\Gamma)$.

<u>Corollary</u> For $n \neq m$, no actions of $\mathbb{F}_m$ and $\mathbb{F}_n$ can be orbit-equivalent.

<u>Note</u> If $\alpha \in Aut(R)$ then α induces a map on cohomology

$$H_k^{(2)}(R) \xrightarrow{\alpha_*} H_k^{(2)}(R)$$

modules over $L^\infty(X) \rtimes \Gamma$ (often finite)

This corresponds to a map $Aut(R) \to Aut_{L^\infty(X) \rtimes \Gamma}(H_k) \simeq M_{p \times p}(L^\infty(X) \rtimes \Gamma)$.

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$$\begin{array}{ccccccc} & & C_0 & \leftarrow & C_1 & \leftarrow & \cdots \\ L^\infty(X) & \leftarrow & A & \leftarrow & A \otimes_{L^\infty(X)} A & \leftarrow & \cdots \end{array}$$

C_k = "span of k-chains"

Example $\mathcal{F} \subseteq \mathcal{M}$ a foliation of a manifold

Given a transversal T, define two points of T to be equivalent if they intersect the same leaf.

This equivalence relation is discrete.

Assume R on T is measure-preserving.

Each orbit of R is a discrete subset of a manifold (the corresponding leaf).

With an appropriate triangulation of the leaves, you get a "simplicial complex over R"

C_k = span of k-chains in the leaves

C_k is acted on by A (holonomies)

These C_k's roughly correspond to **these** ones

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<u>Example</u>

$\mathbb{C}\Gamma \subseteq L(\Gamma)$

$L^\infty(X) \rtimes_{alg} \Gamma \subseteq L^\infty(X) \rtimes \Gamma$

$A \subseteq L^\infty(X) \rtimes \Gamma$

$M \otimes_{alg} M^{op} \subseteq M \bar{\otimes} M^{op}$

Q. Is $\dim_{M_k \bar{\otimes} M_k^{op}} H_k(M \otimes_{alg} M^{op}; L^2(M \bar{\otimes} M^{op}))$ something reasonable? (Open.)

Do know $\dim_{M \bar{\otimes} M^{op}} H_0(M \otimes M^{op}, M \otimes M^{op}) = 0$ unless M's summands are matrix algebras.

(A. Thom)

$$\beta_1^{(2)} = \dim_{M \bar{\otimes} M^{op}} \left(\text{derivations } M \to \mathrm{Aff}(M \bar{\otimes} M^{op}) \right)$$

Unfortunately nontrivial examples are lacking...

<u>Thm</u> (Sukochev and some others...)

$M = L^\infty[0,1]$ $\quad \mathrm{Aff}(M) = \mathcal{M}[0,1]$ so affiliated operators are the non-commutative analogue of measurable functions.

$\exists$ a derivation $\delta : M \to \mathrm{Aff}(M)$ extending $\frac{d}{dt}$ on polynomials.

<u>Fact</u> Let $M = L^\infty([0,1])$.
<u>Any</u> derivation $M \to \mathrm{Aff}(M \bar{\otimes} M^{op}) = \mathcal{M}([0,1]^2)$ is inner.
$(\Rightarrow \beta^{(2)}(M) = 0 \quad)$

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One can prove

1. If $X_1, \ldots, X_n$ generate $L(\mathbb{F}_n)$ are group generators

$$\exists\ \delta_j : \cancel{Alg}\ (X_1, \ldots, X_n) \longrightarrow \mathcal{M}(L(\mathbb{F}_n))$$

$$\delta_j(X_i) = \delta_{ij}$$

(extend from group)

$$\dim_{M \otimes M^{op}} \left(Der\left(Alg(X_1, \ldots, X_n) ; Aff(\ )\right)\right) = n.$$

2. $\forall\ Y_1, \ldots, Y_m \in W^*(X_1, \ldots, X_n)$

$$\dim_{M \otimes M^{op}} Der\left(Alg(X_1, \ldots, X_n, Y_1, \ldots, Y_m) ; Aff(M \otimes M^{op})\right) \geq n.$$

