NOTES ON MATH 223S: TOPICS IN SET THEORY
UCLA FALL 2010

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1. September 24

Prerequisite knowledge: First-order logic; the constructible universe L

Goals of the course: To discuss motivating ideas behind the core model (or inner model) theory project for large cardinals.

Immediate goal: Jensen's "covering lemma": It says (in particular) that if λ is a limit cardinal of uncountable cofinality (eg $\lambda = \aleph_{\omega_1}$) then either $(\lambda^+)^L = \lambda^+$, or L is "very small" (e.g. every uncountable cardinal is inaccessible in L, $|\mathbb{R}^L| = \aleph_0$, $0^\#$ exists).

Consequences: Certain combinatorial statements (such as "PFA", the failure of "SCH") have large cardinal strength.

Other applications of inner model theory ideas: Descriptive set theory (understanding sets more complicated than $\mathbf{\Sigma}^1_2$, e.g. Martin-Solovay tree for $\mathbf{\Sigma}^1_3$), exotic notions of forcing (e.g. Woodin's extender algebra, Prikry forcing...)

- Measurable Cardinals
  - clubs, stationary sets
- Elementary embeddings
  - Scott: $\exists j\colon V \to M$ elementary (and nontrivial), M well founded iff there exists a measurable cardinal;
  - $0^\#$ $\exists$ iff $\exists j\colon L \to L$ elementary (and nontrivial)
- Then: Jensen's covering lemma!

Definition. A cardinal κ is **measurable** if there is $\mu \subseteq \mathcal{P}(\kappa)$ such that

(i) μ is an ultrafilter, i.e.
  - $A, B \in \mu \Rightarrow A \cap B \in \mu$
  - $A \in \mu, B \supset A \Rightarrow B \in \mu$
  - $A \subseteq \kappa \Rightarrow A \in \mu$ or $A^c \in \mu$

(ii) $\{a\} \notin \mu$ all $\alpha \in \kappa$ (i.e. μ is "non-principal")

(iii) if $\delta < \kappa$, $(A_\alpha)_{\alpha\in\delta}$ a sequence of sets in μ, then $\bigcap_{\alpha\in\delta} A_\alpha \in \mu$

(iv) $\kappa > \omega$.

We then say that μ is a **measure on** κ. μ is then said to be **normal** if for any $A \in \mu$, $f\colon A \to \kappa$ with $f(\alpha) < \alpha$ all $\alpha \in A$, we have some $B \subseteq A$, $B \in \mu$, and some $\alpha_0 \in \kappa$, with $f(\alpha) = \alpha_0$ all $\alpha \in B$.

Theorem. *If κ is measurable, then there is a normal measure on κ.*

Proof. Fix a measure ν on κ. For $f, g\colon \kappa \to \kappa$, set

$$f \leq_\nu g$$

iff for some $A \in \nu$ we have

$$f(\alpha) \leq g(\alpha) \text{ all } \alpha \in A.$$

Claim. *$\leq_\nu$ is a "pre-linear" order.*

Proof of claim. Transitivity: Suppose $A_1, A_2 \in \nu$ witness $f_1 \leq_\nu f_2$, $f_2 \leq_\nu f_3$ (i.e. $\forall\alpha \in A_i(f_i(\alpha) \leq_\nu f_{i+1}(\alpha))$). Now take $A = A_1 \cap A_2$. Then $A \in \nu$, and for all $\alpha \in A$, $f_1(\alpha) \leq f_2(\alpha) \leq f_3(\alpha)$.

"*Dichotomy*": Given $f_1, f_2\colon \kappa \to \kappa$ let $A = \{\alpha \in \kappa : f_1(\alpha) \leq f_2(\alpha)\}$ (then $A^c = \{\alpha \in \kappa : f_1(\alpha) > f_2(\alpha)\}$). If $A \in \nu$, then $f_1 \leq_\nu f_2$; if $A \notin \nu$, then $A^c \in \nu$: $f_2 \leq_\nu f_1$. □

Next time: $\leq_\nu$ is a "pre-wellorder". Then: choose $g\colon \kappa \to \kappa$ such that g is $\leq_\nu$-minimal subject to g not constant on any $A \in \nu$. Now: set

$$\mu = \{B \subseteq \kappa : g^{-1}[B] \in \nu\}.$$

2. September 27

Normal measures
- stationary sets

κ a measurable cardinal $\Rightarrow$ indiscernibles for L_κ; in fact, a club of indiscernibles in L_κ.
Theories of indiscernibles: defn of $0^\#$!
$0^\#\exists$ iff $\exists j\colon L \to L$ elementary, $j \neq \text{id}$
Jensen's covering lemma
(Scott: $\exists$ measurable iff $\exists j\colon V \to M$, $j \neq \text{id}$, j elementary, M transitive)

Last time, we began the proof of:

Theorem. *If κ is measurable, then there exists a normal measure on κ.*

Note. a normal measure is one that doesn't allow a nontrivial regressive function: if for any $f\colon \kappa \to \kappa$, $f(\alpha) < \alpha$ all $\alpha \in \kappa$ (or just $\alpha \in B$ some $B \in \mu$), there is $A \in \mu$ s.t. $f|_A$ constant.

Exercise. For ν an ultrafilter on κ, the following are equivalent:

1. for any $f\colon \kappa \to \kappa$, $f(\alpha) < \alpha$ all $\alpha \in \kappa$, $\exists A \in \nu\,(f|_A$ constant)
2. for any $f\colon \kappa \to \kappa, B \in \nu$ with $f(\alpha) < \alpha$ all $\alpha \in B$, $\exists A \in \nu\,(f|_A$ constant).

Proof of theorem (continued): Fix ν a κ-complete, non-principal ultrafilter on κ. For $f, g\colon \kappa \to \kappa$, set $f \leq_\nu g$ if on some $A \in \nu$, $f(\alpha) \leq g(\alpha)$ all $\alpha \in A$.

Claim. *$\leq_\nu$ is a pre-wellorder.*

Proof of claim. Suppose not. I.e., $\exists (f_n)_{n\in\omega}$, $f_{n+1} <_\nu f_n$. Then at each n, $B_n := \{\alpha : f_n(\alpha) \leq f_{n+1}(\alpha)\}$ is *not* in ν. Therefore $A_n := (B_n)^c$ is in ν. Let $A = \bigcap_{n\in\omega} A_n$; then $A \in \nu$. Then on each $\alpha \in A$, each n, $f_{n+1}(\alpha) < f_n(\alpha)$. Thus $(f_n(\alpha))_{n\in\omega}$ is an infinite descending chain of ordinals, contradiction. □

Let $X = \{f\colon \kappa \to \kappa \mid f \text{ is not constant on any } A \in \nu\}$. Let $g \in X$ be $\leq_\nu$-minimal, i.e. $\forall f \in X (g \leq_\nu f)$.

Let $\mu = \{A \subset \kappa \;:\; g^{-1}[A] \in \nu\}$.

Claim. *μ is an ultrafilter.*

Proof of claim. Suppose $A_1, A_2 \in \mu$ (i.e. $g^{-1}[A_i] \in \nu$). Let $B_i = g^{-1}[A_i]$ $(i = 1, 2)$. Then $B_1 \cap B_2 \in \nu$. But $g^{-1}[A_1 \cap A_2] = B_1 \cap B_2 \in \nu$.

If $A \in \mu, B \supseteq A$, then $g^{-1}[B] \supseteq g^{-1}[A] \in \nu$; so $B \in \mu$.

Similarly, if $A \notin \mu$, then $g^{-1}[A] \notin \nu$, so $(g^{-1}[A])^c \in \nu$. But $g^{-1}([A])^c = g^{-1}[A^c]$ therefore $A^c \in \mu$. □

Note. μ is not principal. (For $\alpha_0 \in \kappa$, if $\{\alpha_0\} \in \mu$, then $B = g^{-1}[\{\alpha_0\}] \in \nu$. But then $g|_B$ constant, contradicting $g \in X$.)

For normality, suppose $f\colon \kappa \to \kappa$ has $f(\alpha) < \alpha$ all $\alpha \in \kappa$. Then consider $h = f \circ g\colon \kappa \to \kappa$. $h(\beta) = f(g(\beta)) < g(\beta)$ all $\beta \in \kappa$.

Then $h <_\nu g$: therefore (since $h \notin X$) $\exists A \in \nu$ such that $h|_A$ constant – say $h(\alpha) = \gamma_0$ all $\alpha \in A$.

Let $B = \{\alpha \in \kappa \;:\; f(\alpha) = \gamma_0\}$. $\rho^{-1}[B] = A$, therefore $B \in \mu$. $f|_B$ is constant. □

Remark. In effect, we have formed $\mathrm{Ult}_\nu(\kappa, <)$ and noted it is a wellorder. We (in effect) fixed $\pi\colon \mathrm{Ult}_\nu(\kappa, <) \cong (\lambda, <)$ and let $g\colon \kappa \to \kappa$ have $\pi([g]_\nu) = \kappa$.

Definition. For $(A_\alpha)_{\alpha\in\kappa}$ a sequence of subsets of κ,

$$\mathop{\triangle}_{\alpha\in\kappa} A_\alpha = \{\beta \in \kappa \mid \text{ for all } \alpha < \beta(\beta \in A_\alpha)\}$$

the "**diagonal intersection**" of $(A_\alpha)_{\alpha\in\kappa}$.

Exercise. An ultrafilter μ on κ is normal if it is closed under diagonal intersections. I.e., if $A_\alpha \in \mu$ all $\alpha \in \kappa$, then $\triangle_{\alpha\in\kappa} A_\alpha \in \mu$.

Definition. Let δ be a regular cardinal. $C \subseteq \delta$ is a **club** if:

(i) $\forall \alpha \in \delta\, \exists \beta > \alpha (\beta \in C)$ (C "unbounded")

(ii) if $\lambda \in \delta$ is a limit and $\forall \alpha < \lambda\, \exists \beta \in C (\beta > \alpha, \beta < \lambda)$ then $\lambda \in C$ (C is "closed")

E.g. 1. The set $\{\lambda \in \delta \mid \lambda \text{ is a limit}\}$

E.g. 2. Let $\gamma < \delta$. Suppose at each $\alpha < \gamma$ we have $f_\alpha\colon \delta \to \delta$. Then the set C of closure points, i.e. $\{\lambda \;:\; \forall \alpha \in \gamma\, \forall \beta < \lambda (f_\alpha(\beta) < \lambda)\}$ is a club.

E.g. 3. Let $\mathcal{L}$ be a language of size $< \delta$. Let $\mathcal{M} = (M; R^{\mathcal{M}}, \ldots, f^{\mathcal{M}}, \ldots)$ with $M = \delta$. Then $\{\alpha < \delta \mid (\alpha; R^{\mathcal{M}}, \ldots, f^{\mathcal{M}}) \prec \mathcal{M}\}$ is a club. (Exercises)

3. September 29

Note that some people make use of the notion of club when the ordinal in question may not be regular (indeed, people make use of the notion of club with respect to things besides the ordinals); but clubs in a regular cardinal are what's popular, so that's what we'll stick to.

Lemma. *Let δ be regular. If $\gamma < \delta$, $(C_\alpha)_{\alpha\in\gamma}$ clubs in δ, then $\bigcap_{\alpha\in\gamma} C_\alpha$ is a club.*

Proof. Closed is immediate. For unbounded, fix $\beta \in \delta$. Now, for each $\alpha \in \gamma$, choose $\beta_{0,\alpha} \in C_\alpha, \beta_{0,\alpha} > \beta$. Since δ is regular,

$$\beta_1 = \sup_{\alpha \in \gamma} \beta_{0,\alpha} < \delta.$$

Now again, at each $\alpha \in \gamma$ choose $\beta_{1,n} \in C_\alpha, \beta_{1,n} > \beta_1$. Again,

$$\beta_2 = \sup_{\alpha \in \gamma} \beta_{1,\alpha} < \delta.$$

Repeat.

We get $\beta < \beta_1 < \beta_2 < \beta_3 < \cdots < \delta$, and for each α, n there exists some $\beta_{n,\alpha} > \beta_n$, $\beta_{n,\alpha} < \beta_{n+1}$. Let $\beta_\omega = \sup_{n \in \omega} \beta_n$. Then $\beta_\omega \in C_\alpha$ each α (since $C_\alpha \cap \beta_\omega$ is unbounded). □

Exercise. Let δ be regular. Suppose at each $\alpha \in \delta$ we have $C_\alpha \subseteq \delta$ club. Then $\triangle_{\alpha \in \kappa} C_\alpha$ is again club.

Definition. Let δ be regular. $S \subseteq \delta$ is **stationary** if for any club $C \subseteq \delta$, $C \cap S \neq \emptyset$.

E.g. 1. If $S \subseteq \delta$ is club, then it is stationary.
E.g. 2. If S includes a club, then it is stationary.
E.g. 3. For $\delta = \omega_2$, $\{\alpha < \omega_2 \ : \ \mathrm{cof}(\alpha) = \omega\}$ is stationary.
E.g. 4. $\{\alpha < \omega_2 \ : \ \mathrm{cof}(\alpha) = \omega_1\}$ is stationary in ω_2.

Lemma. *There are $S_1, S_2 \subseteq \omega_1$ stationary, $S_1 \cap S_2 = \emptyset$.*

Proof. Note: For $S \subseteq \omega_1$, either S or S^c is stationary. The failure of the lemma would imply that every stationary set contains a club (since *at most* one of S,S^c could ever be stationary).

Thus we have: S is stationary iff it contains a club. Moreover,

$$\mu = \{S \subseteq \omega_1 \mid S \text{ contains a club}\}$$

is an ultrafilter. By the last lemma, μ is countably complete. So ω_1 is measurable.

Fix distinct elements of 2^ω,

$$(x_\alpha)_{\alpha \in \omega_1} \qquad (2^{\aleph_0} \geq \omega_1).$$

At each n, choose $A_n \in \mu$, $i_n \in \{0,1\}$ such that $x_\alpha(n) = i_n$ all $\alpha \in A_n$. Let $A = \bigcap A_n \in \mu$. A contains a club, therefore $|A| > 1$. Choose $\alpha, \beta \in A$, $\alpha \neq \beta$. Then at each n, $x_\alpha(n) = i_n = x_\beta(n)$; therefore $x_\alpha = x_\beta$, contradiction. □

Remark. This proof used Choice in an essential way. In fact, Solovay showed that ZF plus the Axiom of Determinacy *implies* ω_1 is measurable! In particular, AD implies that there is no ω_1-sequence of distinct reals.

Theorem (“Fodor's lemma”)**.** *Let δ be regular. If $S \subseteq \delta$ is stationary, then for any regressive function $f \colon S \to \delta$, there is a stationary $S_0 \subseteq S$ such that $f|_{S_0}$ is constant.*

Proof. Suppose not. Then for any $\alpha \in \delta$, $f^{-1}[\{a\}]$ is non-stationary, so there exists C_α club in δ such that $f(\beta) \neq \alpha$ all $\beta \in C_\alpha \cap S$. Let $C = \triangle_{\alpha \in \delta} C_\alpha$.

Claim. *C is club.*

Proof of claim. Closure is clear. For unbounded, choose $\beta \in \delta$. Then (exercise) choose $\beta_0 = \beta < \beta_1 < \beta_2 < \dots$ such that at each n, each $\alpha < \beta_n$, $\exists \gamma \in C_\alpha$, $\beta_n < \gamma < \beta_{n+1}$. Then $\beta_\omega = \sup \beta_n \in \triangle_{\alpha \in \delta} C_\alpha$. □

Then for any $\alpha \in C \cap S$ (and such α exists since S is stationary), $f(\alpha) = \beta$ would have $\beta < \alpha$, but then $\alpha \in C_\beta$ (since $\alpha \in C = \triangle_{\gamma \in \delta} C_\gamma$). Therefore $f(\alpha) \neq \beta$, contradiction. $\square$

Lemma. *If $S \subseteq \delta$ is* not *stationary, then there exists $f \colon S \to \delta$, f regressive, f not constant on any unbounded set.*

Proof. Take $C \subseteq \delta$ club, $C \cap S = \emptyset$. For $\alpha \in S$, let $f(\alpha) = \sup\{\beta < \alpha \ : \ \beta \in C\}$. $\square$

Corollary. *Let μ be a measure on κ. If μ is normal, then every element of μ is stationary.*

Proof. Uses μ non-principal, κ-complete $\Rightarrow \forall A \in \mu \ (|A| = \kappa)$. $\square$

Next: if κ is a measurable cardinal, then there will be a club in κ consisting of indiscernibles for L.

4. October 1

Definition. Let $\mathcal{M}$ be a model. Let $I \subseteq M$, and let $<$ be a linear ordering of I. $(I, <)$ is said to be a set of **indiscernibles** for $\mathcal{M}$ if for all n, $a_1 < \cdots < a_n \in I$, $b_1 < \cdots < b_n \in I$, and $\varphi(x_1, \ldots, x_n)$ a first order formula,

$$\mathcal{M} \models \varphi(a_1, \ldots, a_n) \text{ iff } \mathcal{M} \models \varphi(b_1, \ldots, b_n).$$

Exercise. Suppose T is a first order theory with an infinite model. Then T has a model with an infinite set of indiscernibles. (Hint: Let $(c_n)_{n \in \omega}$ be fresh constants. Show that

$$T \ \cup \ \{\varphi(c_{i_1}, c_{i_2}, \ldots, c_{i_n}) \iff \varphi(c_{j_1}, c_{j_2}, \ldots, c_{j_n}) \ : \ i_1 < i_2 < \cdots < i_n, \\ j_1 < j_2 < \cdots < j_n, \varphi \text{ first order}\}$$

is consistent.

To show every finite part is consistent, use Ramsey's Theorem: $[\omega]^n$ =increasing sequences from ω. Given $f \colon [\omega]^n \to \{0, 1\}$, $\exists A \subset \omega$ infinite with $f|_{[A]^n}$ constant.)

Exercise. If T has an infinite model, then T has a non-rigid model, i.e. one which admits a non-trivial automorphism. (Hint: First assume T comes equipped with Skolem functions. Second, get $\mathcal{M}$ with indiscernibles $(C_l)_{l \in \mathbb{Z}}$. Third: Let $\mathbb{N}$ = Skolem Hull (in $\mathcal{M}$) of $\{C_l \mid l \in \mathbb{Z}\}$. Fourth: Any automorphism of $(C_l)_{l \in \mathbb{Z}}$ lifts to $\mathbb{N}$.)

Aside. Leo Harrington proved that Σ^1_1 games are determined if and only if the cardinals (in V) form a class of indiscernibles for L.

We will head to: measurables give clubs of indiscernibles for L.

Definition. Let $(A, <)$ be an ordered set. $[A]^n := \{(a_1, \ldots, a_n) \in A^n \ : \ a_1 < a_2 < \cdots < a_n\}$.

Lemma. *Let κ be a measurable cardinal, μ a normal measure on κ, and $f \colon [\kappa]^n \to \{0, 1\}$. Then there exists $A \in \mu$ such that $f|_{[A]^n}$ is constant.*

Proof. Induction on n. $n = 1$ is trivial. Assume it is true at n. Fix $f \colon [\kappa]^{n+1} \to \{0, 1\}$.

At each $\alpha \in \kappa$, let $f_\alpha \colon [\kappa \setminus \alpha + 1]^n \to \{0, 1\}$ be defined by $\vec{\beta} \mapsto f(\alpha, \vec{\beta})$. By inductive assumption, at each $\alpha \in \kappa$, get $A_\alpha \in \mu$. $f_\alpha|_{[A_\alpha]^n}$ is constant. Let $A = \triangle_{\alpha \in \kappa} A_\alpha$.

Claim. *$A \in \mu$.*

(Why? Otherwise $A^c \in \mu$. At each $\beta \in A^c$, let $g(\beta) =$ least α such that $\beta \notin A_\alpha$. Normality of μ gives $B \subset A^c$, $B \in \mu$, $\alpha_0 \in \kappa$, $g(\beta) = \alpha_0$ all $\beta \in B$. But then: $B \cap A_{\alpha_0} = \emptyset$.)

