PRACTICE PROBLEMS FOR THE FINAL.

Problem 1. Let f, g be continuous functions of period 2π, with Fourier coefficients $c_k$and d_k. Prove that

$$\langle f, g\rangle_{L^2} = \sum_k c_k \bar{d}_k.$$

Note that this gives Parseval's relation in the case that $f = g$.

Solution. Let $S_n(f) = \sum_{-n}^n c_k \exp(ikx)$ and $S_n(g) = \sum_{-n}^n d_k \exp(ikx)$ be the partial sums of the Fourier series for f and g. Thus $\|S_n(f) - f\|_2 \to 0$ and $\|S_n(g) - g\|_2 \to 0$ as $n \to \infty$ by Parselval's theorem. Furthermore, $\|S_n(f)\|_2 \leq \|f\|_2$ and $\|S_n(g)\|_2 \leq \|g\|_2$.

Note that because $\exp(ikx)$, $k \in \mathbb{Z}$ form an orthonormal family, we have that

$$\langle S_n(f), S_n(g)\rangle = \sum_{-n}^n c_k \overline{d_k}.$$

We now compute

$$\begin{aligned} |\langle S_n(f), S_n(g)\rangle - \langle f, g\rangle| &= |\langle S_n(f) - f, S_n(g)\rangle + \langle f, S_n(g)\rangle - \langle f, g\rangle| \\ &\leq |\langle S_n(f) - f, S_n(g)\rangle| + |\langle f, S_n(g) - g\rangle|. \end{aligned}$$

Using the Cauchy-Schwartz inequality, we get

$$\begin{aligned} |\langle S_n(f), S_n(g)\rangle - \langle f, g\rangle| &\leq \|S_n(f) - f\|_2 \|S_n(g)\|_2 + \|f\|_2 \|S_n(g) - g\|_2 \\ &\leq \|S_n(f) - f\|_2 \|g\|_2 + \|S_n(g) - g\|_2 \|f_2\| \to 0 \end{aligned}$$

since $\|S_n(f) - f\|_2 \to 0$ and $\|S_n(g) - g\|_2 \to 0$. Thus

$$\sum_{-n}^n c_k \bar{d}_k = \langle S_n(f), S_n(g)\rangle \to \langle f, g\rangle.$$

Problem 2. Prove the Riemann-Lebesgue lemma: if f is piece-wise continuous on $[a, b]$, then

$$\lim_{n\to\infty} \int_a^b f(t) \exp(int) dt = 0.$$

Solution. See book.

Problem 3. Let f be a complex-valued 2π periodic function, and consider the Fourier coefficients

$$c_k = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(t) \exp(-ikt) dt.$$

Find a condition on c_k that holds if and only if f is real-valued. *Note: we should have required that f be continuously differentiable in this problem!*

Solution. If f is real-valued, we have that

$$\begin{aligned}\overline{c_k} &= \frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi}\overline{f(t)\exp(-ikt)}dt\\ &= \frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi} f(t)\exp(+ikt)dt\\ &= c_{-k}.\end{aligned}$$

Coversely, if $\overline{c_k} = c_{-k}$, from pointwise convergence of the Fourier series of f to f we deduce that

$$\overline{f(x)} = \lim_{n\to\infty}\sum_{-n}^{n}\overline{c_k}\overline{\exp(ikt)} = \lim_{n\to\infty}\sum_{-n}^{n} c_{-k}\exp(-ikt) = f(x),$$

so that f is real-valued.

Problem 4. Find the radius of convergence of the series

$$\sum_{j=0}^{\infty}(j+1)(j+2)x^j.$$

Find the function to which this power series converges.

Solution. Let

$$\rho = \lim_{n\to\infty}(n+1)(n+2)^{1/n} = 1.$$

Thus the radius of convergence of this power series is $1/\rho = 1$.

Let

$$f(x) = \sum_{j=0}^{\infty}(j+1)(j+2)x^j.$$

Let

$$g(x) = \int_0^x f(t)dt.$$

Then we can integrate the power series term-by-term and get that

$$g(x) = \sum_{j=0}^{\infty}(j+2)x^{j+1} = \sum_{j=1}^{\infty}(j+1)x^j.$$

Let

$$h(x) = \int_0^x g(t)dt.$$

We can integrate term-by-term to get that

$$h(x) = \sum_{j=1}^{\infty}x^{j+1} = \sum_{j=2}^{\infty}x^j.$$

Note that

$$1 + x + h(x) = \sum_{j=0}^{\infty}x^j = \frac{1}{1-x}.$$

Thus

$$h(x) = \frac{1}{1-x} - 1 - x.$$

We can now recover g as h' and f as $g' = h''$:

$$g(x) = h'(x) = \frac{1}{(1-x)^2} - 1,$$

$$f(x) = g'(x) = \frac{2}{(1-x)^3}.$$