Q. Is $\mathrm{Der}(M; \mathrm{Aff}(M\bar{\otimes}M^{op})) \neq 0$ $\quad M = L(\mathbb{F}_n)$

To answer "yes" want to do:

$$V(X_1,\dots,X_n,Y_1,\dots,Y_m) = \{(\delta(X_1),\dots,\delta(X_n)) \in L^2(M)\otimes L^2(M):$$
$$X_1,\dots,X_n,Y_1,\dots,Y_m \in M \qquad \delta \in \mathrm{Der}(\mathrm{Alg}(X_1,\dots,X_n,Y_1,\dots,Y_m), L^2(M)\otimes L^2(M))\}$$

$$\Delta(X_1,\dots,X_n,Y_1,\dots,Y_m) = \dim_{M\bar{\otimes}M^{op}} V(X_1,\dots,X_n,Y_1,\dots,Y_m)$$

$$\dim_{M\otimes M^{op}}(\mathrm{Der}(M; \mathrm{Aff}(M\bar{\otimes}M^{op}))) = \sup_{n,X_1,\dots,X_n} \ \inf_{m,Y_1,\dots,Y_m} \Delta(X_1,\dots,X_n,Y_1,\dots,Y_m)$$

Want lower bound on Δ.

Theorem $L(\mathbb{F}_n) = W^*(\underbrace{X_1,\dots,X_n}_{\text{free semicircular family}})$

$$\Rightarrow \Delta(X_1,\dots,X_n,Y_1,\dots,Y_m) \geq n \qquad \forall Y_1,\dots,Y_m.$$

Pf Sketch
(1) $\Delta(X_1',\dots,X_n') \geq \delta^*(X_1',\dots,X_n')$
(2) $\delta^*(X_1',\dots,X_n') \geq \delta(X_1',\dots,X_n')$
(3) $\delta(X_1',\dots,X_n') \geq \delta_0(X_1',\dots,X_n')$
(4) $\delta_0(X_1,\dots,X_n,Y_1,\dots,Y_m) \geq \delta_0(X_1,\dots,X_n)$ if $Y_1,\dots,Y_m \in W^*(X_1,\dots,X_n)$.

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So what are these things?

$\delta_0(Z_1, \ldots, Z_n)$ = "Free entropy dimension"

$\Gamma(Z_1, \ldots, Z_n, k, l, \varepsilon) = \{ (z_1, \ldots, z_n) \in (M_{k \times k}^{SA})^n$ s.t.

$$\left| \frac{1}{k} \mathrm{Tr}(p(z_1, \ldots, z_n)) - \tau(p(Z_1, \ldots, Z_n)) \right| < \varepsilon$$

$\forall p$ monomial of degree $\leq l \}$.

$$\delta_0(Z_1, \ldots, Z_n) = \limsup_{\delta \downarrow 0} \inf_{l, \varepsilon} \limsup_{k \to \infty} \frac{1}{k^2} \log K_\delta(\Gamma(Z_1, \ldots, Z_n), k, l, \varepsilon).$$

Here $K_\delta(X)$ = min # δ-balls required to cover X

$$\limsup_{\delta \downarrow 0} \frac{\log K_\delta(X)}{\log \delta} = \text{"covering dimension" of } X$$

ICBST (e.g. use HW from 259A) that

$\delta_0(X_1, \ldots, X_n) = n$ if $X_1, \ldots, X_n$ semicircular.

Now $\pi_X(\Gamma(X_1, \ldots, X_n, Y_1, \ldots, Y_m)) \subseteq \Gamma(X_1, \ldots, X_n)$

In fact they're equal (use the fact that $Y_i \in W^*(X_1, \ldots, X_n)$ so they can be approximated by polynomials in $X_1, \ldots, X_n$).

Then you just use the fact that projections cannot increase covering numbers.