At each $\alpha \in A$, $A \setminus \alpha + 1 \subseteq A_\alpha$. Then $f_\alpha|_{[A\setminus\alpha+1]^n}$ is constant. Let $g(\alpha) \in \{0,1\}$ be this constant value. We can find $B \subseteq A$, $B \in \mu$ such that $g|_B$ is constant. Fix $i_0 \in \{0,1\}$, $g(\alpha) = i_0$ all $\alpha \in B$. Then for any $\alpha_1 < \alpha_2 < \cdots < \alpha_n < \alpha_{n+1} \in B$,

$$f(\alpha_1, \ldots, \alpha_n, \alpha_{n+1}) = f_{\alpha_1}(\alpha_2, \ldots, \alpha_n, \alpha_{n+1}).$$

But since $\alpha_2, \ldots, \alpha_n, \alpha_{n+1} > \alpha_1$, and $\alpha_1, \ldots, \alpha_{n+1} \in A$, we have

$$f_{\alpha_1}(\alpha_2, \ldots, \alpha_n, \alpha_{n+1}) = \text{ constant value for } f_{\alpha_1} = g(\alpha_1) = i_0.$$

□

Theorem. *Let κ be a measurable cardinal. Then there is a club $C \subseteq \kappa$ of indiscernibles for L_κ.*

Proof. Let $(\varphi(x_1, \ldots, x_{k(n)}))_{n\in\omega}$ enumerate the formulas of set theory. At each n, let $f_n\colon [\kappa]^{k(n)} \to \{0,1\}$ by

$$\vec{\alpha} \mapsto \begin{cases} 1 & \text{if } L_\kappa \models \varphi_n(\vec{\alpha}) \\ 0 & \text{if } L_\kappa \models \neg\varphi_n(\vec{\alpha}). \end{cases}$$

At each n, use last lemma to get $A_n \in \mu$, $f_n|_{[A_n]^{k(n)}}$ constant.

Let $A = \bigcap_{n\in\omega} A_n$. Then $A \in \mu$, A is a set of indiscernibles for L_κ. Since μ is normal, A is stationary.

Let $h_n\colon L_\kappa \to L_\kappa$ by

$$\vec{a} \mapsto \begin{cases} <_L \text{-least } b \text{ s.t. } L_\kappa \models \varphi_n(\vec{a}, b) & \text{if such } b \text{ exists} \\ 0 & \text{otherwise.} \end{cases}$$

So h_n is the "n-th definable Skolem function over L".

Let $(\rho_n)_{n\in\omega}$ enumerate all possible compositions of the h_n.

Exercise. This is unnecessary: the h_n's are closed under composition.

These are still all definable over L_κ. Let

$$\begin{aligned} M &= \text{Skolem Hull}_{L_\kappa}(A) \\ &= \{g_n(\vec{a}) \ : \ n \in \omega, \vec{\alpha} \in A^{<\omega}\}. \end{aligned}$$

$M \prec L_\kappa$ and $|M| = \kappa$. Fix $\pi\colon M \cong L_\alpha$. Note $\alpha = \kappa$.

Next time: $\pi[A]$ is closed in κ.

5. October 4

Last time, we proved:

Theorem. *If κ is measurable, then there is a stationary set $S \subset \kappa$ consisting of indiscernibles for L_κ.*

We resume the proof of:

Theorem. *Let κ be a regular cardinal, with $S \subseteq \kappa$ a stationary set of indiscernibles for L_κ. Then there is $C \subseteq \kappa$ such that*

(i) *C is club*
(ii) *C is a set of indiscernibles for L_κ*
(iii) *$C \cap \delta$ is club for all $\delta < \kappa$ regular.*

Proof (continued): Let M be the closure of S under functions definable over L_κ. Then (since L_κ has definable Skolem functions), $M \prec L_\kappa$. $(M, \in)$ is wellfounded and extensional, so by Mostowski,

$$\exists \pi\colon (M, \in) \cong (N, \in)$$

N transitive. By condensation, $N = L_\delta$ some δ.

Claim. *$\delta = \kappa$.*

Proof of claim. Since $M \subseteq L_\kappa$, $\mathrm{ot}(M \cap \mathrm{Ord}) \leq \kappa$; therefore $\delta \leq \kappa$. But $S \subseteq M$; therefore $|M \cap \mathrm{Ord}\,| \geq \kappa$, and $\delta \geq \kappa$. □

Let $C = \pi[S]$.

Note. Since S is a set of indiscernibles for L_κ and $M \prec L_\kappa$, C is a set of indiscernibles for $N = L_\kappa$. Moreover: since M is the Skolem hull of S, $N = L_\kappa$ is the Skolem hull of C.

Claim. *$S_0 := \{\alpha \in S \,:\, \pi(\alpha) = \alpha\}$ is stationary.*

Proof of claim. Suppose not, then $S_1 := S \setminus S_0$ is stationary. (In general: No stationary set is the union of two non-stationaries.) But then $\pi(\alpha) < \alpha$ all $\alpha \in S_1$, and π is $1-1$; but this contradicts Fodor's lemma. □

$S_0 \subseteq C$, C is stationary.

Claim. *C is club (in κ).*

Proof of claim. Suppose $\lambda < \kappa$ is a limit ordinal, $C \cap \lambda$ unbounded in λ, $\lambda \notin C$. Then there exist $\alpha_1 < \cdots < \alpha_n < \beta_1 < \cdots < \beta_m \in C$), f definable over L_κ, such that $\alpha_n < \lambda < \beta_1$ and $L_\kappa \models f(\vec{\alpha}, \vec{\beta}) = \lambda$. Without loss of generality, β_1 is the least element of $C > \lambda$. In particular, we have:

(A) $$(\forall \delta \in C)[(\delta < \beta_1) \Rightarrow (L_\kappa \models \delta < f(\vec{\alpha}, \vec{\beta}) < \beta_1)].$$

Consider some $\overline{\lambda} \in C$ with $\overline{\lambda}$ a limit, $C \cap \overline{\lambda}$ unbounded, $\overline{\lambda} > \lambda$ (such $\overline{\lambda}$ exists – let $D = \{\gamma \in \kappa \mid C \cap \gamma$ is unbounded in $\gamma\}$; D is club; then take $\overline{\lambda} \in C \cap D \setminus (\lambda + 1)$).

Let $\gamma_1 = \overline{\lambda} < \gamma_2 < \cdots < \gamma_m$ be in C. From (A) and indiscernibility we get

(B) $$(\forall \delta \in C)[(\delta < \gamma_1 = \overline{\lambda}) \Rightarrow (L_\kappa \models \delta < f(\vec{\alpha}, \vec{\gamma}) < \gamma_1 = \overline{\lambda})].$$

Therefore since $C \cap \overline{\lambda}$ is unbounded, $f(\vec{\alpha}, \vec{\gamma}) \geq \overline{\lambda} = \gamma_1$. But this contradicts (B). □

Claim. *If $\delta < \kappa$ regular, then $|C \cap \delta| = \delta$.*

Proof of claim. Let $\{\gamma_k\}_{k \in \omega}$ be the first ω elements of $C \setminus \delta$ in increasing order. The result follows if we can show that

$$\delta \subseteq \text{Skolem Hull}_{L_\kappa}((C \cap \delta) \cup \{\gamma_n\}_{n \in \omega}).$$

For this, it is sufficient to show that if

$$\alpha_1 < \cdots < \alpha_n < \lambda_1 < \lambda_2 < \beta_1 < \cdots < \beta_m$$

are elements of C, $\gamma_1 < \lambda_1$, and if f is an ordinal-valued L_κ-definable function such that

$$f(\vec{\alpha}, \lambda_2, \vec{\beta}) < \gamma_1,$$

then

$$f(\vec{\alpha}, \lambda_1, \vec{\beta}) = f(\vec{\alpha}, \lambda_2, \vec{\beta}).$$

(To see that this is sufficient, suppose we have this result. Any ordinal in δ is $f(\vec{\alpha}, \vec{\beta})$ for some $\alpha_1 < \cdots < \alpha_n < \gamma_1 \leq \beta_1 < \cdots < \beta_m$ in C and some L_κ-definable function f. Applying the result and a straightforward induction, we have $f(\vec{\alpha}, \vec{\beta}) = f(\vec{\alpha}, \gamma_1, \ldots, \gamma_m)$. Hence any element of δ is definable from $(C \cap \delta) \cup \{\gamma_k\}_{k \in \omega}$.)

Now, it cannot be the case that $f(\vec{\alpha}, \lambda_1, \vec{\beta_1}) > f(\vec{\alpha}, \lambda_2, \vec{\beta})$. For were this so, we could fix $\lambda_2 < \eta_1 < \cdots < \eta_m \in C$ with $C \cap \eta_1$ unbounded in η_1, and let

$$\lambda_1 < \lambda_2 < \lambda_3 < \cdots < \lambda_l < \lambda_{l+1} < \cdots < \eta_1$$

be in C. By indiscernibility, we would have, for all $l \in \omega$,

$$f(\vec{\alpha}, \lambda_l, \vec{\eta}) > f(\vec{\alpha}, \lambda_{l+1}, \vec{\eta}),$$

contradicting wellfoundedness of the class of ordinals.

In a similar vein, we show that we cannot have $f(\vec{\alpha}, \lambda_1, \vec{\beta}) < f(\vec{\alpha}, \lambda_2, \vec{\beta})$. For in this case, let $(\vec{\eta}_\xi)_{\xi \in \kappa}$ be a coordinatewise increasing sequence of $(m+1)$-tuples in C, such that for all $\xi \in \kappa$,

$$\gamma_1 < \eta_{\xi,1} < \cdots < \eta_{\xi,m} < \eta_{\xi,m+1}.$$

It then follows by indiscernibility that, for all $\xi \in \kappa$,

$$f(\vec{\alpha}, \vec{\eta}_\xi) < f(\vec{\alpha}, \vec{\eta}_{\xi+1}) < \gamma_1,$$

so that $(f(\vec{\alpha}, \vec{\eta}_\xi))_{\xi \in \kappa}$ is an increasing sequence in γ_1 of length κ, a contradiction. □

This claim completes the proof of the theorem. □

6. October 6

Remark. In the above theorem, [the existence of a club satisfying] (iii) follows from [the existence of one satisfying] (i) and (ii).

There is a club $D \subseteq \kappa$ such that $L_\alpha \prec L_\kappa$ for all $\alpha \in D$. Thus, by indiscernibility [and absoluteness of the satisfaction relation], for every $\alpha \in C$, $L_\alpha \prec L_\kappa$. So in particular, for any regular $\delta < \kappa$, $L_\delta \prec L_\kappa$, and $C \cap \delta$ will be a club of indiscernibles for L_δ with the same theory.

Definition. Suppose there is some regular κ with a club $C \subseteq \kappa$ of indiscernibles for L_κ. Then let $(c_n)_{n \in \omega}$ be the first ω elements of C (in increasing order). Then

$$0^\# = \mathrm{Th}(L_\kappa; \in, c_0, c_1, \ldots)$$

(By the remarks above, this is well-defined.)

When we say "$0^\#$ exists", we mean that there exists such a regular cardinal κ and club C of indiscernibles for L_κ, and $0^\#$ is the theory just defined.

Broader remarks (a little imprecise): In effect, $0^\# \in 2^\omega \sim \mathcal{P}(\omega)$. Can we define $0^\#$ "more internally"? We can, and there are two parts to this:

(1) Various syntactical demands: $T = 0^{\#}$ implies: (Assume T complete)

$$T \supseteq \mathrm{ZFC} + V = L$$

for φ first order, $n_1 < \cdots < n_l, m_1 < \cdots < m_l$,

$$\varphi(c_{n_1}, \ldots, c_{n_l}) \iff \varphi(c_{m_1}, \ldots, c_{m_l}) \in T.$$

(2) "Remarkability." Things like

$$f(c_0, c_1, \ldots, c_l, c_{l+1}, c_{l+3}, c_{l+4}, \ldots) < c_l$$
$$\implies f(c_0, c_1, \ldots, c_\ell, c_{l+2}, c_{l+3}, \ldots) = f(c_0, c_1, \ldots, c_l, c_{l+1}, c_{l+3}, \ldots)$$

and

$$f(c_0, \ldots, c_n, c_{n+2}, c_{n+3}, \ldots) < c_{n+1} \implies f(c_0, \ldots, c_n, c_{n+2}, c_{n+3}, \ldots) \leq c_n$$

for f a definable function.

(3) "Wellfoundedness." For any (regular) cardinal κ, if we "stretch" or "expand" T along $(c_\alpha)_{\alpha\in\kappa}$ as indiscernibles, then the result is wellfounded. In fact, a Skolem hull argument can be used to show that if "stretching" T along $(c_\alpha)_{\alpha\in\lambda}$ is wellfounded for all limit $\lambda < \omega_1$, then T stretched along any $(c_\alpha)_{\alpha\in\kappa}$ (κ a limit ordinal) will be wellfounded.

(1) and (2) are Δ^1_1 conditions. (3) is Π^1_2. These together (if phrased precisely) have at most one solution.

Thus: $0^{\#}$ (if it exists) is a Π^1_2 singleton.

Aside. The study of Π^1_2 singletons has been a whole research area in its own right. *cf.* "Π^1_2 Singletons & $0^{\#}$" Harrington, Kechris Fund. Math. Vol. 95 (1977) pp. 167-171. They show: If $0^{\#}\exists$, and $x^{\#}$ exists ($x \in 2^\omega$) (relatives $0^{\#}$ to $L[x]$) and $\{x\}$ is Π^1_2, then either $0^{\#} \in L[x]$ or $x^{\#} \in L[0^{\#}]$.

Definition. For M, N transitive sets, $j\colon M \to N$ a function, which is at least Σ_0-elementary, $\mathrm{cp}(j)$ (the **critical point** of j) is the least ordinal α (if it exists) in M with $j(\alpha) > \alpha$.

Remark. If M is a transitive class, $M \neq V$, $j\colon V \to M$, j nonidentity and elementary, then $\mathrm{cp}(j)$ exists. (Exercise.)

Theorem (Scott)**.** *Let κ be a cardinal. Then the following are equivalent:*

(I) *κ is measurable.*

(II) *for all $\gamma \geq \kappa + \omega$, there is a transitive set M and an elementary map $j\colon V_\gamma \to M$,* $\mathrm{cp}(j) = \kappa$

(III) *for some $\gamma \geq \kappa + \omega$, there is transitive M and an elementary map $j\colon V_\gamma \to M$,* $\mathrm{cp}(j) = \kappa$.

Corollary. *If a measurable exists, $V \neq L$.*

7. October 8

Proof. (I)⇒(II): Fix μ a measure on κ. Fix γ. For $f, g\colon \kappa \to V_\gamma$, set $f \sim_\mu g$ if $\{\alpha \in \kappa \mid f(\alpha) = g(\alpha)\} \in \mu$. (This *is* an equivalence relation.) $[f]_\mu = \{g \mid g \sim_\mu f\}$. Set $[f]_\mu \in_\mu [g]_\mu$ if $\{\alpha \mid f(\alpha) \in g(\alpha)\} \in \mu$ (This is well-defined, since μ is closed under intersections).

$$\mathcal{N} = (\{[f]_\mu \mid f\colon \kappa \to V_\gamma\}, \in_\mu)$$

Claim (Łos's Theorem)**.** *For $\varphi(\vec{x})$ first order, $f_1, \dots, f_n\colon \kappa \to V_\gamma$,*

$$\mathcal{N} \models \varphi([f_1]_\mu, \dots) \textit{ iff } \{\alpha \in \kappa \mid V_\gamma \models \varphi(f_1(\alpha), \dots)\} \in \mu.$$

Proof of claim. For φ atomic ("$x_1 \in x_2$" or "$x_1 = x_2$") this is immediate from construction. We then prove it for all (first order) φ by induction on logical complexity.

For negation: $\{\alpha \in \kappa \mid V_\gamma \models \neg\varphi(f_1(\alpha), \dots)\} \in \mu$ iff $\{\alpha \in \kappa \mid V_\gamma \models \varphi(f_1(\alpha), \dots)\} \notin \mu$.

For conjunction: $\{\alpha \in \kappa \mid V_\gamma \models \varphi_1(f_1(\alpha), \dots) \wedge \varphi_2(f_1(\alpha) \dots)\} \in \mu$ iff $\{\alpha \in \kappa \mid V_\gamma \models \varphi_1(f_1(\alpha), \dots)\} \in \mu$ and $\{\alpha \in \kappa \mid V_\gamma \models \varphi_2(f_1(\alpha), \dots)\} \in \mu$.

For existential quantification: The critical step: If $A \in \mu$ and at all $\alpha \in A$,

$$V_\gamma \models \exists x \varphi(f_1(\alpha), \dots, f_n(\alpha), x)$$

then at each $\alpha \in A$ fix some $a_\alpha \in V_\gamma$ such that

$$V_\gamma \models \varphi(f_1(\alpha), \dots, f_n(\alpha), a_\alpha).$$

Then let $g\colon \kappa \to V_\gamma$

$$\alpha \mapsto \begin{cases} a_\alpha & \text{for } \alpha \in A \\ 0 & \text{for } \alpha \notin A \end{cases}$$

Then by inductive assumption,

$$\mathcal{N} \models \varphi([f_1]_\mu, \dots, [f_n]_\mu, [\rho]_\mu)$$
$$\mathcal{N} \models \exists x \varphi([f_1]_\mu, \dots, [f_n]_\mu, x).$$

□

Claim. *$\mathcal{N}$ is wellfounded.*

Proof of claim. Instead, let $[f_{n+1}]_\mu \in_\mu [f_n]_\mu$ for all n. Let $A_n = \{\alpha \mid f_{n+1}(\alpha) \in f_n(\alpha)\}$. Assumption gives each $A_n \in \mu$. Since $\bigcap_{n\in\omega} A_n \in \mu$ (μ a measure) choose $\alpha \in \bigcap_{n\in\omega} A_n$. Then $f_{n+1}(\alpha) \in f_n(\alpha)$ all n, a contradiction. □

For $\alpha \in V_\gamma$, let $c_a\colon \kappa \to V_\gamma$ by $\alpha \mapsto a$.

Claim. *$V_\gamma \to \mathcal{N}$ by $a \mapsto [c_a]_\mu$ is elementary.*

Proof. Since $\mathcal{N} \models \varphi([c_{a_1}]_\mu, \dots, [c_{a_n}]_\mu)$ iff $\{\alpha \in \kappa \mid V_\gamma \models \varphi(c_{a_1}(\alpha), \dots, c_{\alpha_n}(\alpha))\} \in \mu$ iff $\{\alpha \in \kappa \mid V_\gamma \models \varphi(a_1, \dots, a_n)\} \in \mu$ iff $V_\gamma \models \varphi(a_1, \dots, a_n)$. □

Let $\pi\colon \mathcal{N} \cong (M, \in)$ be the Mostowski collapse. Then $j\colon V_\gamma \to M$ defined by $a \mapsto \pi([c_a]_\mu)$ is elementary.

Claim. *$j(\beta) = \beta$ all $\beta < \kappa$.*

Proof of claim. It suffices to show that if $[f]_\mu \in_\mu [c_\beta]_\mu$, and $\beta < \kappa$, then $\exists \beta_0 < \beta$ such that

$$[f]_\mu = [c_{\beta_0}]_\mu$$

(since if we have this, we can argue by transfinite induction that $j(\beta) = \beta$ all $\beta < \kappa$). Suppose then $[f]_\mu \in_\mu [c_\beta]_\mu$. Then $A = \{\alpha \in \kappa \mid f(\alpha) \in c_\beta(\alpha) = \beta\} \in \mu$. At each $\delta < \beta$, let $A_\delta = \{\alpha \in \kappa \mid f(\alpha) = \delta\}$. Then $A = \bigcup_{\delta<\beta} A_\delta$. Since $\beta < \kappa$, some $A_\delta \in \mu$. Then $[f]_\mu = [c_\delta]_\mu$. □

Claim. *$j(\kappa) > \kappa$.*

Proof of claim. Let $\Delta\colon \kappa \to \kappa$ by $\alpha \mapsto \alpha$. For $\beta \in \kappa$,

$$\{\alpha \mid c_\beta(\alpha) \in \Delta(\alpha)\} = \kappa \setminus \beta + 1 \in \mu.$$

But $\alpha = \Delta(\alpha) \in c_\kappa(\alpha) = \kappa$ *all* $\alpha \in \kappa$. For all $\alpha \in \kappa$,

$$j(\alpha) = \alpha < \pi([\Delta]_\mu) < j(\kappa).$$

□

(II)⇒(III): Trivial.
(III) ⇒(I): Fix $j\colon V_\gamma \to M$ as above. Let $\mu = \{A \subseteq \kappa \mid \kappa \in j(A)\}$.

Claim. *This is an ultrafilter.*

Proof of claim.

$$\begin{aligned} A \in \mu &\text{ iff } \kappa \in j(A) \\ &\text{ iff } \kappa \notin j(\kappa) \setminus j(A) \\ &\text{ iff } \kappa \notin j(\kappa \setminus A) \ (j \text{ elementary}) \\ &\text{ iff } \kappa \setminus A \notin \mu. \end{aligned}$$

For $A, B \subset \kappa$,

$$\begin{aligned} A \cap B \in \mu &\text{ iff } \kappa \in j(A \cap B) = j(A) \cap j(B) \\ &\text{ iff } \kappa \in j(A) \ \&\ \kappa \in j(B) \\ &\text{ iff } A \cap B \in \mu. \end{aligned}$$

□

Claim. *μ is non-principal.*

Proof of claim. Fix $\alpha < \kappa$, then $\{\alpha\} \in \mu$ iff $\kappa \in j(\{\alpha\})$ iff $\kappa \in \{j(\alpha)\} = \{\alpha\}$, since $\mathrm{cp}(j) = \kappa > \alpha$. □

Claim. *μ is κ-complete.*

Proof of claim. Fix $\delta < \kappa$, $\vec{A} = (A_\beta)_{\beta\in\delta}$ all in μ. $\bigcap_{\beta\in\delta} A_\beta \in \mu$ iff $\kappa \in j(\bigcap_{\beta\in\delta} A_\beta)$.

Note: For $\beta < \delta$, $j(A_\beta) = j(\vec{A})_{j(\beta)} = j(\vec{A})_\beta$. But $j(\bigcap_{\beta\in\delta} A_\beta) = \bigcap_{\beta\in\delta} j(A_\beta)$ since $\mathrm{cp}(j) = \kappa$, which contains κ. □

This completes the proof of the theorem. □

8. October 11

Recall: The following are equivalent:

(I) κ measurable
(II) $\exists j_\gamma \colon V_\gamma \to M_\gamma$ elementary, $\mathrm{cp}(j_\gamma) = \kappa$, M_γ is a transitive set.

If μ is a measure on κ, then $M_\gamma \cong \mathrm{Ult}(V_\gamma, \mu)$ via $\pi_\gamma \colon \mathrm{Ult}(V_\gamma, \mu) \cong M_\gamma$.

Lemma. *For $\gamma < \gamma'$, $M_\gamma \subseteq M_{\gamma'}$ and $j_{\gamma'}|_{V_\gamma} = j_\gamma$.*

Proof. For $a \in V_\gamma$, $c_a \colon \kappa \to \{a\}$ by $\alpha \mapsto a$. Then $j_\gamma \colon V_\gamma \to M_\gamma$ via $a \mapsto \pi_\gamma([c_a]_\mu)$.

It suffices to show that if $f \colon \kappa \to V_{\gamma'}$, $[f]_\mu \in_\mu [c_a]_\mu$, $a \in V_\gamma$, then $\exists g \colon \kappa \to V_\gamma$ with $[f]_\mu = [g]_\mu$. But given such an f,

$$\{\alpha \in \kappa \mid f(\alpha) \in c_a(\alpha) = a\} \in \mu.$$

Hence if we let $g \colon \kappa \to V_\gamma$,

$$\alpha \mapsto \begin{cases} f(\alpha) & \text{when } f(\alpha) \in a \\ 0 & \text{when } f(\alpha) \notin a \end{cases}$$

then $[f]_\mu = [g]_\mu$. □

Hence we can take

$$j = \bigcup_{\gamma \in \mathrm{Ord}} j_\gamma, \quad M = \bigcup_{\gamma \in \mathrm{Ord}} M_\gamma$$

and obtain an embedding.

Theorem (Scott)**.** *The following are equivalent.*

(I) *κ is measurable;*
(II) *$\exists$ transitive M and an elementary embedding $j \colon V \to M$, $\mathrm{cp}(j) = \kappa$.*

Proof. (II)⇒(I): clearly such a j gives $j_{\kappa+\omega} \to M_{j(\kappa+\omega)}$, and apply last theorem.

(I)⇒(II): We need to check the $j = \bigcup_{\gamma \in \mathrm{Ord}} j_\gamma$ as above is elementary. Fix $\varphi(\vec{x})$ first order, $\vec{a} \in V^{<\infty}$. Say $\varphi(\vec{x})$ is Σ_n.

Recall: Reflection gives there is a club class C of ordinals δ such that $V_\delta \prec_{\Sigma_n} V$. Similarly, there is a club class $D \subseteq \mathrm{Ord}$ of δ such that

$$\bigcup_{\alpha < \delta} M_\alpha \prec M = \bigcup_{\alpha \in \mathrm{Ord}} M_\alpha.$$

So there exists δ such that

(I) $\vec{a} = a_1, \ldots, a_l \in V_\delta$
(II) $\mathrm{cof}(\delta) > \kappa$
(III) $V_\delta \prec_{\Sigma_n} V$
(IV) $\bigcup_{\alpha < \delta} M_\alpha \prec_{\Sigma_n} M$.

Claim. $M_\delta = \bigcup_{\alpha \in \delta} M_\alpha$.

Proof of claim. It suffices to show that $\mathrm{Ult}_\mu(V_\delta) = \bigcup_{\alpha \in \delta} \mathrm{Ult}_\mu(V_\alpha)$. But given $f \colon \kappa \to V_\delta$, at each $\alpha \in \kappa$ we can let $h(\alpha)$ be the least $\beta < \delta$ such that

$$f(\alpha) \in V_\beta.$$

Since $\mathrm{cof}(\delta) > \kappa$, $\exists \gamma_0 < \delta$ such that $h(\alpha) \in \gamma_0$ all $\alpha \in \kappa$. Then $f \colon \kappa \to V_{\gamma_0}$. □

Then

$$\begin{aligned} V \models \varphi(\vec{a}) &\text{ iff } V_\delta \models \varphi(\vec{a}) && \text{by (III)} \\ &\text{iff } M_\delta \models \varphi(j(\vec{a})) && \text{(construction of } j) \\ &\text{iff } \bigcup_{\alpha\in\delta} M_\alpha \models \varphi(j(\vec{a})) && \text{(claim)} \\ &\text{iff } M \models \varphi(j(\vec{a})) && \text{(III).} \end{aligned}$$

□

Corollary. *If there exists a measurable, then $V \neq L$.*

Proof. Suppose not. Let κ be the least measurable. Form $j\colon V \to M$ elementary, $\mathrm{cp}(j) = \kappa$. Then $V = L$, $M \subseteq L = V$, $\mathrm{Ord} \subseteq M$, $M \models \mathrm{ZFC}$. Therefore $M = L$. Now $j\colon L \to L$ with $j(\kappa) > \kappa$; so $L \models \exists$ a measurable $< j(\kappa)$ and $L \models \exists$ a measurable $< \kappa$ (j elementary), contradiction. □

So far: If $\exists$ a measurable, in particular $\exists j\colon L \to L$ elementary, $j \neq \mathrm{id}$. So $L \models \nexists$ measurable. Therefore if $\exists j\colon L \to L$ elementary then $V \neq L$.

Question. Can we characterize when $\exists j\colon L \to L, j \neq \mathrm{id}$?

Answer. Yes: Iff $\exists 0^\#$.

Definition. Let κ be an ordinal. Then $\mu \subseteq \mathcal{P}(\kappa)^L = \{A \subseteq \kappa \mid A \in L\}$ is an ***L*-ultrafilter** if

(i) $A, B \in \mu \Longrightarrow A \cap B \in \mu$
(ii) $A \subseteq B, A \in \mu, B \in \mathcal{P}(\kappa)^L \Longrightarrow B \in \mu$
(iii) For all $A \in \mathcal{P}(\kappa)^L$, either $A \in \mu$ or $\kappa \setminus A \in \mu$.
(iv) $\emptyset \notin \mu$.

Note. We do *not* require $\mu \in L$.

Definition. Let μ be an L-ultrafilter on κ. Let $\delta \geq \kappa$. Then for $f, g\colon \kappa \to L_\delta$, $f, g \in L$, set

$$[f]_\mu = [g]_\mu \text{ if } \{\alpha \in \kappa \mid f(\alpha) = g(\alpha)\} \in \mu$$
$$[f]_\mu \in_\mu [g]_\mu \text{ if } \{\alpha \in \kappa \mid f(\alpha) \in g(\alpha)\} \in \mu$$

Remark. This is well-defined.

Then $\mathrm{Ult}_\mu(L_\delta, L) = (\{[f]_\mu \;:\; f \in L, f\colon \kappa \to L_\delta\}, \in_\mu)$.

9. October 13

Theorem. *Let $\mu \subseteq \mathcal{P}(\kappa)^L$ be an L-ultrafilter. Let δ be an ordinal, $f_1, \ldots, f_n\colon \kappa \to L_\delta$, $f_1, \ldots, f_n \in L$, $\varphi(\vec{x})$ a formula. Then*

$$\mathrm{Ult}_\mu(L_\delta, L) \models \varphi([f_1]_\mu, \ldots, [f_n]_\mu) \textit{ iff } \{\alpha \in \kappa \mid L_\delta \models \varphi(f_1(\alpha), \ldots, f_n(\alpha))\} \in \mu.$$

(The proof is basically the same as the proof of Łos's Theorem.)

Lemma. *Let $\lambda \geq \kappa^+$ and μ an L-ultrafilter on κ. If $\mathrm{Ult}_\mu(L_{\kappa^+}, L)$ is wellfounded, then so is $\mathrm{Ult}_\mu(L_\lambda, L)$.*

Proof. Suppose $\mathrm{Ult}_\mu(L_\lambda, L)$ is illfounded. Let $(f_n)_{n\in N}$ be a sequence of functions such that

(i) each $f_n \in L$
(ii) each $f_n \colon \kappa \to L_\lambda$
(iii) $\{\alpha \mid f_{n+1}(\alpha) \in f_n(\alpha)\} \in \mu$ at all n.

Fix $\delta \geq \lambda$ with each $f_n \in L_\delta$. Let M be the Skolem Hull of $\kappa + 1 \cup \{f_n\}_{n\in\mathbb{N}}$ in L_δ. Then let $\pi \colon M \cong L_\alpha$ be the collapsing map (condensation). Note: $\pi(\beta) = \beta$ all $\beta \leq \kappa$ (since $\kappa \subseteq M$). Note also: $|M| < \kappa^+$, therefore $\alpha < \kappa^+$. At each n, let $g_n = \pi(f_n)$,

(i) $g_n \in L$
(ii) $g_n \colon \kappa \to L_\alpha \subseteq L_{\kappa^+}$
(iii) $\{\alpha \mid g_{n+1}(\alpha) \in g_n(\alpha)\} = \{\alpha \mid f_{n+1}(\alpha) \in f_n(\alpha)\}$ since $\pi|_\kappa = \mathrm{id}$.

□

Lemma. *Let $\lambda \geq \kappa^+$, $j \colon L_{\kappa^+} \to L_\lambda$ elementary with $\mathrm{cp}(j) = \kappa$. Then if we let*

$$\mu = \{A \subseteq \kappa \mid A \in L, \kappa \in j(A)\},$$

then

(i) *μ is an L-ultrafilter, and*
(ii) *$\mathrm{Ult}_\mu(L_{\kappa^+}, L)$ is wellfounded.*

Proof. (i) is exactly as in Scott's theorem.

For (ii), define $\rho\colon \mathrm{Ult}_\mu(L_{\kappa^+}, L) \to L_\lambda$ by $[f]_\mu \mapsto jf(\kappa)$.

Claim. *This is well-defined.*

Proof of claim. Suppose $f_1 \sim_\mu f_2$. Then $\{\alpha \mid f_1(\alpha) = f_2(\alpha)\} \in \mu$; therefore

$$\kappa \in j(\{\alpha \mid f_1(\alpha) = f_2(\alpha)\}) = \{\alpha \mid (jf_1)(\alpha) = (jf_2)(\alpha)\}$$

since j is elementary. □

Claim. *If $[f_1]_\mu \in_\mu [f_2]_\mu$, then $\rho(f_1) \in \rho(f_2)$.*

Proof of claim.

$$\begin{aligned}[f_1]_\mu \in_\mu [f_2]_\mu &\implies \{\alpha \in \kappa \mid f_1(\alpha) \in f_2(\alpha)\} \in \mu \\ &\implies \kappa \in j(\{\alpha \mid f_1(\alpha) \in f_2(\alpha)\} = \{\alpha \mid (jf_1)(\alpha) \in (jf_2)(\alpha)\}.\end{aligned}$$

□

This completes the proof of (ii) and so of the lemma. □

Definition. For δ a regular cardinal, $S \subseteq \delta$ stationary, we say that $T \subseteq S$ is **relatively club** if $\exists C \subseteq \delta$ club, $T = S \cap C$.

Remark. T relatively club $\implies T \neq \emptyset$ (in fact, T unbounded). The relatively club sets (in S stationary) are closed under $< \delta$ intersections.

Definition. For $\gamma < \delta$ both regular cardinals, $\mathrm{Cof}_\gamma := \{\alpha < \delta \mid \text{cofinality of } \alpha = \gamma\}$.

10. October 18

Let μ be an L-ultrafilter on κ non-principal with the property that if $(B_\beta)_{\beta\in\alpha} \in L$, each $B_\beta \in \mu$, then $\bigcap_{\beta\in\alpha} B_\beta \in \mu$. Let $j_\mu\colon L_{\kappa^{++}} \to \mathrm{Ult}_\mu(L_{\kappa^{++}}, L)$ be the ultrapower map. Then:

(1) if $\alpha < \kappa$, and $[g]_\mu \in_\mu j_\mu(\alpha)$, then $\exists\beta < \alpha$ such that $[g]_\mu = j_\mu(\beta)$
 (Proof: Wlog, $g\colon \kappa \to \alpha$. At each $\beta < \alpha$, let $A_\beta = \{\gamma < \kappa \mid g(\gamma) = \beta\}$. Each $A_\beta \in L$, since $g \in L$.
 $\kappa = \bigcup_{\beta<\alpha} A_\beta$. Therefore at some β, $A_\beta \in \mu$. Hence $[g]_\mu = [c_\beta]_\mu = j_\mu(\beta)$.)

(2) If $g\colon \kappa \to \gamma$ some ordinal $\gamma < \kappa^{++}$, $g \in L$,
$$|\{[f]_\mu \ : \ [f]_\mu \in_\mu [g]_\mu\}| \le \max(\kappa^+, |\gamma|).$$
(Proof: If $[f]_\mu \in_\mu [g]_\mu$ then without loss of generality $f(\alpha) \in g(\alpha)$ all $\alpha \in \kappa$. Therefore $f \in ((\gamma)^\kappa)^L$ and (since γ is *not* a cardinal $> \kappa$ with $\mathrm{cof}(\kappa)$ with cofinality κ) $|((\gamma)^\kappa)^L| \le \max(\kappa^+, |\gamma|)$.)

(3) If $\gamma < \kappa^{++}$, $\mathrm{cof}(\gamma) = \kappa^+$ and $[g]_\mu \in_\mu j_\mu(\gamma)$, then $\exists\delta < \gamma$ such that $[g]_\mu \in_\mu j_\mu(\delta)$.
 (Proof: Wlog $g\colon \kappa \to \gamma$. But then $\mathrm{cof}(\gamma) > \kappa$ therefore g is bounded. Hence (i.e.) there is some $\delta < \gamma$ such that $g(\alpha) \in \gamma = c_\gamma(\alpha)$ all $\alpha \in \kappa$; hence $[g]_\mu \in_\mu [c_\gamma]_\mu = j_\mu(\gamma)$.)

Recall: If $j\colon L_\delta \to L_\lambda$ elementary, $\mathrm{cp}(j) = \kappa$, and if we let $\mu = \{A \in \mathcal{P}(\kappa)^L \ : \ \kappa \in j(A)\}$, then for all $\alpha < \kappa$, $(B_\beta)_{\beta<\alpha}$ in L, if each $B_\beta \in \mu$, then $\bigcap_{\beta<\alpha} B_\beta \in \mu$.

(Proof: $j(\bigcap_{\beta\in\alpha} B_\beta) = \bigcap_{\beta\in\alpha} j(B_\beta)$ (since $\alpha < \mathrm{cp}(j)$) therefore $\kappa \in \bigcap_{\beta\in\alpha} j(B_\beta) \implies \bigcap_{\beta\in\alpha} B_\beta \in \mu$.)

Lemma. *Let μ be a non-principal L-ultrafilter on κ. Assume whenever $\alpha < \kappa$ and $(B_\beta)_{\beta\in\alpha} \in L$ is a sequence of sets in μ, we have $\bigcap_{\beta\in\alpha} B_\beta \in \mu$. Assume $\mathrm{Ult}_\mu(L_{\kappa^{++}}, L)$ is wellfounded. Then*

(I) $\mathrm{Ult}_\mu(L_{\kappa^{++}}, L) \cong L_{\kappa^{++}}$

(II) *if $\pi\colon \mathrm{Ult}_\mu(L_{\kappa^{++}}, L) \cong L_{\kappa^{++}}$ is the Mostowski collapsing map, then if we let $j = \pi \circ j_\mu\colon L_{\kappa^{++}} \to L_{\kappa^{++}}$ then $\{\delta < \kappa^{++} \mid j(\delta) = \delta\}$ is relatively club in Cof_{κ^+}.*

Proof. Any ordinal in $\mathrm{Ult}_\mu(L_{\kappa^{++}}, L)$ will have $< \kappa^{++}$ predecessors by (2). Therefore the ultrapower, if wellfounded, will be isomorphic to $L_{\kappa^{++}}$.

(3) immediately gives fixed points relatively closed in Cof_{κ^+}. To show the fixed points are unbounded, begin with some $\delta = \delta_0 < \kappa^{++}$. Successively choose $(\delta_\alpha)_{\alpha\le\kappa^+}$ such that $\delta_{\alpha+1} > j(\delta_\alpha)$ and $\delta_\lambda = \bigcup_{\alpha<\lambda} \delta_\alpha$ for λ a limit. Then by (3),
$$j(\delta_{\kappa^+}) = \lim_{\alpha\to\kappa^+} j(\delta_\alpha) = \lim_{\alpha\to\kappa^+} \delta_\alpha = \delta_{\kappa^+}.$$
□

Theorem. *Let $\kappa < \kappa^+ \le \delta < \lambda$ be ordinals. Let $i\colon L_\delta \to L_\lambda$ be elementary with $\mathrm{cp}(i) = \kappa$. Then $0^\#$ exists.*

Proof. By the lemma today and last time, we obtain $j\colon L_{\kappa^{++}} \to L_{\kappa^{++}}$ such that

(i) $\mathrm{cp}(j) = \kappa$

(ii) if $\gamma < \kappa^{++}$ has $\mathrm{cof}(\gamma) = \kappa^+$, then it is a "continuity point of the embedding" i.e. $j(\gamma) = \sup_{\alpha<\gamma} j(\alpha)$.

(iii) $\{\gamma < \kappa^{++} \mid j(\gamma) = \gamma\}$ is relatively club in Cof_{κ^+}.

Let $C_0 = \{\beta \geq \kappa \mid j(\beta) = \beta\}$. Given C_α, $\alpha < \kappa^{++}$, $C_{\alpha+1} = \{\beta \in C_\alpha \mid \text{ot}(C_\alpha \cap \beta) = \beta\}$. At λ a limit $< \kappa^{++}$,

$$C_\lambda = \bigcap_{\alpha<\lambda} C_\alpha.$$

Let κ_α = least element of C_α.

We will show the κ_α's form a stationary set of indiscernibles.

The plot: Carefully choosing Skolem Hulls, we will show that for $\alpha < \beta$ there exists

$$\pi_{\alpha,\beta}\colon \text{Skolem Hull}_{L_{\kappa^{++}}}(\kappa_\alpha \cup C_\beta) \to L_{\kappa^{++}}$$

such that

(i) $\pi_{\alpha,\beta}(\kappa_\beta) = \kappa_\alpha$
(ii) for $\gamma < \alpha$ or $\beta < \gamma$, $\pi_{\alpha,\beta}(\kappa_\gamma) = \kappa_\gamma$.