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Free entropy is

$$\chi(A_1,\dots,A_n) = \inf_{\ell,\varepsilon} \limsup_{k\to\infty} \frac{1}{k^2}\log \mathrm{Vol}\left(\Gamma(A_1,\dots,A_n),k,\ell,\varepsilon + \frac{n}{2}\log k\right)$$

$$\delta(Z_1,\dots,Z_n) = n - \limsup_{t\downarrow 0} \frac{\chi(Z_1+tS_1,\dots,Z_n+tS_n)}{\log t}$$

$S_1,\dots,S_n$ free semicircular, free from $Z_1,\dots,Z_n$

$\delta \geq \delta_0$

Use L'Hôpital to get $\quad n - \lim \dfrac{\frac{d}{ds}\Big|_{s=t} \chi(Z_1+sS_1,\dots,Z_n+sS_n)}{1/t}$

How to compute this derivative?

$\mathfrak{A} = \mathrm{Alg}(A_1,\dots,A_n)$

$\partial_j : L^2(\mathfrak{A}) \longrightarrow L^2(\mathfrak{A})\otimes L^2(\mathfrak{A})$ densely defined on $\mathfrak{A}$

$\partial_j(A_i) = \delta_{ij}\, 1\otimes 1$

If $1\otimes 1 \in$ domain ∂_j^*

$$\Phi^*(A_1,\dots,A_n) \stackrel{\text{def}}{=} \sum_{j=1}^{n} \|\partial_j^*(1\otimes 1)\|_2^2 \qquad \text{"free Fisher information"}$$

<u>Th (Biane / Capitain / Guillonet)</u>

$$\frac{d}{ds}\Big|_{s=t} \chi(Z_1+sS_1,\dots,Z_n+sS_n) \geq \Phi^*(Z_1+tS_1,\dots,Z_n+tS_n)$$

This is a hard theorem.

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Define

$$\delta^*(Z_1,\dots,Z_n) = n - \limsup_{t\downarrow 0} t\Phi^*(Z_1+tS_1,\dots,Z_n+tS_n)$$

Then (2) in the proof works.

$$\delta^*(\underbrace{X_1,\dots,X_n}_{\text{free semicircular.}}) = n$$

$X_j = l_j + l_j^*$

l_j = left creation ops on $\bigoplus_{N\geq 0} (\mathbb{C}^n)^{\otimes N}$

r_j = right creation operator

$[r_j, l_i] = 0 \qquad [r_j, l_i^*] = -\delta_{ij} P_\Omega$

This is Hilbert-Schmidt (it's in $L^2(M)\bar{\otimes} L^2(M)$ where $M = W^*(X_1,\dots,X_n)$)

$$\partial_j(x) = [x, r_j] : \underset{\text{Alg}(X_1,\dots,X_n)}{\overset{\shortparallel}{\mathcal{O}\!\mathcal{L}}} \to L^2(M)\bar{\otimes} L^2(M)$$

If $\partial_j = [\,\cdot\,, B_j]$ then $\partial_j^*(1\otimes 1) = (B_j - JB_j^*J)\Omega$

$W^*(Z_1+tS_1,\dots,Z_n+tS_n) \subseteq W^*(Z_1,\dots,Z_n,S_1,\dots,S_n)$
$\subseteq \underset{\tilde{r}_j}{B}\left(L^2(W^*(Z_1,\dots,Z_n)) * L^2(W^*(S_1,\dots,S_n))\right)$

$r_j \in B(L^2(W^*(S_1,\dots,S_n)))$

$[\tilde{r}_j, x] = 0 \quad \forall x \in W^*(Z_1,\dots,Z_n)$

$[S_i, \tilde{r}_j] = \delta_{ij} P_1$

$B_j^t = \frac{1}{t} E_t \tilde{r}_j E_t$

Z's satisfy algebraic relations $\Rightarrow$ $Z_j + tS_j$ almost satisfy them $\Rightarrow$ yay.