11. October 20

We have $j\colon L_{\kappa^{++}} \to L_{\kappa^{++}}$, $\text{cp}(j) = \kappa$, j elementary, such that the fixed points for j are relatively club in $\text{Cof}_{\kappa^+} \subseteq \kappa^{++}$.

$C_0 = \{\delta > \kappa \mid j(\delta) = \delta$.

$C_{\alpha+1} = \{\delta \in C_\alpha \mid \text{ot}(C_\alpha \cap \delta) = \delta\}$

$C_\lambda = \bigcap_{\alpha<\lambda} C_\alpha$ for λ a limit.

κ_α = least element of C_α ($\alpha < \kappa^{++}$).

Let $M_\alpha = \text{Skolem Hull}_{L_{\kappa^{++}}}(\kappa \cup C_\alpha)$ $(= \{f(\vec{\beta}) \ : \ f \text{ is definable over } L_{\kappa^{++}}, \vec{\beta} \in (\kappa \cup C_\alpha)^{<\omega}\})$

Claim. *Each C_α is relatively club in Cof_{κ^+} all $\alpha < \kappa^{++}$ (i.e. $\exists D$ club s.t. $D \cap \text{Cof}_{\kappa^+} = C_\alpha \cap \text{Cof}_{\kappa^+}$).*

Proof of claim. For D club, $\{\delta \in D \mid \text{ot}(D \cap \delta) = \delta\}$ is again club. Thus, if C_α is relatively club in Cof_{κ^+}, then so is $C_{\alpha+1}$. For $\lambda < \kappa^{++}$ a limit, use that the intersection of $< \kappa^{++}$ clubs is club. □

Claim. *$\{\kappa_\alpha \mid \alpha < \kappa^{++}\}$ is relatively club in Cof_{κ^+}.*

Proof of claim. Clearly increasing, therefore unbounded. Now if $\delta(\alpha) \to \delta$, $\alpha \in \kappa^+$, increasing, $\text{cof}(\delta) = \kappa^+$, then $\kappa_{\delta(\alpha)} \in C_{\delta(\alpha)}$ and therefore $\kappa_{\delta(\alpha)} \in C_\beta$ all $\beta \leq \delta(\alpha)$.

Hence, $\forall \beta < \delta \, \exists \alpha_0 \, \forall \alpha > \alpha_0 (\kappa_{\delta(\alpha)} \in C_\beta)$. Hence

$$\lim_{\alpha\to\kappa^+} \kappa_{\delta(\alpha)} \in \bigcap_{\beta<\delta} C_\beta = C_\delta$$

(since each C_β is relatively club in Cof_{κ^+}), and hence $\kappa_\delta = \lim_{\alpha\to\kappa^+} \kappa_{\delta(\alpha)}$.

In particular, this set is stationary in κ^{++}. □

Hence (claim before last) $|M_\alpha| = \kappa^{++}$ all $\alpha < \kappa^{++}$. Hence $\exists \pi_\alpha\colon M_\alpha \cong L_{\kappa^{++}}$.

Claim. *At each $\alpha < \kappa^{++}$,*

(i) *$M_\alpha \cap \text{Ord} = \kappa \cup C_\alpha$,*
(ii) *$C_{\alpha+1} = \{\beta > \kappa \mid \pi_\alpha(\beta) = \beta\}$.*

Proof of claim. Simultaneous induction on α.

$\alpha = 0$. $C_0 \cup \kappa =$ fixed points of j. If $x \in M_0 =$ Skolem $\text{Hull}_{L_{\kappa^{++}}}(C_0 \cup \kappa)$, then $j(x) = x$. Hence $M_0 \cap \text{Ord} = \kappa \cup C_0$.

But then (ii) follows, since for $\delta \in \text{Ord} \cap M_0$,

$$\pi_0(\delta) = \text{ot}(M_0 \cap \delta) = \text{ot}((C_0 \cup \kappa) \cap \delta).$$

Thus for $\delta > \kappa$, $\pi_0(\delta) = \delta$ iff $\text{ot}(\delta \cap M_0) = \text{ot}(\delta \cap (C_0 \cup \kappa)) = \delta$ iff $\delta \in C_1$.

Note then in general by the same argument at α:

$$M_\alpha \cap \text{Ord} = \kappa \cup C_\alpha \Longrightarrow C_{\alpha+1} = \{\beta > \kappa \mid \pi_\alpha(\beta) = \beta\}.$$

Hence we have (ii) at α.

We now need to show (i) at $\alpha + 1$. Suppose $\beta \in M_{\alpha+1} \cap \text{Ord}$. Then $\exists \vec{\gamma} \in \kappa \cup C_{\alpha+1}$, f definable over $L_{\kappa^{++}}$ such that $\beta = f(\vec{\gamma})$. But then $\pi_\alpha(\beta) = \pi_\alpha(f(\vec{\gamma})) = f(\pi_\alpha(\vec{\gamma})) = f(\vec{\gamma})$. Hence by (ii) at α, we have $\beta \in C_{\alpha+1} \cup \kappa$.

Suppose (i) is true at all $\alpha < \lambda$, λ a limit. Let $\beta \in M_\lambda$. So $\beta = f(\vec{\gamma})$ some $\vec{\gamma} \in \kappa \cup C_\lambda$. But then $\vec{\gamma} \in \kappa \cup C_\alpha$ all $\alpha < \lambda$, hence $\beta \in M_\alpha$ all $\alpha < \lambda$. Hence by inductive assumption, $\beta \in C_\alpha \cup \kappa$ all $\alpha < \lambda$. Therefore $\beta \in C_\lambda \cup \kappa$. □

From this claim, $\pi_\alpha(\kappa_\alpha) = \kappa$ all $\alpha < \kappa^{++}$. [From this point on, keep in mind that the κ_α's are stationary, and remember how we defined the M_α's.]

Now for $\alpha < \beta < \kappa^{++}$, let

$$M_{\alpha,\beta} = \text{Skolem Hull}_{L_{\kappa^{++}}}(\kappa_\alpha \cup C_\beta).$$

Let $\pi_{\alpha,\beta} \colon M_{\alpha,\beta} \cong L_{\kappa^{++}}$.

Claim. *For $\alpha < \beta < \kappa^{++}$, $\pi_{\alpha,\beta}(\kappa_\beta) = \kappa_\alpha$.*

Proof of claim. Note, $\pi_\alpha(\vec{\gamma}) = \vec{\gamma}$ all $\vec{\gamma} \in \kappa \cup C_\beta$. To prove the claim, we just need to check that $M_{\alpha,\beta} \cap \kappa_\beta = \kappa_\alpha$. If not, then there exist f (definable over $L_{\kappa^{++}}$), $\vec{\alpha} < \kappa_\alpha$), and $\vec{\gamma} \in C_\beta$ such that

$$L_{\kappa^{++}} \models f(\vec{\alpha}, \vec{\gamma}) \in [\kappa_\alpha, \kappa_\beta).$$

If so,

$$L_{\kappa^{++}} \models (\exists \vec{\beta} < \kappa_\alpha)(f(\vec{\beta}, \vec{\gamma}) \in [\kappa_\alpha, \kappa_\beta)).$$

Apply π_α to this situation. We get

$$L_{\kappa^{++}} \models (\exists \vec{\beta} < \pi_\alpha(\kappa_\alpha))(f(\vec{\beta}, \vec{\gamma}) \in [\pi_\alpha(\kappa_\alpha), \kappa_\beta))$$

i.e.

$$L_{\kappa^{++}} \models (\exists \vec{\beta} < \kappa)(f(\vec{\beta}, \vec{\gamma}) \in [\kappa, \kappa_\beta)).$$

But this contradicts the fact that $M_\beta \cap [\kappa, \kappa_\beta) = \emptyset$. □

12. October 22

Recall from last time: we had $\rho\colon L_{\kappa^+} \to L_\lambda$ elementary, with $\mathrm{cp}(j) = \kappa$. We then got $j\colon L_{\kappa^{++}} \to L_{\kappa^{++}}$ with $\mathrm{cp}(j) = \kappa$ and the set of fixed points relatively club in Cof_{κ^+}. And we obtained

$$\begin{aligned} C_0 &= \{\delta \mid (\text{fixed points of } j \cap \delta) \text{ order type } \delta\} \\ C_{\alpha+1} &= \{\delta \mid C_\alpha \cap \delta \text{ order type } \delta\} \\ C_\lambda &= \bigcap_{\alpha<\lambda} C_\alpha,\ \lambda \text{ limit.} \end{aligned}$$

Then κ_α = least point in C_α.

We saw $\{\kappa_\alpha \mid \alpha \in \kappa^{++}\}$ stationary for $\alpha < \beta$. Obtained $M_{\alpha,\beta} = \text{Skolem Hull}_{L_\kappa^{++}}(\kappa_\alpha \cup C_\beta)$, $\pi_{\alpha,\beta}\colon M_{\alpha,\beta} \cong L_{\kappa^{++}}$, and $\pi_{\alpha,\beta}(\kappa_\beta) = \kappa_\alpha$. Note that for $\gamma > \beta, \gamma \in \kappa^{++}$, $\mathrm{ot}(C_\beta \cap \kappa_\gamma) = \kappa_\gamma$; $\kappa_\gamma \in C_{\beta+1}$, therefore $\mathrm{ot}(\kappa_\gamma \cap C_\beta) = \kappa_\gamma$.

Let $\rho_{\alpha,\beta}\colon L_{\kappa^{++}} \to L_{\kappa^{++}}$ be defined by

$$\rho_{\alpha,\beta}(a) = \pi^{-1}_{\alpha,\beta}(a).$$

Then $\rho_{\alpha,\beta}$ is elementary, $\rho_{\alpha,\beta}(\kappa_\alpha) = \kappa_\beta$; $\rho_{\alpha,\beta}(\kappa_\gamma) = \kappa_\gamma$ for $\gamma > \beta$, and for $\gamma < \alpha$.

Claim. *$\{\kappa_\alpha \mid \alpha \in \kappa^{++}\}$ forms a set of indiscernibles for $L_{\kappa^{++}}$.*

Proof. Fix $\varphi(x_1, \dots, x_n)$, $\alpha_0 < \alpha_1 < \cdots < \alpha_n$. We want

$$L_{\kappa^{++}} \models \varphi(\kappa_0, \kappa_1, \dots, \kappa_n) \text{ iff } L_{\kappa^{++}} \models \varphi(\kappa_{\alpha_0}, \dots, \kappa_{\alpha_n}).$$

Let $f\colon L_{\kappa^{++}} \to L_{\kappa^{++}}$

$$f = \rho_{0,\alpha_0} \circ \rho_{1,\alpha_1} \circ \cdots \circ \rho_{n,\alpha_n}.$$

Easily checked (from properties above) that f is as required to have $f(\kappa_i) = \kappa_{\alpha_i}$. □

So we have a stationary set of indiscernibles, and by a previous (much earlier) argument we get a club. □

The same argument gives:

Theorem. *If $\exists\rho\colon L_{\kappa^+} \to L_\lambda$, $\mathrm{cp}(\rho) = \kappa$, ρ elementary, then at every regular $\delta > \kappa^+$ there is a club of indiscernibles for L_δ.*

Theorem. *The following are equivalent.*

(1) *$0^\#$ exists*
(2) *at every regular δ, there is a club of indiscernibles for L*
(3) *there is some regular δ and $j\colon L_\delta \to L_\lambda$ elementary, $j \neq \mathrm{id}$.*
(4) *at every regular δ, $\exists j\colon L_\delta \to L_\lambda$ elementary, $j \neq \mathrm{id}$.*
(5) *$\exists j\colon L \to L$ elementary, $j \neq \mathrm{id}$.*

Proof. (2) $\Longrightarrow$ (1): immediate.
(1) $\Longrightarrow$ (3): Homework Q1 part (v).
(3) $\Longrightarrow$ (2): using the proof of the last theorem.
(2) $\Longrightarrow$ (4): Homework Q1 part (v).
(4) $\Longrightarrow$ (3): Clearly.
Clearly (5) $\Longrightarrow$ (3).

(3) $\Longrightarrow$ (5): Take the induced L-measure on $\mathrm{cp}(j) = \kappa$.

Recall: $\mathrm{Ult}_\mu(L_\gamma, L)$ will be wellfounded all γ, since $\delta \geq \kappa^+$. But then these fit together to give $j\colon L \to L$. □

Aside. If C, D are two clubs of indiscernibles for L_κ in κ that generate all of L_κ, then in fact $C = D$. Hence if $0^\#$ exists we can obtain in a coherent way a club class of indiscernibles for L. [?]

Theorem (Jensen's "covering lemma")**.** *Assume $0^\#$ does not exist. Then for any ordinal α and $X \subseteq \alpha$, $\exists Y \in L$ with $X \subseteq Y$, $|Y| \leq |X| + \aleph_1$.*

Corollary. *If $0^\#$ does not exist, then at every singular cardinal λ, $(\lambda^+)^L = \lambda^+$.*

Proof. Fix $\lambda < \alpha < \lambda^+$, α a limit. Take $X \subseteq \alpha$, X cofinal, $|X| < \lambda$. Choose $Y \in L$, $X \subseteq Y \subseteq \alpha$, $|Y| \leq |X| + \aleph_1 < \lambda$. Then $|Y|^L < \lambda$. So $L \models \alpha$ not a regular cardinal. □

Structure of Proof of Covering Lemma:

Fix α and assume covering lemma holds at all $\alpha' < \alpha$.

Note. α must be a cardinal in L for there to be any chance that the covering lemma fails for the first time at α.

Fix $X \subseteq \alpha$ (a possible counterexample), X cofinal in α, $|X| < |\alpha|$. We choose $M \prec L_\alpha$, $|X| \subseteq M$, $|M| \leq |X| + \aleph_1$. (M chosen carefully!!)

Then fix $L_\beta \cong M$, and $\rho\colon L_\beta \to L_\alpha$ the inverse of the collapse. $|\beta| < |\alpha|$.

Case (1). β is a cardinal in L.

Extend ρ to $\rho_\gamma\colon L_\gamma \to L_{\alpha(\gamma)}$ all $\gamma \geq \beta \Longrightarrow 0^\#$.

Case (2). β is not a cardinal in L.

Go to first n, γ such that $\exists \eta < \beta$ with $\beta \subseteq \Sigma_n$-Skolem $\mathrm{Hull}_{L_\gamma}(\eta \cup p)$, some finite $p \in L_\gamma$.

Extend ρ to $\rho^*\colon L_\gamma \to L_\delta$, some δ, ρ^* Σ_n-elementary. But then still gets

$$\rho^*[\beta] \subseteq \Sigma_n\text{-Skolem Hull}(\rho^*[\eta] \cup \rho^*(p)).$$

13. October 25

Model Theory Review:

If $M_0 \prec M_1 \prec M_2 \prec \ldots M_n \prec M_{n+1} \prec \ldots$ then $M_\omega = \bigcup_{n\in\omega} M_n$ has $M_n \prec M_\omega$ all n.

Generalization: Suppose $(M_n)_{n\in\omega}$ is a sequence of models, and at each n we have $f_n\colon M_n \to M_{n+1}$ elementary.

In this case, we can naturally define $M_\omega = \mathrm{DirLim}_{n\in\omega}(M_n, \rho_n)$. Then we have at each n some $\rho_{n,\infty}\colon M_n \to M_\omega$ elementary, $\rho_{n+1,\infty} \circ \rho_n = \rho_{n,\infty}$.

Proof. Let $X = \{(n, a) \mid a \in M_n\}$. Set $(n, a) \sim (n + \ell, b)$ if $\rho_{n,n+\ell} = \rho_{n+\ell-1} \circ \rho_{n+\ell-2} \circ \ldots \rho_n(a) = b$.

Note. This is an equivalence relation. (Critical idea: $(n, a) \sim (m, b)$ iff $\forall^\infty k$ $\rho_{n,k}(a) = \rho_{m,k}(b)$.)

Let M_ω have as its underlying set $\{[(n,a)]_\sim \mid n \in \omega, a \in M_n\}$. Then for $[(n_1,a_1)]_\sim$, $[(n_2,a_2)]_\sim$, ..., $[(n_k,a_k)]_\sim \in M_\omega$, set $M_\omega \models \varphi([(n_1,a_1)]_\sim, \ldots, [(n_k,a_k)]_\sim)$ iff

$$\forall^\infty \ell \ M_\ell \models \varphi(\rho_{n_1,\ell}(a_1), \ldots, \rho_{n_k,\ell}(a_k)).$$

(One can check this works out and is well-defined...) □

Recall: $X \subseteq \alpha$, α an ordinal, X cofinal in α, α a cardinal in L. We want: Either $0^\#$ exists, or there is a $Y \supseteq X$, $Y \in L$ with $|Y| \leq |X| + \aleph_1$. We may also assume the covering lemma holds for all $\alpha' < \alpha$.

First: Let's prove this under the simplifying assumption that $|X|^\omega = |X|$.

Note. We may assume $|X| < |\alpha|$ (otherwise, it is trivial).

(This is not too bizarre! E.g. $\mathrm{cof}\,|X| > \omega$, $V \models \mathrm{GCH}$, *or* $|X| = \lambda$, λ a strong limit with $\mathrm{cof}(\lambda) > \omega$.)

Let θ be a strong limit cardinal, $\theta > \alpha$, $\mathrm{cof}(\theta) > \omega$. Let $H(\theta) = \{a \mid |\mathrm{TC}(a)| < \theta\}$.

Note. $H(\theta)$ is transitive; $H(\theta)^\omega \subseteq H(\theta)$; $H(\theta) \models \mathrm{ZFC} - \mathrm{Replacement}$.

Claim. *There exists $N \prec H(\theta)$ such that*

(i) $X \subseteq N$, $X, \alpha, L_\alpha \in N$
(ii) $N^\omega \subseteq N$.
(iii) $|N| = |X|$.

Proof of claim. Start with $N_0 \prec H(\theta)$ satisfying (i) and (iii). Let $N_1 \prec H(\theta), {N_0}^\omega \subseteq N_1$, $|N_1| = |X|^\omega = |X|$. In general, $N_{\alpha+1} \prec H(\theta)$, ${N_\alpha}^\omega \subseteq N_{\alpha+1}$, $|N_{\alpha+1}| = |X|^\omega = |X|$. At limit λ, $N_\lambda = \bigcup_{\alpha<\lambda} N_\alpha$. $N = N_{\omega_1}$. □

Let M be the transitive collapse of N. Let $\rho\colon N \cong M$ be the collapsing map. Let $L_\beta = \rho(L_\alpha)$. Define $\pi\colon L_\beta \to L_\alpha$ by $a \mapsto \rho^{-1}(a)$.

Remarks. $|\beta| \leq |M| = |N| = |X| < |\alpha|$
$X \subseteq \pi[\beta] := \{\pi(\beta') \ : \ \beta' \in \beta\}$
$|\beta| = |X|$.
Moreover, since $X \subseteq \alpha$ is cofinal, $|\alpha| > |X|$, $\pi \neq \mathrm{id}$, $\mathrm{cp}(\pi) < \beta$.

Remark. $M^\omega \subseteq M$ (since $N^\omega \subseteq N$).

Case (1). β is a cardinal in L.

We will show $0^\#$ exists by constructing an elementary embedding $\pi^*\colon L_{\beta^{++}} \to L_\lambda$ some λ, $\pi^* \supseteq \pi$. (Recall: such an embedding gives $0^\#$.)

Now let $D = \{(\eta, p) \ : \ \eta < \beta, p \subseteq L_{\beta^{++}}, p \text{ is finite}\}$.

For $(\eta,p),(\eta',p') \in D$, set $(\eta,p) \leq (\eta',p')$ if $\eta \leq \eta'$, $p \subseteq p'$.

Let $M_{(\eta,p)} = \text{Skolem Hull}^{L_{\beta^{++}}}(\eta \cup p)$.

Note. $M_{(\eta,p)} \in L$.

Let $\gamma\colon D \to \mathrm{Ord}$ be so that for each $(\eta,p) \in D$, $\pi_{(\eta,p)}\colon M_{(\eta,p)} \cong L_{\gamma(\eta,p)}$ is the Mostowski collapse.

Note. $\pi_{\eta,p} \in L$.

Since β is a cardinal in L, $\gamma(\eta,p)<\beta$.

For $(\eta,p)\le(\eta',p')$, let

$$j_{(\eta,p),(\eta',p')}\colon L_{\gamma(\eta,p)}\to L_{\gamma(\eta',p')}$$

by $\pi_{(\eta',p')}\circ{\pi_{(\eta,p)}}^{-1}=j_{(\eta,p),(\eta',p')}$.

Exercise. Each $j_{(\eta,p),(\eta',p')}\in L_\beta$.

14. October 27

α a cardinal in L;
$X\subseteq\alpha$, $|X|<|\alpha|$, X cofinal;
$|X^\omega|=|X|$;
θ strong limit, $\mathrm{cof}(\theta)>\omega$;
$N\prec H(\theta)$, $N^\omega\subseteq N$, $X,\alpha,L_\alpha\in N$; $\rho\colon N\cong M$, M transitive;
$L_\beta=\rho(L_\alpha)$, $\pi\colon L_\beta\to L_\alpha$, $\mathrm{cp}(\pi)<\beta$, $X\subseteq\pi[\beta]=\{\pi(\beta')\ :\ \beta'<\beta\}$.

Case (1). β is a cardinal in L.

$D=\{(\eta,p)\ :\ \eta<\beta, p\subseteq L_{\beta^{++}}, p\text{ finite}\}$. Set $(\eta,p)\le(\eta',p')$ if $\eta<\eta', p\subseteq p'$. Then at each $(\eta,p)\in D$,

$$M_{(\eta,p)}=\text{Skolem Hull}^{L_{\beta^{++}}}(\eta\cup p),$$

$\pi_{(\eta,p)}\colon M_{(\eta,p)}\cong L_{\gamma(\eta,p)}$. Then for $d\le d'$ in D,

$$j_{d,d'}\colon L_{\gamma(d)}\to L_{\gamma(d')},\qquad a\mapsto\pi_{d'}({\pi_d}^{-1}(a)).$$

By cardinality considerations, each $\gamma(\eta,p)<\beta$; therefore $L_{\gamma(\eta,p)}\in L_\beta$.

Claim. *For $d=(\eta,p)\le d'=(\eta',p')$, $j_{(\eta,p),(\eta',p')}\in L_\beta$.*

Proof of claim. Note $\pi_d,\pi_{d'}\in L$, therefore $j_{d,d'}\in L$. By cardinality considerations, $j_{d,d'}\in L_\beta$. □

[This is easy, but in some sense isn't the correct proof, since it doesn't generalize to Case (2).]

Another proof of claim. Note $\pi_d|_\eta=\mathrm{id}, \pi_{d'}|_{\eta'}=\mathrm{id}$. Let

$$p=\{a_1,\dots,a_n\},\qquad p'=\{a_1,\dots,a_n,a_{n+1},\dots,a_m\}.$$

Let $b_i=\pi_d(a_i), i\le n$, $c_i=\pi_{d'}(a_i), i\le n$. Then for $f\colon L_{\gamma(d)}\to L_{\gamma(d)}$ a function definable over $L_{\gamma(d)}$, $\eta_0,\dots,\eta_\ell\in\eta$, it follows from elementarity of $\pi_d,\pi_{d'}$ that

$$j_{d,d'}(f(\eta_0,\dots,\eta_\ell,b_0,\dots,b_n))=f(\eta_0,\dots,\eta_\ell,c_0,\dots,c_n).$$

□

Let $\mathcal{A}=\mathrm{DirLim}_{d\le d'\in D}(L_{\gamma(d)},j_{d,d'})$ (see Homework 2).

$j_{d,\infty}\colon L_{\gamma(d)}\to\mathcal{A}$ elementary all $d\in D$. $\mathcal{A}=(A,\in^{\mathcal{A}})$, $A=\{[(a,d)]_\sim\ :\ d\in D, a\in L_{\gamma(d)}\}$, $[(a,d)]_\sim=[(a',d')]_\sim$ if $\exists d''\ge d,d'$, $j_{d,d''}(a)=j_{d',d''}(a')$; $[(a,d)]_\sim\in^{\mathcal{A}}[(a',d')]_\sim$ if $\exists d''\ge d,d'$ such that $j_{d,d''}(a)\in j_{d',d''}(a')$.

Note. $\mathcal{A}$ is isomorphic to $L_{\beta^{++}}$. For $[(a,d)]_\sim\in\mathcal{A}$, let $\psi([(a,d)]_\sim)={\pi_d}^{-1}(a)$ so that $\psi\colon\ \mathrm{DirLim}(L_{\gamma(d)},j_{d,d'})\cong L_{\beta^{++}}$.

Note. This is well defined, because ${\pi_{d'}}^{-1} \circ j_{d,d'} = {\pi_d}^{-1}$ for $d \leq d'$.

Given this, it is easily checked to be elementary and onto.

At each $d \leq d'$ in D, let $N_d = \pi(L_{\gamma(d)})$, $i_{d,d'} = \pi(j_{d,d'})$. This still forms a directed system of models and elementary maps (since each finite part is in L_β, and π is elementary).

Claim. $\mathrm{DirLim}_{d \leq d'}(N_d, i_{d,d'})$ *is well founded.*

Proof of claim. Suppose not. Then $\exists (d_n)_{n\in\omega}$, $(b_n)_{n\in\omega}$ with $d_n \leq d_{n+1}$ all n, $b_n \in N_{d_n}$ and $b_{n+1} \in i_{d_n,d_{n+1}}(b_n)$.

Then, $(d_n)_{n\in\omega} \in N$. But $(N_{d_n}, i_{d_n,d_{n+1}})_{n\in\omega} \in N$ (since $N^\omega \subseteq N$!!) [This is the part where we jump up and down because something seriously magical is happening.]

Hence, $(L_{\gamma(d_n)}, j_{d_n,d_{n+1}})_{n\in\omega} \in M$. And, by elementarity,

$$M \models \exists (a_n)_{n\in\omega} \text{ such that } \forall n (a_{n+1} \in j_{d_n,d_{n+1}}(a_n)).$$

Hence $\mathrm{DirLim}_{d\leq d'}(L_{\gamma(d)}, j_{d,d'})$ is wellfounded. □

Now let $\hat{\rho}\colon \mathrm{DirLim}_{d\leq d'}(N_d, i_{d,d'}) \cong L_\gamma$ be the Mostowski collapse.

Given $a \in L_{\beta^{++}}$, say $a \in M_d$, we let

$$\hat{\pi}\colon a \mapsto \pi_d(a) \mapsto \pi(\pi_d(a)) \to i_{d,\infty}(\pi(\pi_d(a))) \to \hat{\rho}(i_{d,\infty}(\pi(\pi_d(a))))$$

Claim. $\hat{\pi}|_{L_\beta} = \pi$.

15. October 29

Recall the setup: $X \subseteq \alpha$, α a cardinal in L. X cofinal in α; $\pi\colon L_\beta \to L_\alpha$ by $X \subseteq \pi[\beta]$. We started to take the direct limit: $D = \{(\eta, p) \ : \ \eta < \beta, p \text{ finite}\}$. $M_{(\eta,p)} = \text{Skolem Hull}^{L_{\beta^{++}}}(\eta \cup p)$; $\pi_{(\eta,p)}\colon M_{(\eta,p)} \cong L_{\gamma(\eta,p)}$. For $d \leq d'$, $j_{d,d'}\colon M_d \to M_{d'}$ by $a \mapsto \pi_{d'} \circ \pi_d^{-1}$.

Case (1): β is a cardinal in L. $\psi\colon \mathrm{DirLim}_{d\leq d' \in D}(L_{\gamma(d)}) \cong L_{\beta^{++}}$

(maps to via π^* domain of) $\hat{\rho}\colon \mathrm{DirLim}_{d\leq d'\in D}(\pi(L_{\gamma(d)}), \pi(j_{d,d'})) \cong L_\lambda$ is wellfounded (note existence of elementary map π^* is essentially what homework 2 is about)

Defined $\hat{\pi}\colon L_{\beta^{++}} \to L_\lambda$ via $a \mapsto \hat{\rho} \circ \pi \circ \psi^{-1}(a)$.

Then $\hat{\pi}$ is elementary.

Claim. $\hat{\pi}|_\beta = \pi|_\beta$.

Proof of claim. Fix $\beta_0 < \beta$. Let $\eta > \beta_0$. Then for all $d' \geq d = (\eta, p)$, $\pi_{d'}|_\eta = \mathrm{id}$. So $j_{d,d'}|_\eta = \mathrm{id}$ all $d' \geq d$. Then it follows $\psi \circ j_{d,\infty}|_\eta = \mathrm{id}$. (Technically: One would prove by induction on $\eta' \leq \eta$ that $\psi \circ j_{d,\infty}|_{\eta'} = \mathrm{id}$.)

Similarly, for all $d' \geq d$,

$$\pi(j_{d,d'})|_{\pi(\eta)} = i_{d,d'}|_{\pi(\eta)} = \mathrm{id}$$

by elementarity of π. Thus $\hat{\rho} \circ i_{d,\infty}|_{\pi(\eta)} = \mathrm{id}$.

Thus: given $\eta < \beta$, $\hat{\pi}|_\eta = \pi|_\eta$ since $\hat{\pi} = \hat{\rho} \circ \pi^* \circ \psi^{-1}$.

For $\eta' < \eta$, $\pi^* \circ \psi^{-1}(\eta') = i_{d,\infty} \circ \pi \circ j_{d,\infty}^{-1}(\psi^{-1}(\eta'))$ where d is chosen so that $\psi^{-1}(\eta') \in \mathrm{im}(j_d, \infty)$. But if $d = (\eta, p)$, $\eta > \eta'$ we saw that $\psi \circ j_{d,\infty}|_\eta = \mathrm{id}$, hence $\pi^* \circ \psi^{-1}(\eta') = i_{d,\infty} \circ \pi(\eta')$.

But $\hat{\rho} \circ i_{d,\infty} \circ \pi(\eta') = \pi(\eta')$ since $\hat{\rho} \circ i_{d,\infty}|_{\pi(\eta')} = \mathrm{id}$. □

Since $\pi|_\beta \neq \mathrm{id}$, $\mathrm{cp}(\hat{\pi}) < \beta$, and we have $0^\#$.

Case (2). β is not a cardinal in L.

Note. For $1 \leq j < \omega$, $\delta \in \text{Ord}$, $A \subseteq L_\delta$, $H_j^{L_\delta}(A)$ is the Σ_j-skolem hull of A in L_δ. I.e. it is the smallest set such that

(i) $A \subseteq H_j^{L_\delta}(A)$;
(ii) if $\vec{a} \in H_j^{L_\delta}(A)$, $\varphi \in \Sigma_j$, $L_\delta \models \exists x \varphi(x, \vec{a})$, then for $b <_L$-least in L_δ with $L_\delta \models \varphi(b, \vec{a})$, we have $\delta \in H_j^\delta(A)$.

$H_\omega^{L_\delta}(A)$ =full Skolem hull of A in L_δ.

Facts. If δ is an ordinal,

(i) $<_L |_{L_\delta}$ is Σ_1 over L_δ
(ii) if $M \prec_{\Sigma_1} L_\delta$, then $M \cong L_\gamma$ some γ
(iii) L_δ is closed under Σ_0-comprehension

(Here Σ_0 means Δ_0.)

Remarks. (iii) is an exercise in the definition of L. Precise proofs of (i) & (ii) postponed.

Let δ be least such that there is some j, $1 \leq j \leq \omega$, some $\eta < \beta$, $p \subseteq L_\delta$ such that $H_j^{L_\delta}(\eta \cup p) \supseteq \beta$. (Exercise in condensation: Case assumption gives such δ, η, p, j exist.)

For this δ, now minimize j. So we have fixed lexicographically least (δ, j).

Case (2a). $j = \omega$

Case (2b). $i < j < \omega$

Case (2c). $j = 1$, δ a limit.

Case (2d). $j = 1$, δ a successor.

Next time: case (2a): Will let $D = \{(\eta, p, n) \ : \ n \in \omega, \eta < \beta, p \subseteq L_\delta \text{ finite}\}$. $M_{(\eta,p,n)} = H_\eta^{L_\delta}(\eta \cup p)$.

16. November 1, 2010

We've worked up to:

- $\pi \colon L_\beta \to L_\alpha$, α a cardinal in L
- $X \subseteq \pi[\beta]$
- and: covering lemma holds at all $\alpha' < \alpha$.

Case (2). β is not a cardinal in L.

[Magic moment: we will locate the first time in the history of the universe that β gets to not be a cardinal in L.] Minimize δ, and then j ($1 \leq j \leq \omega$) such that

$$\beta \subseteq H_j^{L_\delta}(\eta \cup p)$$

some $\eta < \beta$, $p \subseteq L_\delta$ finite.

Case (2a). $j = \omega$.

Let $D = \{(\eta, p, n) \ : \ \eta < \beta, p \subseteq L_\delta \text{ finite}, n \in \omega\}$. For $(\eta, p, n) \in D$,

$$M_{(\eta,p,n)} = H_n^{L_\delta}(\eta \cup p).$$

$\pi_{(\eta,p,n)} \colon M_{(\eta,p,n)} \cong L_{\gamma(\eta,p,n)}$.

Claim. $\gamma(\eta, p, n) < \beta$.

Proof. Note $\gamma(\eta, p, n) \leq \delta)$.

Note also: $L_{\gamma(\eta,p,n)} \subseteq H_n^{L_{\gamma(\eta,p,n)}}(\eta \cup \pi_{\gamma(\eta,p,n)}(p))$.

But if $\beta \leq \gamma(\eta, p, n)$, then $\beta \subseteq H_n^{L_{\gamma(\eta,p,n)}}(\eta \cup p')$ some finite p'. Using $\gamma(\eta, p, n) \leq \delta$, this contradicts minimality of δ or of j. □

Let $\pi_{d,d'} = \pi_{d'} \circ \pi_d^{-1}$ all $d \leq d'$ in D.

Claim. *Each* $\pi_{d,d'} \in L_\beta$.

Proof. For $d = (\eta, p, n)$, $d' = (\eta', p', n')$, f a Σ_n-definable function, $\pi_{(\eta,n,p)}(p) = (p_0, \ldots, p_\ell)$, $\pi_{(\eta',n',p')}(p) = (p'_0, \ldots, p'_\ell)$, $\eta_0, \ldots, \eta_k \in \eta$,

$$\pi_{d,d'}(f^{L_{\gamma(\eta,p,n)}}(\eta_0, \ldots, \eta_k, p_0, \ldots, p_\ell)) = f^{L_{\gamma(\eta',p',n')}}(\eta_0, \ldots, \eta_k, p'_0, \ldots, p'_\ell)$$

Hence $\pi_{d,d'}$ as a subset of $L_{\gamma(\eta,p,n)} \times L_{\gamma(\eta',p',n')}$ is (first order) definable from $\eta, p_0, \ldots, p_\ell$, $p'_0, \ldots, p'_\ell$. Hence it is in $L_{\beta'}$ any $\beta' > \gamma(\eta, p, n), \gamma(\eta', p', n')$. □

As in last argument,

$$\psi_0 \colon \operatorname*{DirLim}_{d \leq d' \in D}(L_{\gamma(d)}, \pi_{d,d'}) \cong L_\delta$$

for $d = (\eta, p, n)$, $j_{d,\infty} \colon L_{\gamma(d)} \to \mathrm{DirLim}(\cdots)$ will be Σ_n-elementary.

Note. For any $d_0 \in D$, $\mathrm{DirLim}_{d \leq d' \in D}(L_{\gamma(d)}, \pi_{d,d'})$ is maximally $\cong$ to $\mathrm{DirLim}_{d_0 \leq d \leq d' \in D}(L_{\gamma(d)}, \pi_{d,d'})$.

As before, $\mathrm{DirLim}_{d \leq d' \in D}(\pi(L_\gamma(d)), \pi(\pi_{d,d'}))$ is wellfounded, say

$$\psi_1 \colon \operatorname*{DirLim}_{d \leq d' \in D}(\pi(L_{\gamma(d)}), \pi(\pi_{d,d'})) \cong L_\lambda.$$

[Fixing $i_{d_0,\infty} \colon L_{\gamma(d_0)} \to \mathrm{DirLim}_{d \leq d' \in D}(\pi(L_{\gamma(d)}), \pi(\pi_{d,d'}))$] Then also as before, get

$$\rho \colon L_\delta \to L_\lambda \text{ with } \rho \circ \psi_0 \circ j_{d,\infty} = \psi_1 \circ i_{d,\infty} \circ \pi|_{L_{\gamma(d)}}.$$

Note. ρ is fully elementary (since $\pi|_{\gamma(d)}$ is elementary all d, and $j_{d,\infty}$ is Σ_n-elementary all d sufficiently large) Then for all $\alpha_0 \in X$, let $\beta_0 < \beta$ have $\pi(\beta_0) = \alpha_0$.

Also as before: $\rho|_{L_\beta} = \pi$.

Fix $\eta < \beta$, $p \subseteq L_\delta$ finite such that $\beta \subseteq H_\omega^{L_\delta}(\eta \cup p)$. ["fixing the way we see β is smaller"] Then for all $\alpha_0 \in X$, choose $\beta_0 < \beta$ with $\pi(\beta_0) = \alpha_0$. Then there is Σ_n (some n) definable f, $\eta_0, \ldots, \eta_k \in \eta$, with $f^{L_\delta}(\eta_0, \ldots, \eta_k, p) = \beta_0$. But by elementarity of ρ,

$$f^{L_\lambda}(\pi(\eta_0), \ldots, \pi(\eta_k), \pi(p)) = \pi(\beta_0) = \alpha_0.$$

Thus, $X \subseteq H_\omega^{L_\lambda}(\pi(\eta) \cup \pi(p))$. ($\pi(\eta) < \alpha$, α a cardinal in L.)

Hence for $Z = H_\omega^{L_\lambda}(\pi(\eta) \cup \pi(p))$ we have that $Z \in L$, $|Z|^L < \alpha$, $X \subseteq Z$. But now assuming inductive hypothesis (covering holds at all $\alpha' < \alpha$) we obtain the covering lemma for X.

17. November 3

Last time we finished Case (2a). We've worked our way up to

Case (2b). $1 < j < \omega$.

Technical background facts.

(i) each L_α has a Σ_1-definable wellordering.
(ii) if $M \prec_{\Sigma_1} L_\alpha$, then $M \cong L_{\overline{\alpha}}$ some $\overline{\alpha}$.
(iii) L_α closed under Σ_0-definability.

Recall: we fixed the least δ, j such that $\exists p \subseteq L_\delta$ finite, $\eta < \beta$, $\beta \subseteq H_j^{L_\delta}(\eta \cup p)$.

Claim. *$j \neq 0$.*

Proof of claim. Suppose, for simplicity, $p = L_{\delta'}$, some $\delta' < \delta$; can assume $\delta' \geq \beta$. Let $\varphi(L_{\delta'}, x, y)$ be Σ_0, and suppose $\eta < \beta$ has $\forall \beta' < \beta\ \exists \eta' < \eta$ such that β' is $<_L$-least with $\varphi(L_{\delta'}, \beta', \eta')$.

Let ψ be Σ_n (some n) such that $\forall x, y \in L_\delta$,

$$L_{\delta'} \models \psi(x, y) \text{ iff } \varphi(L_{\delta'}, x, y)$$

e.g. if $(*)$ $\varphi(L_\delta, x, y)$ is "$x \in y \wedge \exists z \in L_\delta(\varphi_0(z, y))$", $\varphi \in \Sigma_0$, then $\psi(x, y)$ is "$x \in y \wedge \exists z(\varphi_0(z, y))$".

A slightly more general case: $A = \{x \in L_{\delta'} \mid L_{\delta'} \models \theta(x)\}$, and suppose $\forall \beta' < \beta\ \exists \eta' < \eta$ such that β' is $<_L$-least such that $\varphi(A, \beta', \eta')$ where φ is as above $(*)$. Then $\psi(x, y)$ is "$x \in y \wedge \exists z(\theta(z) \wedge \varphi_0(z, y))$" □

Then let $D = \{(\eta, p) \ :\ p \subseteq L_\delta \text{ finite}, \eta < \beta\}$, $(\eta, p) \leq (\eta', p')$ if $\eta \leq \eta'$, $p \subseteq p'$. For $(\eta, p) \in D$,

$$M_{\eta,p} = H_{j-1}^{L_\delta}(\eta \cup p), \qquad \pi_d \colon M_d \cong L_{\gamma(d)} \text{ for } d = (\eta, p).$$

Let $j_{d,d'} \colon L_{\gamma(d)} \to L_{\gamma(d')}$ by $x \mapsto \pi_{d'} \circ \pi_d^{-1}(x)$. Then $\mathrm{DirLim}_{d \leq d'}(L_{\gamma(d)}, j_{d,d'}) \cong L_\delta$. Each $j_{d,d'}$ is Σ_{j-1} elementary, since $j - 1 < j$, and j was minimal for case (2b).

$\beta \nsubseteq H_{j-1}^{L_{\gamma(d)}}(\eta \cup \pi_d(p))$. *But* $L_{\gamma(d)} \subseteq H_{j-1}^{L_{\gamma(d)}}(\eta \cup \pi_d(p))$ since $M_d = H_{j-1}^{L_\delta}(\eta \cup p)$. Therefore $\gamma(d) < \beta$.

Hence $\pi(L_{\gamma(d)}) = N_d$, $\pi(j_{d,d'}) = i_{d,d'}$ are well-defined.

Similarly (to last case) $\mathrm{DirLim}_{d \leq d'}(N_d, i_{d,d'})$ well founded. Say $\cong L_\lambda$ some λ.

The natural embedding

$$\operatorname*{DirLim}_{d \leq d'}(L_{\gamma(d)}, j_{d,d'}) \to \operatorname*{DirLim}_{d \leq d'}(N_d, i_{d,d'})$$

is Σ_j-elementary (homework). So, as before, we get $\hat{\pi} \colon L_\delta \to L_\lambda$ Σ_j elementary with $\hat{\pi}|_\beta = \pi$. Fix $\eta < \beta, p \subseteq L_\delta$ finite, such that $\beta \subseteq H_j^{L_\delta}(\eta \cup p)$. Therefore $\forall \alpha_0 \in X$, $\exists \beta_0 \in \beta(\pi(\beta_0) = \alpha_0)$. $\beta_0 \in H_j^{L_\delta}(\eta \cup p)$, therefore $\alpha_0 \in H_j^{L_\lambda}(\pi(\eta) \cup \pi(p))$ (by Σ_j elementarity of $\hat{\pi}$). Thus, $X \subseteq H_j^{L_\lambda}(\pi(\eta) \cup \pi(p))$. This has cardinality $< \alpha$ in L, so it finishes as before.

Case (2c). $j = 1$, δ a limit.

$D = \{(\eta, p, \overline{d}) \ : \ \eta < \beta, p \subseteq L_{\overline{d}}$ finite, $\overline{d} < \delta\}$. Let $M_{(\eta,p,\overline{d})} = H_\omega^{L_{\overline{\delta}}}(\eta \cup p)$. $\pi_d \colon M_d \cong L_{\gamma(d)}$. For $d \le d'$, $j_{d,d'}$ is Σ_0-elementary.

$$\operatorname*{DirLim}_{d \le d'}(L_{\gamma(d)}, j_{d,d'}) \cong L_\delta$$

because δ is a limit. Then as before. $[(p, \eta, \overline{\delta}) \le (p', \eta', \overline{\delta}')$ if $p \subseteq p'$, $\eta \le \eta'$, $\overline{\delta} = \overline{\delta}'$ or $\overline{\delta} \in p'$.]

18. November 5

Recall we had a map $\pi \colon L_\beta \to L_\alpha$ with $X \subseteq \pi[\beta]$. We minimized δ, j such that $\exists p \subseteq L_\delta$ finite, $\eta < \beta$, $\beta \subseteq H_j^{L_\delta}(\eta \cup p)$.

Case (2d). $j = 1$, δ a successor: $\delta = \overline{\delta} + 1$.

Fact. If $\psi_0 \in \Sigma_0$, $\psi_1 \in \Sigma_n$, then there exists $\psi_2 \in \Sigma_m$ (some m) such that if $q \in L_\delta$, $A = \{x \in L_{\overline{\delta}} \ : \ L_{\overline{\delta}} \models \psi_1(x, q)\}$, and $a \in L_{\overline{\delta}}$

$$L_{\overline{\delta}+1} \models \psi_0(A, a) \text{ iff } L_{\overline{\delta}} \models \psi_2(p, a).$$

(Why? Go through $\psi_0(A, a)$ and replace all occurrences of "$x \in A$" by "$\psi_1(x, q)$".)

Given this it follows that is $\beta \subseteq H_1^{L_{\overline{\delta}+1}}(\eta \cup p)$, $\eta < \beta$, $p \subseteq L_{\overline{\delta}}$ finite, then $\beta \subseteq H_\omega^{L_{\overline{\delta}}}(\eta \cup p)$.

Why? Let $\exists y\, \psi_0(x, z, y)$ be Σ_1, $\psi_0 \in \Sigma_0$. Suppose $\forall \beta' < \beta\ \exists \eta' < \eta$ such that β' is least such that

$$L_{\overline{\delta}+1} \models \exists y\, \psi_0(\beta', \eta', y).$$

Fix β' and then η' as above (i.e. β' is least such that $L_{\overline{\delta}+1} \models \exists y\, \psi_0(\beta', \eta', y)$). Now fix the y, and assume

$$y = \{x \in L_{\overline{\delta}} \ : \ L_{\overline{\delta}} \models \psi_1(x, q)\}$$

for $q \in L_{\overline{\delta}}$. Then apply the fact to get ψ_2 as given there. Then

$$L_{\overline{\delta}} \models \psi_2(\beta', \eta', q).$$

Claim. *β' is least such that $\exists q$ with $L_{\overline{\delta}} \models \psi_2(\beta', \eta', q)$.*

Proof of claim. Suppose instead $\beta'' < \beta'$, $q' \in L_{\overline{\delta}}$ and $L_{\overline{\delta}} \models \psi_2(\beta'', \eta', q')$. Then for $y' = \{x \in L_{\overline{\delta}} \ : \ L_{\overline{\delta}} \models \psi_1(x, q')\}$, we have (by assumption on ψ_2) $L_{\overline{\delta}+1} \models \psi_0(\beta'', \eta', y')$ – contradicting the minimality assumption of β' subject to $L_{\overline{\delta}+1} \models \exists y\, \psi_0(\beta', \eta', y)$. $\square$

Thus we have $\beta \subseteq H_\omega^{L_{\overline{\delta}}}(\eta)$ given $\beta \subseteq H_1^{L_{\overline{\delta}+1}}(\eta)$. The argument is completely similar if we drag along the parameter p.

Remarks. **On How to Work Without $|X|^\omega = |X|$.**

Lemma. *Suppose $\pi \colon L_\beta \to L_\alpha$ is elementary. Suppose $\gamma \le \gamma'$, and at each $\eta < \beta$, $p \subseteq L_\gamma$ finite, we let*

$$M_{(\eta,p)} = H_\omega^{L_\gamma}(\eta \cup p)$$

and have $\pi_{(\eta,p)} \colon M_{(\eta,p)} \cong L_{\gamma(d)}$ and $\pi_{d,d'} \colon L_{\gamma(d)} \to L_{\gamma(d')}$, $\pi_{d,d'} = \pi_{d'} \circ \pi_d^{-1}$.

Similarly, let $M'_{(\eta,p)} = H_\omega^{L_{\gamma'}}(\eta \cup p)$ where $p \subseteq L_{\gamma'}$ is finite and $\eta < \beta$. Put $\pi'_{(\eta,p)} \colon M'_{(\eta,p)} \cong L_{\gamma'(d)}$, $\pi'_{d,d'} \colon L_{\gamma'(d)} \to L_{\gamma'(d')}$, $\pi'_{d',d} = \pi'_{d'} \circ (\pi'_d)^{-1}$.

Suppose (for all such suitable $d = (\eta, p)$) that $L_{\gamma(d)}, \pi_{d,d'}, L_{\gamma'(p)}, \pi'_{d',d} \in L_\beta$. Then if $\mathrm{DirLim}_{d\leq d'}(\pi(\pi'_{d',d}), \pi(L'_\gamma(d)))$ is wellfounded, so it $\mathrm{DirLim}_{d\leq d'}(\pi(\pi_{d',d}), \pi(L_{\gamma(d)}))$.

Idea of proof: If $\gamma \in p'$, $p \subseteq p'$, $\eta \leq \eta'$, then there is a natural (Σ_0-elementary) map from $M_{(\eta,p)} \to M_{(\eta',p')}$ (inclusion!); and then from $L_{\gamma(\eta,p)}$ to $L_{\gamma'(\eta',p')}$.

Similar lemmas hold for the other types of direct limits appearing in Case (1), Cases (2a-c). Now, given $X \subseteq L_\alpha$, α a cardinal in L, θ as before, we build $N \prec H(\theta)$ in ω_1-steps. $\pi_0 L_{\beta_0} \to L_\alpha$.

Begin with $N_0 \prec H(\theta)$. $\alpha, X \in N_0$, $X \subseteq N_0$. Now go through and consider all direct limits for some γ with illfounded limit under π.

Then minimize the γ.

And then throw into N_1 the witnesses to illfoundedness of the direct limit. Iterate $N_\alpha \to N_{\alpha+1}$ (throw in witnesses to illfoundedness). $N_{\omega_1} = \bigcup_{\alpha<\omega_1} N_\alpha$. Then

$$M_{\omega_1} = \text{Transitive Collapse}(N_{\omega_n}), \qquad \hat{\pi}\colon N_{\omega_1} \cong M_{\omega_1}.$$

$L_\beta = \hat{\pi}(L_\alpha)$. Then $\pi\colon L_\beta \to L_\alpha$, $a \mapsto \hat{\pi}'(a)$.

I claim: The direct limits needed to remain wellfounded under π.

19. November 8

We now head towards: $0^\#$ *exists iff* Σ^1_1*-determinacy.* $\Longrightarrow$ due to Martin; $\Longleftarrow$ due to Harrington.

Definition. $A \subseteq (\omega^\omega)^\ell \times \omega^k$ is Π^0_1, $\ell \geq 1$, if there is $T \subseteq (\omega^{<\omega})^\ell \times \omega^k$ such that

(i) if $(s_1, \dots, s_\ell, \vec{p}) \in T$, then $\mathrm{lh}(s_1) = \mathrm{lh}(s_2) = \cdots = \mathrm{lh}(s_\ell)$
(ii) if $(s_1, \dots, s_\ell, \vec{p}) \in T$, then $\forall m < \mathrm{lh}(s_1)((s_1|_m, \dots, s_\ell|_m, \vec{p}) \in T)$
(iii) $A = \{(\vec{x}, \vec{p}) \in (\omega^\omega)^\ell \times \omega^k \ : \ \forall m((x_1|_m, \dots, x_\ell|_m, \vec{p}) \in T)\}$
(iv) T is recursive (in the Gödel codes: $\{\ulcorner \vec{s}, \vec{p} \urcorner \ : \ (\vec{s}, \vec{p}) \in T\}$ is recursive)

Definition. If $T \subseteq (\omega^{<\omega})^\ell \times \omega^k$ satisfies (i) and (ii), then we say that T is a **tree**.

If T satisfies (i),(ii),(iii), but is (only) recursive in z, then we say that A is $\Pi^0_1(z)$.

Definition. If $T \subseteq (\omega^{<\omega})^\ell \times \omega^k$ is a tree, then

$$\{(\vec{x}, \vec{p}) \in (\omega^\omega)^\ell \times \omega^k \ : \ (\forall m)(x_1|_m, \dots, x_\ell|_m, \vec{p}) \in T\}$$

is denoted by $[T]$, and is called the **set of branches through** T.

Lemma. *Let* $T \subseteq (\omega^{<\omega})^\ell \times \omega^k$ *be* Π^0_1 *(in the Gödel codes),* T *a tree. Then* $[T]$ *is* Π^0_1.

Proof. Say $T = \{(\vec{s}, \vec{p}) \ : \ \forall m\, R(\ulcorner \vec{s}, \vec{p} \urcorner, m)\}$ where R is recursive. Then let

$$\hat{T} = \{(s_1, \dots, s_\ell, \vec{p}) \ : \mathrm{lh}(s_1) = \cdots = \mathrm{lh}(s_\ell), \\ \forall m < \mathrm{lh}(s_1) \forall q < \mathrm{lh}(s_1)\, R(\ulcorner (s_1|_m, \dots, s_\ell|_m, \vec{p}) \urcorner, q)\}.$$

$\hat{T}$ is recursive, and $[\hat{T}] = [T]$. □

Examples. (i) $2^\omega = \{0,1\}^\omega$ is a Π^0_1 subset of ω^ω. (Take $T = 2^{<\omega}$.)
(ii) For $x \in 2^\omega$, let $\in_x = \{(n,m) \ : \ x(2^n 5^m) = 1\}$. Then

$$\{x \in 2^\omega \ : \in_x \ \text{is a linear ordering of } \omega\}$$

is Π^0_1 as a subset of ω^ω.

$$(T = \{s \in 2^{<\omega} : \forall n, m < \mathrm{lh}(s)\, n \neq m \wedge 2^n5^m < \mathrm{lh}(s) \wedge 2^m5^n < \mathrm{lh}(s) \implies \text{exactly one of } s(2^n5^m) = 1 \text{ and } s(2^m5^n) = 1 \text{ holds; and } \forall n_1, n_2, n_3 < \mathrm{lh}(s)\, \neg(s(2^{n_1}5^{n_2}) = s(2^{n_2}5^{n_3}) = 1 \wedge s(2^{n_1}5^{n_3}) = 0)\}.)$$

(iii) Let R be recursive. Then

$$A := \{x \in \omega^\omega \mid \forall n[(\neg\exists m\, R(n, m) \wedge x(n) = 0) \vee (x(n) = m \text{ where } m \text{ is least such that } R(n, m))]\}$$

holds. Then A is Π^0_1. (Let

$$T = \{s \in \omega^{<\omega} : \forall n, m < \mathrm{lh}(s)(\text{if } R(n, m) \text{ then } R(n, s(n)) \text{ and } s(n) \leq m), \text{ and } [\forall m < \mathrm{lh}(s)\neg R(n, m)] \implies s(n) = 0.\})$$

Exercise. There is a recursive tree $T \subseteq \omega^{<\omega}$ such that
(i) $|[T]| = 1$, and
(ii) if $x \in [T]$, then $x \in 2^\omega$, $x \equiv_T 0'$.

Definition. For $A \subseteq (\omega^\omega)^{\ell+1} \times \omega^k$,

$$p[A] = \{(\vec{x}, \vec{p}) : \exists y(y, \vec{x}, \vec{p}) \in A\}.$$

We say that $B \subseteq (\omega^\omega)^\ell \times \omega^k$ is Σ^1_1 if there is a Π^0_1 set $A \subseteq (\omega^\omega)^{\ell+1} \times \omega^k$ with $B = p[A]$.

Definition. For $B \subseteq \omega^\omega$, G_B is the following infinite game between players, **I** & **II**, who alternate playing elements of ω:

I	n_0		n_2		$\dots$
II		n_1		n_3	$\dots$

We say that **I wins** if $x = (n_0, n_1, \dots) \in B$; otherwise **II** wins.

We say that **I has a winning strategy** if there is some function $\sigma : \omega^{<\omega} \to \omega$ such that whenever $x \in \omega^\omega$, with $x(2n) = \sigma(x(0), x(1), \dots, x(2n-1))$, then $x \in B$.

II has a winning strategy if there is $\sigma : \omega^{<\omega} \to \omega$ such that $\forall x \in \omega^\omega$ if at all n

$$x(2n + 1) = \sigma(x(0), x(1), \dots, x(2n)),$$

then $x \notin B$.

We say that B is **determined** if one of the two players has a winning strategy.

Definition. Σ^1_1-determinacy is the statement that whenever $B \subseteq \omega^\omega$ is Σ^1_1, then G_B is determined.

20. November 10

Recall. $A \subseteq (\omega^\omega)^k \times \omega^m$ is Σ^1_1 if there is a recursive tree $T \subseteq (\omega^{<\omega})^{k+1} \times \omega^m$ such that

$$A = p[T] := \{(\vec{x}, \vec{p}) : \exists y(y, \vec{x}, \vec{p}) \in [T](\text{i.e. } \forall \ell \in \omega(y|_\ell, x_0|_\ell, \dots, x_{k-1}|_\ell, \vec{p}) \in T)\}.$$

Definition. B is said to be Π^1_n if B^c is Σ^1_n, and C is Σ^1_{n+1} if there is $B \in \Pi^1_n$ with

$$C = \{(\vec{x}, \vec{p}) : \exists y((y, \vec{x}, \vec{p}) \in B)\}.$$

$\Delta^1_n = \Sigma^1_n \cap \Pi^1_n$.

Recall we defined, for $x \in 2^\omega$, $\in_x = \{(n,m) \ : \ x(2^n 5^m) = 1\}$.

Examples (of Σ^1_1 sets).

(i) $A = \{x \in 2^\omega \mid \ \in_x \ \text{is illfounded}\}$.
$B = \{(y,x) \in \omega^\omega \times \omega^\omega \ : \ x \in 2^\omega, x(2^{y(n+1)}5^{y(n)}) = 1 \text{ all } n \in \omega\}$ is Π^0_1: take

$$T = \{(s,t) \in \omega^{<\omega} \times \omega^{<\omega} \ : t \in 2^{<\omega}, \forall n < \mathrm{lh}(t) = \mathrm{lh}(s), \text{ if } 2^{s(n+1)}5^{s(n)} < \mathrm{lh}(t), \text{ then } t(2^{s(n+1)}5^{s(n)}) = 1\}.$$

$B = [T]$, $A = p[B]$.

(ii) $A = \{(x,y) \in 2^\omega \times 2^\omega \ : \ (\omega, \in_x) \cong (\omega, \in_y)\}$;

$$B = \{(z,x,y) \in \omega^\omega \times 2^\omega \times 2^\omega \ : z(2^{z(3^n)}) = z(3^{z(2^n)}) = n \text{ all } n \in \omega, \text{ and } \forall n \forall m \in \omega \ x(2^n 5^m) = y(2^{z(2^n)}5^{z(2^m)})\}$$

i.e. $n \mapsto z(2^n)$ defines a bijection from $\omega \to \omega$ conjugating $\in_x, \in_y$.

Exercise. $B \in \Pi^0_1$, $A \in \Sigma^1_1$.

(iii) $A = \{(x,y) \in 2^\omega \times 2^\omega \ : \in_x, \in_y$ are linear orderings of ω, and $(\omega, \in_x)$ is isomorphic to an *initial segment* of $(\omega, \in_y)\}$.
Then A is again Σ^1_1.

Lemma. *If $A_1, A_2 \in \Sigma^1_1$. Then*

(i) $A_1 \cup A_2 \in \Sigma^1_1$
(ii) $A_1 \cap A_2 \in \Sigma^1_1$

Proof. Say $A_1, A_2 \subseteq \omega^\omega$ (for notational simplicity). Say $A_1 = p[T_1]$, $A_2 = p[T_2]$, T_1, T_2 recursive.
(i) Let $S = T_1 \cup T_2$. Then $p[S] = p[T_1] \cup p[T_2]$.
(ii) Let

$$B = \{(y,x) \in \omega^\omega \times \omega^\omega \ : \forall \ell((y(2), y(2^2), \ldots, y(2^\ell)), (x(0), \ldots, x(\ell-1)) \in T_1 \wedge ((y(3), y(3^2), \ldots, y(3^\ell)), (x(0), \ldots, x(\ell-1))) \in T_2)\}.$$

Then $p[B] = A_1 \cap A_2$, $B \in \Pi^0_1$ (exercise). □

Lemma. *Σ^1_1 is closed under number quantification, i.e., if $A \subseteq (\omega^\omega)^k \times \omega^{m+1}$ is Σ^1_1, then*

(i) $\{(\vec{x}, \vec{p}) \mid \forall n \, (\vec{x}, n, \vec{p}) \in A\} \in \Sigma^1_1$
(ii) $\{(\vec{x}, \vec{p}) \mid \exists n \, (\vec{x}, n, \vec{p}) \in A\} \in \Sigma^1_1$.

Proof. Assume $k = 1$, $m = 0$, and $A = p[T]$ where $T \subseteq (\omega^{<\omega})^2 \times \omega$ is recursive.
(i) Let

$$S = \{(s,t) \in (\omega^{<\omega})^2 \ : \ \mathrm{lh}(s) = \mathrm{lh}(t), \forall n, \ell < \mathrm{lh}(s), \text{ if } 2^n 5^\ell < \mathrm{lh}(s), \text{ then } ((s(2^n), s(2^n 5), \ldots, s(2^n 5^\ell)), (t(0), \ldots, t(\ell)), n) \in T\}.$$

Then $p[S] = \{x \mid \forall n \exists y((y,x,n) \in [T])\}$.

(ii) Now take

$$U = \{(s,t) \in (\omega^{<\omega})^2 \ : \ \mathrm{lh}(s) = \mathrm{lh}(t), \forall \ell < \mathrm{lh}(s)((s(1),\dots,s(\ell)),(t(0),\dots,t(\ell-1)),s(0)) \in T\}.$$

Then $(y,x) \in [U]$ iff $((y(1),y(2),\dots),x,y(0)) \in [T]$. □

Corollary. *Let $\varphi(x_0,\dots,x_n)$ be a formula of set theory. Let $m_0,\dots,m_n \in \omega$. Then $\{x \in 2^\omega \mid (\omega,\in_x) \models \varphi(m_0,\dots,m_n)\}$ is Σ^1_1.*

Proof. Clearly true for $\varphi(\vec{x})$ quantifier free. Now note that up to logical equivalence, all formulas can be built up from q.f. formulas using $\wedge,\vee,\exists x,\forall x$. Now it follows by induction on logical complexity. The last lemma handles $\exists x,\forall x$ (since there will be quantifying over ω), and the lemma before $\wedge,\vee$. □

Corollary. *(φ as above) $\{x \in 2^\omega \mid (\omega,\in_x) \models \varphi(\vec{m})\} \in \Pi^1_1$, and $\in \Delta^1_1$.*

Exercise (not completely trivial). $\{\ulcorner\varphi(\vec{m})\urcorner \ : \ \varphi \in \mathcal{L}(\mathrm{PA}), (\mathbb{N};+,\cdot,0,1,S) \models \varphi(\vec{m})\} \in \Sigma^1_1$ (and therefore Δ^1_1).

[Comment: for any $\alpha < {\omega_1}^{\mathrm{CK}}$, the theory of L_α is Σ^1_1!]

21. November 12

Definition. For $T \subseteq \omega^{<\omega} \times \omega^{<\omega}$ a tree,

$$p[T] = \{x \mid \exists y \forall n (y|_n, x|_n) \in T\}$$

We say that $A \subseteq \omega^\omega$ is $\boldsymbol{\Sigma}^1_1$ if there is a tree $T \subseteq \omega^{<\omega} \times \omega^{<\omega}$ with $A = p[T]$.

Caution: Lightface (Σ,Π) versus boldface $(\boldsymbol{\Sigma},\boldsymbol{\Pi})$ is a minor typographical distinction that belies a massive conceptual one!

Remark. $\boldsymbol{\Sigma}^1_1 = \bigcup_{x \in 2^\omega} \Sigma^1_1(x)$.

Definition. For $T \subseteq \omega^{<\omega} \times \omega^{<\omega}$ a tree and $x \in \omega^\omega$,

$$T_x = \{s \in \omega^{<\omega} \ : \ (s, x|_{\mathrm{lh}(s)}) \in T\}.$$

Remarks.

(i) if T is recursive, then T_x is (uniformly) recursive in x.
(ii) $[T_x] \neq \emptyset$ iff $x \in p[T]$.

Definition. For $s,t \in \omega^{<\omega}$, set $s <_{\mathrm{KB}} t$ if either

(i) $s \supsetneq t$, or
(ii) there is some n with $s(n) \neq t(n)$, and for n least with this property, $s(n) < t(n)$.

Lemma. *$<_{\mathrm{KB}}$ linearly orders $\omega^{<\omega}$.*

Proof. Exercise. □

Theorem. *For $T \subseteq \omega^{<\omega}$ a tree, $[T] = \emptyset$ iff $<_{\mathrm{KB}}$ wellorders T.*

Proof. ($\Longleftarrow$): If $y \in [T]$, then at all n, $y|_{n+1} <_{\mathrm{KB}} y|_n$.

($\Longrightarrow$): $(s_n)_{n\in\omega}$ is an infinite descending sequence in $<_{\mathrm{KB}}$. Then find N_0 s.t. $\forall n, m > N_0\,(s_n(0) = s_m(0))$. Similarly we can then find $N_1 > N_0$ s.t. $\forall n, m > N_1\,(s_n(1) = s_m(1))$, etc. We get $(N_i)_{i\in\omega}$, $(k_i)_{i\in\omega}$ such that $\forall n > N_i\,(s_n(i) = k_i)$. Then at each ℓ, $(k_0, k_1, \ldots, k_\ell) \in T$. ☐

Comment. If $A = p[T]$, $A \in \Sigma^1_1$, T recursive, then for all $x \in \omega^\omega$, $x \notin A$ iff $<_{\mathrm{KB}}$ wellorders T_x.

Note also: for $s \in \omega^{<\omega}$ with $\mathrm{lh}(s) = \ell$, whether $s \in T_x$ just depends on $x|_\ell$ (so T_x depends "continuously" on x).

Definition. We equip ω^ω with the product topology.

For $s \in \omega^{<\omega}$, $N_s = \{x \in \omega^\omega \mid s \subseteq x\}$.

Thus $\{N_s \mid s \in \omega^{<\omega}\}$ forms a basis of ω^ω.

Theorem. *Let A be $\mathbf{\Sigma^1_1}$. Let $T \subseteq \omega^{<\omega} \times \omega^{<\omega}$ be a tree with $A = p[T]$. Then either*

(I) $|A| \le \aleph_0$, *or*

(II) *A contains a homeomorphic copy of* 2^ω.

Proof. Let $T_0 = T$. For α an ordinal, let

$$T_{\alpha+1} = \{(s,t) \in T_\alpha \ : \exists s_0, s_1, t_0, t_1 \text{ s.t.}$$
$$\text{(i) } s_0, s_1 \supseteq s$$
$$\text{(ii) } t_0, t_1 \supseteq t$$
$$\text{(iii) } t_0(n) \neq t_1(n) \text{ some } n \text{ ("}t_0 \perp t_1\text{")}$$
$$\text{(iv) } (s_0, t_0), (s_1, t_1) \in T_\alpha\}$$

At λ a limit, $T_\lambda = \bigcap_{\alpha<\lambda} T_\alpha$.

Claim. $\exists \delta < \omega_1\,(T_\delta = T_{\delta+1})$.

Proof of claim. T_0 is countable, and $T_\alpha \subseteq T_\beta$ for $\alpha \ge \beta$. ☐

Case (I). $T_\delta = \emptyset$.

Fix $(y, x) \in [T]$. Let α be least such that $(y|_n, x|_n) \notin T_\alpha$, some n. Then $\alpha = \beta + 1$, some β. Fix m with $(y|_m, x|_m) \notin T_\alpha$. Let $s = y|_m, t = x|_m$. Then $\forall s_0, s_1 \supseteq s\,\forall t_0, t_1 \supseteq t$ with $(s_0, t_0), (s_1, t_1) \in T_\beta$, we must have (iii) fails, i.e. $t_0(n) = t_1(n)$ all $n < \mathrm{lh}(t_0), \mathrm{lh}(t_1)$. This means, for all $(s,t) \in T_\beta$, $s \supseteq y|_m$, $t \supseteq x|_m$, we must have $t \subseteq x$. This defines x from $(y|_m, t|_m)$ over T_β. Since $\omega^{<\omega} \times \omega^{<\omega}$ and δ both countable, there can be at most $\aleph_0$ many such x's.

Case (II). $T_\delta \neq \emptyset$.

Start with any $(s_0, t_0) \in T_\delta$. Since $(s_0, t_0) \in T_{\delta+1} = T_\delta$, choose $s_{\langle 0\rangle}, s_{\langle 1\rangle} \supsetneq s_0$ and $t_{\langle 0\rangle}, t_{\langle 1\rangle} \supsetneq t_1$, $(s_{\langle 0\rangle}, t_{\langle 0\rangle}), (s_{\langle 1\rangle}, t_{\langle 1\rangle}) \in T_\delta$, $t_{\langle 0\rangle} \perp t_{\langle 1\rangle}$.

Continue in this fashion; we get $(s_u, t_u)_{u\in 2^{<\omega}}$ such that:

(a) $s_u \subseteq s_{u'}$ and $t_u \subseteq t_{u'}$ for $u \subseteq u'$

(b) $t_{u^\frown 0} \perp t_{u^\frown 1}$ all $u \in 2^{<\omega}$.

Then define $f : 2^\omega \to [T]$ by

$$f(w) = \bigcup_{\ell \in \omega} t_{w|_\ell}.$$

Then f effects a homeomorphism between 2^ω and a (compact) subset of $[T]$. □

Theorem. *Let $T \subseteq \omega^{<\omega} \times \omega^{<\omega}$ be a tree, $A = p[T]$. Let $\mathcal{M}$ be a transitive model of the first 10 billion axioms of ZFC. Suppose $T \in \mathcal{M}$. Then either*

(I) *$A \subseteq \mathcal{M}$, or*
(II) *A contains a homeomorphic copy of 2^ω.*

Proof. Note in the last argument, $\mathcal{M}$ correctly calculates T_α, all α. □

22. November 15

Theorem. *Let $\mathcal{O} \subseteq \omega^\omega$ be open. Then $G_\mathcal{O}$ is determined.*

Proof. The game proceeds as follows:

I	n_0		n_2		...
II		n_1		n_3	...

I wins $G_\mathcal{O}$ if $(n_0, n_1, n_2, \dots) \in \mathcal{O}$.

Suppose **I** does not have a winning strategy. Then given n_0 as **I**'s first move, then let **II** respond with n_1 such that **I** does not have a winning strategy in the game from that position; i.e. so that **I** does not have a winning strategy in $G_{\{x \mid n_0 ^\frown n_1 ^\frown x \in \mathcal{O}\}}$.

In general, given $p \in \omega^\omega$ of odd length, let **II** respond with $n_{\mathrm{lh}(p)}$ so that **I** does not have a winning strategy in $G_{\{x \mid p ^\frown n_{\mathrm{lh}(p)} ^\frown x \in \mathcal{O}\}}$. In the end, we have $x = (n_0, n_1, \dots) \in \omega^\omega$.

I claim $x \notin \mathcal{O}$. Otherwise, there is ℓ with $N_{x|_\ell} \subseteq \mathcal{O}$. But then **I** definitely had a winning strategy at $(n_0, n_1, \dots, n_{\ell-1})$. □

Remark. $C \subseteq \omega^\omega$ is closed iff there is a tree $T \subseteq \omega^{<\omega}$ s.t. $[T] = C$.

(*Proof.* ($\Longrightarrow$) Given C, let $T = \{s \mid N_s \cap C \neq \emptyset,$ i.e. $N_s \not\subseteq \omega^\omega \setminus C\}$.

($\Longleftarrow$) [T] is clearly closed by definition.)

Definition. For κ an ordinal, $T \subseteq \kappa^{<\omega} \times \omega^{<\omega}$ is a **tree** if

(i) $(u, s) \in T \Longrightarrow \mathrm{lh}(u) = \mathrm{lh}(s)$
(ii) $(u, s) \in T, \ell < \mathrm{lh}(u) \Longrightarrow (u|_\ell, s|_\ell) \in T$.

We then let G_T be the following game:

I	α_0		α_1		α_2		...
II		n_0		n_1		n_2	...

I plays $\alpha_i \in \kappa$, **II** plays $n_i \in \omega$. **I** wins if at some ℓ,

$$((\alpha_0, \dots, \alpha_{\ell-1}), (n_0, \dots, n_{\ell-1})) \notin T.$$

Theorem. *For $T \subseteq \kappa^{<\omega} \times \omega^{<\omega}$ a tree, G_T is determined (i.e. one of the players has a winning strategy).*

Proof. Same argument as before. □

Theorem. *If $A \subseteq \omega^\omega$ is $\mathbf{\Sigma}^1_1$, then there is a tree $S \subseteq {\omega_1}^{<\omega} \times \omega^{<\omega}$ such that*

$$A^c = p[S] := \{x \in \omega^\omega \mid \exists f \in (\omega_1)^\omega\, \forall \ell\, ((f|_\ell, x|_\ell) \in S)\}.$$

Proof. Fix $T \subseteq \omega^{<\omega} \times \omega^{<\omega}$ a tree with $p[T] = A$. Then for $x \in \omega^\omega$, $x \in A^c$ iff $<_{\mathrm{KB}}$ wellorders T_x (last week).

Let $(s_n)_{n\in\omega}$ enumerate $\omega^{<\omega}$ with $\mathrm{lh}(s_n) \leq n$ all $n \in \omega$. Let

$$\begin{aligned} S = \{(u, s) \in {\omega_1}^{<\omega} \times \omega^{<\omega} \ :\ & \mathrm{lh}(u) = \mathrm{lh}(s), \text{ and for all } m, m' < \mathrm{lh}(u) \\ & \text{with } \mathrm{lh}(s_m) = k, \mathrm{lh}(s_{m'}) = k', \text{ if } (s_m, s|_k), (s_{m'}, s|_{k'}) \in T \\ & \text{and } s_m <_{\mathrm{KB}} s_{m'}, \text{ then } u(m) < u(m')\}. \end{aligned}$$

Suppose $(f, x) \in [S]$. (Want: T_x wellordered by $<_{\mathrm{KB}}$.) At each $s_m \in T_x$, let $\rho(s_m) = f(m)$.

If $s_m <_{\mathrm{KB}} s_{m'}$, $s_m, s_{m'} \in T_x$, with $\mathrm{lh}(s_m) = k$, $\mathrm{lh}(s_{m'}) = k'$, then $(s_m, x|_k), (s_{m'}, x|_{k'}) \in T$ (by definition of T_x) and so $f(m) < f(m')$ by definition of S. Hence $\rho(s_m) < \rho(s_{m'})$. So $(T_x, <_{\mathrm{KB}})$ is wellordered; therefore $[T_x] = \emptyset$, and $x \notin A = p[T]$.

Conversely, suppose $x \notin A$, so $(T_x, <_{\mathrm{KB}})$ is wellordered. Let $\rho : T_x \to \omega_1$ such that

$$s <_{\mathrm{KB}} s' \Longrightarrow \rho(s) < \rho(s').$$

Then given $m \in \omega$, if $s_m \in T_x$, then set $f(m) = \rho(s_m)$; if $s_m \notin T_x$, set $f(m) = 0$.

Suppose $m, m' \in \omega$, $\mathrm{lh}(s_m) = k$, $\mathrm{lh}(s_{m'}) = k'$, then $s_m \in T_x$ iff $(s_m, x|_k) \in T$. Hence $\forall \ell \in \omega$ if $m, m' < \ell$ and we let $u = f|_\ell$, $s = x|_\ell$, then $(s_m, s|_k), (s_{m'}, s_{k'}) \in T$ iff $s_m, s_{m'} \in T_x$. Hence if $(s_m, s|_k), (s_{m'}, s|_{k'}) \in T$, then $f(m) < f(m')$ iff $\rho(s_m) < \rho(s_{m'})$ iff $s_m <_{\mathrm{KB}} s_{m'}$; so $(f, x) \in [S]$. □

Theorem. *If $0^\#$ exists, $A \subseteq \omega^\omega$ is Σ^1_1, then G_A is determined.*

Idea. Fix $T \subseteq \omega^{<\omega} \times \omega^{<\omega}$, $T \in L$, $p[T] = A$. Define [a game with more information] G^*_T by

I	n_0		n_2		$\dots$
II		n_1, α_0		n_3, α_1	$\dots$

II wins iff at all ℓ

$$((n_0, \dots, n_{\ell-1}), (\alpha_0, \dots, \alpha_{\ell-1})) \in S$$

(S defined as above, but in L). This is determined, and if **II** wins G^*_T, then **II** wins G_A. If **I** wins, take a strategy in L, and make **II** play in the indiscernibles.

23. November 17

Theorem. *If $A \subseteq \omega^\omega$ is Σ^1_1 and $0^\#$ exists, then G_A is determined.*

Proof. Say $A = p[T]$, $T \subseteq \omega^{<\omega} \times \omega^{<\omega}$ a recursive tree. Let $(p_n)_{n\in\omega}$ enumerate the primes in increasing order. For $m = p_0^{n_0+1} p_1^{n_1+1} \dots p_\ell^{n_\ell+1}$, let $s_m = (n_0, n_1, \dots, n_\ell)$. For m *not* of that form, $s_m = \emptyset$.

Note. $T \in L$, $m \mapsto s_m \in L$, and $\mathrm{lh}(s_m) \leq m$ all m.

Let $\kappa = {\omega_1}^V$ $(> {\omega_1}^L!!)$

Let

$$\begin{aligned} S = \{(u, s) \in \kappa^{<\omega} \times \omega^{<\omega} \ :\ & \mathrm{lh}(u) = \mathrm{lh}(s) \text{ and } \forall m, m' < \mathrm{lh}(u)\ \mathrm{lh}(s_m) = k, \mathrm{lh}(s_{m'}) = k' \\ & \text{if } (s_m, s|_k), (s_{m'}, s|_{k'}) \in T \text{ and } s_m <_{\mathrm{KB}} s_{m'} \text{ then } u(m) < u(m')\}. \end{aligned}$$

Note. $S \in L$.

Let G_S^* be the following game:

$$\begin{array}{c|cccccccc} \mathbf{I} & n_0 & & n_2 & & n_4 & & & \dots \\ \hline \mathbf{II} & & n_1, \alpha_0 & & n_3, \alpha_1 & & n_5, \alpha_2 & & \dots \end{array}$$

where each $n_i \in \omega, \alpha_i \in \kappa$.

II loses if we ever have some $((\alpha_0, \dots, \alpha_\ell), (n_0, \dots, n_\ell)) \notin S$.

Note. This is determined (since if **I** does not have a winning strategy, **II** just plays to avoid slipping into a situation where **I** does).

In what follows, "**I** wins in L" means **I** has a winning strategy σ for the game in question, and $\sigma \in L$; similarly for **II**, V, etc.

Claim. *If* **II** *wins in* L*, then* **II** *wins in* V*.*

Proof of claim. Since **II** wins iff **II** has not lost at any finite position, and the finite positions are all in L. ☐

Claim. *If* **I** *wins in* L*, then* **I** *wins in* V*.*

Proof of claim. Let σ be a winning strategy for **I** in L. Let

$$S^\sigma = \{p \mid p \text{ is a play in } G_S^* \text{ with } \mathbf{II} \text{ to move } (\mathrm{lh}(p) \text{ is odd}) \\ \text{and } p \text{ is in accord with } \sigma, \text{ i.e. } \forall \ell < \mathrm{lh}(p), \ell \text{ even}, p(\ell) = \sigma(p|_\ell)\}.$$

Now let

$$S_0^\sigma = \{p \in S^\sigma \ : \ \mathbf{II} \text{ has already lost – i.e., fallen out of } S\}.$$

Given S_α^σ, let

$$S_{\alpha+1}^\sigma = \{p \in S^\sigma \ : \ \forall a \text{ with } p^\frown a \in G_S^*, \ p^\frown a^\frown \sigma(p^\frown a) \in S_a^\sigma\}.$$

For λ a limit,

$$S_\lambda^\sigma = \left\{ p \in S^\sigma \ : \ \forall a \text{ with } p^\frown a \in G_S^*, \ p^\frown a^\frown \sigma(p^\frown a) \in \bigcup_{\alpha<\lambda} S_\alpha^\sigma \right\}.$$

Eventually get to some $\delta \in \kappa^+$ such that

$$\bigcup_{\alpha\in\delta} S_\alpha^\sigma = \bigcup_{\alpha\in\mathrm{Ord}} S_\alpha^\sigma.$$

Note. If $\bigcup_{\alpha\in\delta} S_\alpha^\sigma \neq S^\sigma$, then **II** has a winning strategy – stay *outside* $\bigcup_{\alpha\in\delta} S_\alpha^\sigma$ (in L!!).

So: $\bigcup_{\alpha\in\delta} S_\alpha^\sigma = S^\sigma$. Suppose there was a winning run for **II** against σ, say

$$\begin{array}{c|cccccccc} \mathbf{I} & n_0 & & n_2 & & n_4 & & & \dots \\ \hline \mathbf{II} & & n_1, \alpha_0 & & n_3, \alpha_1 & & n_5, \alpha_2 & & \dots \end{array}.$$

Then at each ℓ, let β_ℓ be least such that for

$$p_\ell = \begin{array}{|ccccc} n_0 & & \dots & & n_{2\ell} \\ \hline & n_1, \alpha_0 & \dots & n_{2\ell-1}, \alpha_{\ell-1} & \end{array},$$

we have $p_\ell \in S_{\beta_\ell}^\sigma$. Since this is a winning run for **II**, β_ℓ is never 0. But then $\beta_{\ell+1} < \beta_\ell$ for all ℓ, a contradiction. ☐

[Comment: This (?) is basically the proof of Shoenfield absoluteness.]

Claim. *If* **II** *wins* G_S^*, *then* **II** *wins* G_A.

Proof of claim. Let σ win G_S^*. Then given n_0, let $\hat{\sigma}(n_0) = n_1$ where $\sigma(n_0) = (n_1, \alpha_0)$. Then $\hat{\sigma}(n_0, n_1, n_2) = n_3$ where $\sigma((n_0, (n_1, \alpha_0), n_2)) = (n_3, \alpha_1)$, etc.

But then, if $(n_0, n_1, \dots)$ is an infinite play according to $\hat{\sigma}$, then there is a sequence $(\alpha_0, \alpha_1, \dots)$ such that $((\alpha_0, \dots)(n_0, \dots)) \in [S]$. For $x = (n_0, n_1, \dots)$, $f = (\alpha_0, \alpha_1, \dots)$, f witnesses $(T_x, <_{\mathrm{KB}})$ is wellordered. This implies $x \notin A$, so that **II** wins G_A. ☐

Claim. *If* **I** *wins* G_S^*, *then* **I** *wins* G_A.

Proof of claim. Let $\sigma \in L$ be a winning strategy for **I** in G_S^*. We may assume there is a club $C \subseteq {\omega_2}^V$ of indiscernibles for L, $\kappa = {\omega_1}^V \in C$. Without loss of generality, σ is definable from κ in L. We define $\hat{\sigma}$ in G_A for **I**...

24. November 19

Recall: we had $A = p[T]$, $T \subseteq \omega^{<\omega} \times \omega^{<\omega}$ recursive. Want: G_A determined.

We put $\kappa = {\omega_1}^V$ and assume $0^\#$ exists, with $C \subseteq \kappa$ a club of indiscernibles for L. We defined the game G_S^* by

I	n_0		n_2		$\dots$
II		n_1, α_0		n_3, α_1	$\dots$

where $n_i \in \omega$, $\alpha_i \in \kappa$, $x = (n_0, n_1, \dots)$ and $f = (\alpha_0, \alpha_1, \dots)$. **II** is obligated to use f to witness wellfoundedness of $(T_x, <_{\mathrm{KB}})$ (it is a closed game for **II**).

G_S^* must be determined. We saw last time that **I** wins in V iff **I** wins in L. If **II** wins G_S^*, then **II** wins G_A. Now: let σ be a winning strategy in L for **I**. We will define (in V!) a derived strategy $\hat{\sigma}$ for **I** in G_A.

G_A:

n_0		n_2	
	n_1		n_3

G_S^*:

n_0		n_2
	n_1, α_0^1	

If $(s_0, (n_0, n_1)|_{\mathrm{lh}(s_0)}) \in T$ let $\alpha_0 \in C$. Let $n_2 = \sigma(n_0, (n_1, \alpha_0))$. Otherwise, let $\alpha_0 = 0$. (Note: $\sigma(n_0, (n_1, \alpha_0))$ does *not* depend on which $\alpha_0 \in C$ we choose, by indiscernibility.)

Now if $(s_1, (n_0, n_1, n_2)|_{\mathrm{lh}(s_1)}) \in T$, replace α_0 by α_0^1 in C, and then play α_1^1 as required. Then play $n_4 = \sigma(n_0, (n_1, \alpha_0^1), n_2, (n_3, \alpha_1^1))$, etc.

In general, given

n_0		n_2	$\dots$	n_{2k}	
	n_1		$\dots$		n_{2k+1}

a play according to $\hat{\sigma}$, we choose $(\alpha_i^k)_{i<k}$ such that

(a) if $(s_i, (n_0, \dots)|_{\mathrm{lh}(s_i)}) \notin T$, let $\alpha_i^k = 0$.

(b) for the rest, choose the α_i^k's in C so that $\alpha_i^k < \alpha_j^k$ iff $s_i <_{\mathrm{KB}} s_j$.

By indiscernibility, $\hat{\sigma}$ does not (then) depend on further specifics of how we chose the α_i^k's.

$$\hat{\sigma}(n_0, \dots, n_{2k+1}) = \sigma(n_0, (n_1, \alpha_0^k), n_2, (n_3, \alpha_1^k), \dots, (n_{2k+1}, \alpha_k^k)).$$

Claim. *If* $x = (n_0, n_1, \dots)$ *is an infinite play according to* $\hat{\sigma}$, *then* $x \in A$.

Proof of claim. Suppose for a contradiction that $(T_x, <_{\mathrm{KB}})$ is wellordered. Let

$$f : (T_x, <_{\mathrm{KB}}) \to C \cap \kappa$$

be order preserving ($|C|$ uncountable). Then at each i, let $\alpha_i = 0$ if $s_i \notin T_x$, $= f(s_i)$ if $s_i \in T_x$. Then

$$\begin{array}{|cccccc} n_0 & & n_2 & & \dots & \\ \hline & n_1, \alpha_0 & & n_3, \alpha_1 & \dots & \end{array}$$

is a run in G^*_S where **II** defeats σ, a contradiction. □

This concludes the proof that if **I** wins G^*_S, then **I** wins $G_A \dots$ □

thus completing the proof of the theorem. □

We now head towards

Theorem (Harrington)**.** *Σ^1_1 determinacy implies $0^\#$ exists.*

Preview of proof: we define a game

$$\begin{array}{c|ccccc|l} \mathbf{I} & n_0 & & n_1 & & \dots & x \in 2^\omega \\ \hline \mathbf{II} & & m_0 & & m_1 & \dots & y \in 2^\omega, \end{array}$$

I must play a code for a wellordering isomorphic to $(\alpha, \in)$. Then **II** must play a model of ZFC* $+ V = L$, which has L_α as an initial segment. By "boundedness," **I** cannot have a winning strategy. So: suppose **II** has a winning strategy coded by z. Will show: $\forall \alpha < \delta\, L_\delta[z] \models \mathrm{ZFC}^* \Longrightarrow \mathcal{P}(a)^L \subseteq L_\gamma[z]$. (This gives for λ a limit cardinal, $(\lambda^+)^L < (\lambda^+)^{L[z]}$, from which $0^\#$ follows.)

25. November 22

Last time we started to give a preview of the proof of

$$\Sigma^1_1 \text{ Determinacy} \Longrightarrow 0^\# \text{ exists.}$$

We will define a game

$$\begin{array}{c|ccccc|l} \mathbf{I} & n_0 & & n_1 & & \dots & x \\ \hline \mathbf{II} & & m_0 & & m_1 & \dots & y \end{array}$$

I must play $x \in 2^\omega$ with $\in_x$ a wellorder of ω. If so, then **II** must play some y with

(i) $M_y \models \mathrm{ZFC}^*$
(ii) $(\omega; \in_x) \cong$ to an initial segment of Ord^{M_y}
(iii) $M_y \models V = L$.

Today we show: **I** cannot have a winning strategy.

Definition. A relation R on X is **wellfounded** if for every non-empty $A \subseteq X$, $\exists a \in A$ s.t. $\forall d \in A\ (\neg bRa)$.

Theorem. *For R a relation on X, the following are equivalent:*

(i) *R is wellfounded;*
(ii) *$\exists \rho\colon X \to \delta$ some ordinal δ s.t. $\forall aRb\, (\rho(a) \in \rho(b))$*
(iii) *there does not exist $(a_n)_{n\in\omega}$ with $a_{n+1} R a_n$ all n.*

Proof. (i) $\implies$ (ii): Without loss of generality, R is transitive. At each $a \in X$, we set $X_a = \{b \mid bRa, \text{ or } a = b\}$. Let

$$A = \{a \in X \mid \exists \rho_a : X_a \to \delta_a \text{ some ordinal } \delta_a, \text{ with } b_1 R b_2 \implies \rho_a(b_1) \in \rho_a(b_2)\}$$

Claim. *$\exists \rho\colon A \to \delta$, some δ, such that $\rho(a) = \sup\{\rho(b) + 1 \ : \ bRa\}$.*

Proof of claim. At each $a \in A$, let $\alpha_a = \inf\{\rho_a(a) \ : \ \rho_a\colon X_a \to \delta_a \text{ is as above}\}$. Note: $\alpha_a = \sup\{\alpha_b + 1 \ : \ bRa\}$. Set $\rho(a) = \alpha_a$. □

Claim. *$A = X$.*

Proof of claim. Otherwise choose $a \in A^c$, $\forall b\,(bRa \implies b \in A)$. Then let $\rho^+\colon A \cup \{a\} \to \delta + 1$, $\rho^+|_A = \rho$, $\rho(a) = \sup\{\rho(\lambda) + 1 \ : \ bRa\}$. □

(ii) $\implies$ (iii): clear (since $\in$ is wellfounded).

(iii) $\implies$ (i): If $\neg$(i), say $A \subseteq X$, $A \neq \emptyset$, with no R-minimal element, choose $a_0 \in A_1$, then $a_{n+1} \in A$, $a_{n+1} R a_n$ (always possible, since a_n not R-minimal). □

Definition. For R a wellfounded relation on a set X, $a \in X$,

$$\text{rank}_R(a) := \inf\{\rho(a) \ : \ \rho\colon X \to \delta \text{ some ordinal } \delta,\ \forall b_1, b_2 \in X,\ (b_1 R b_2 \implies \rho(b_1) \in \rho(b_2))\}.$$

Then $\text{rank}(R) := \sup\{\text{rank}_R(a) + 1 \ : \ a \in X\}$.

Theorem (Kunen-Martin)**.** *If R is a $\mathbf{\Sigma}^1_1$ wellfounded relation on ω^ω, then* $\text{rank}(R) < \omega_1$.

Proof. Say $T \subseteq \omega^{<\omega} \times \omega^{<\omega} \times \omega^{<\omega}$ is a tree with $R = p[T]$. Let

$$X = \{((s_m)_{m \le n+1}, (u_m)_{\le n}) \in (\omega^{<\omega})^{<\omega} \times (\omega^{<\omega})^{<\omega} \ : \ n \in \omega \text{ and } \forall m \le n((u_m, s_{m+1}, s_m) \in T, \ \text{lh}(u_m) = \text{lh}(s_{m+1}) = \text{lh}(s_m) = n)\}.$$

Let R^* be defined on X by

$$((t_m)_{m \le k+1}, (v_m)_{m \le k}) \ R^* \ ((s_m)_{m \le n+1}, (u_m)_{m \le n})$$

if $k > n$ and $t_m \supseteq s_m$ all $m \le n+1$, $v_m \supseteq u_m$ all $m \le n$.

Claim. *R^* is wellfounded.*

Proof of claim. Suppose instead

$$\big((s^i_m)_{m \le n(i)+1}, (u^i_m)_{m \le n(i)}\big)_{i \in \omega}$$

is an R^* infinite descending chain.

Note. $\forall i \le j$, $m \le n(i)$, $s^i_m \subsetneq s^j_m$, $u^i_m \subsetneq u^j_m$.

Let $x_m = \bigcup_{i \in \omega} s^i_m$, $y_m = \bigcup_{i \in \omega} u^i_m$. Then $(y_m, x_{m+1}, x_m) \in [T]$ all m, contradicting wellfoundedness of R. □

Since X is countable, $\text{rank}(R^*) < \omega_1$. Let

$$\hat{X} = \{((x_m)_{m \le n+1}, (y_m)_{m \le n}) \ : \ n \in \omega, \text{ s.t. } \forall m \le n\,(y_m, x_{m+1}, x_m) \in [T]\}.$$

Define $((x_m)_{m \le n+1}, (y_m)_{m \le n}) \ \hat{R} \ ((\overline{x}_m)_{m \le \overline{n}+1}, (\overline{y}_m)_{m \le \overline{n}})$ if $n > \overline{n}$ and $x_m = \overline{x}_m$ all $m \le \overline{n} + 1$, $y_m = \overline{y}_m$ all $m \le \overline{n}$.

Note. For $x \in \omega^\omega$, $\text{rank}_R(x) = \text{rank}_{R^*}((x), \emptyset)$.

It suffices to see that $\mathrm{rank}_{\hat{R}} \le \mathrm{rank}_{R^*}$. But given $((x_m)_{m\le n+1}, (y_m)_{m\le n}) \in \hat{X}$, let

$$\theta((x_m)_{m\le n+1}, (y_m)_{m\le n}) = ((x_m|_n)_{m\le n+1}, (y_m|_n)_{m\le n}).$$

So $\theta\colon \hat{X} \to X$ with $a\hat{R}b \Longrightarrow \theta(a)R\theta(b)$; so $\mathrm{rank}(\hat{R}) \le \mathrm{rank}(R)$. □

Definition. $\mathrm{WO} = \{x \in 2^\omega \mid \in_x \text{ wellorders } \omega\}$, and for $x \in \mathrm{WO}$, $\|x\| = \text{unique } \alpha \in \mathrm{Ord}$ s.t. $(\alpha; \in) \cong (\omega; \in_x)$.

Corollary. *If $A \subseteq \mathrm{WO}$ is $\mathbf{\Sigma}^1_1$, then $\exists \delta \in \omega_1$ such that $\|x\| \le \delta$ all $x \in A$.*

26. November 24

This property of A is sometimes called "boundedness."

Proof. Set $(x,m)R(y,n)$ iff
 (i) $x = y \in A$
 (ii) $m \in_x n$.

This is wellfounded, and the δ from Kunen-Martin bounds $\{\|x\| : x \in A\}$. □

Recall we defined the game

$$\begin{array}{c|cccccc} \mathbf{I} & n_0 & & n_1 & & \dots & x \\ \hline \mathbf{II} & & m_0 & & m_1 & \dots & y \end{array} \qquad n_i, m_i \in \{0,1\}.$$

If $x \notin \mathrm{WO}$, **II** wins. If $x \in \mathrm{WO}$, **II** wins if $(\omega; \in_x)$ is isomorphic to an initial segment of Ord^{M_y} (where $M_x := (\omega; \in_x)$), and $M_y \models \mathrm{ZFC}^* \wedge V = L$.

Claim. **I** *does not have a winning strategy for G_L.*

Proof of claim. Let $\sigma\colon \omega^{<\omega} \to \omega$ be a winning strategy for **I**. Let

$$A = \{x \in 2^\omega \mid \exists y \in 2^\omega \text{ s.t. } x(0) = \sigma(\emptyset),\ x(n+1) = \sigma(x(0), y(0), \dots, y(n))\}.$$

Then $A \subseteq \mathrm{WO}$, $A \in \mathbf{\Sigma}^1_1$. Therefore there exists $\delta < \omega_1$ such that $\|x\| \le \delta$ for all $x \in A$.

Let $\gamma > \delta$ be countable, $L_\gamma \models \mathrm{ZFC}^*$. Choose $\pi : \omega \to L_\gamma$ a bijection. Define $y \in 2^\omega$, $y(2^n 3^m) = 1$ if $\pi(n) \in \pi(m)$, and $y(k) = 0$ in all other cases. Then $(\omega; \in_y) \models \mathrm{ZFC}^* + V = L$ ($(\omega; \in_y) \cong L_\gamma$). And $\|x\| \le \delta < \gamma$ for all $x \in A$; in particular this is true for x the response to y indicated by σ. So: **II** wins with y. □

Fix $\sigma : \omega^{<\omega} \to \omega$ a winning strategy for **II**. Let $z \in 2^\omega$, $z(\ulcorner s, n \urcorner) = 1$ if $\sigma(s) = n$; $z(k) = 0$ all other k (so z "encodes" σ).

It suffices to show: if $L_\alpha[z] \models \mathrm{ZFC}^*$ and $\delta < \alpha$, then $\mathcal{P}(\delta)^L \in L_\alpha[z]$. (Fix α with $\aleph_\omega < \alpha < {\aleph_\omega}^+$, $L_\alpha[z] \models \mathrm{ZFC}^*$. Then $\mathcal{P}(\aleph_\omega)^L \in L_\alpha[z]$; therefore $((\aleph_\omega)^+)^L < (\aleph_\omega)^+$, so that $0^\#$ exists, by Jensen's covering lemma.)

Definition. $(A; <)$ is a **pseudo-wellordering** if there exists $M \models \mathrm{ZFC}^*$ such that

$$(\mathrm{Ord}^M; \in^M) \cong (A; <).$$

Lemma. *If $(A; <)$ is a pseudo-wellordering then either*
 (i) *$<$ is a wellorder*
 (ii) *there is an ordinal α such that $(A; <) \cong ((1 + \mathbb{Q}) \times \alpha; <_{\mathrm{lex}})$.*

Here $1+\mathbb{Q}$ is the order type of a first element followed by the rationals; it is isomorphic to $([0,1)\cap\mathbb{Q};<)$. (The order type of the rationals is also sometimes called η.)

Let $S_\infty = \{\tau\colon \omega\to\omega \mid \tau \text{ is a permutation}\}$. This is a G_δ subset of ω^ω. (It is a Polish space.) Given $x\in 2^\omega$ with $x(k)=0$ all k not of the form 2^m3^n and $\tau\in S_\infty$, $\tau\cdot x\in 2^\omega$ is defined by

$$\tau\cdot x(2^n3^m) = x(2^{\tau^{-1}(n)}3^{\tau^{-1}(m)}).$$

Fact. For $A\subseteq 2^\omega\times 2^\omega$, $A\in\mathbf{\Sigma}^1_1$,

$$\{(x,y)\in 2^\omega\times 2^\omega \mid \forall^*\tau\in S_\infty((\tau\cdot x,y)\in A)\}$$

is again $\mathbf{\Sigma}^1_1$. Here $\forall^*\tau\in S_\infty(\varphi(\tau,\dots))$ means there is a relatively comeager $B\subseteq S_\infty$ such that $\forall\tau\in B(\varphi(\tau,\dots))$.

Consider the set of $a\in 2^\omega$ such that $\exists x\in 2^\omega$ with $(\omega;\in_x)$ a pseudo-wellorder, and $\forall^*\tau\in S_\infty$

$$\left(\begin{array}{ll} & \sigma*(\tau\cdot x) \longleftarrow \ \sigma\text{'s response to } \mathbf{I} \text{ playing } \tau\cdot x \\ a\in\mathcal{P}(\omega) & \end{array}\right)$$

This is $\Sigma^1_1(z)$; therefore $\subseteq L_\alpha[z]$ if $L_\alpha[z]\models \mathrm{ZFC}^*$ does *not* contain a perfect set ($|C|\le\aleph_1$).

27. November 29

Definition. For X a Polish space (i.e. a separable, completely metrizable space) $B\subseteq X$ is **meager** if it is included in a countable union of closed nowhere dense sets.

(Baire Category Theorem: No non-empty open $U\subseteq X$ is meager.)

B is **comeager** if B^c is meager.

$B\subseteq X$ has the **property of Baire** if there is an open $\mathcal{O}$ such that $B\,\Delta\,\mathcal{O} := B\setminus\mathcal{O}\cup\mathcal{O}\setminus B$ is meager.

Fact. The sets with the property of Baire form a σ-algebra. (Exercise.)

Definition. For $\varphi(\dots)$ some property, X Polish, we write $\forall^*x\in X\,\varphi(x)$ if $\{x\in X\mid\varphi(x)\}$ is comeager, and $\exists^*x\in X\,\varphi(x)$ if that set is non-meager.

Definition. $S_\infty := \{\sigma:\omega\to\omega \mid \sigma \text{ is a permutation (i.e. a bijection)}\}$.

For $\vec{a}\in\omega^{<\omega}$, let $W_{\vec{a}} := \{\sigma\in S_\infty \mid \sigma(i)=a_i \text{ all } i<\mathrm{lh}(\vec{a})\}$.

We take $\{W_{\vec{a}}\mid \vec{a}\in\omega^{<\omega}\}$ as the basis for the topology on S_∞.

Lemma. *S_∞ is a Polish space.*

Proof. For $\sigma,\tau\in S_\infty$, $\sigma\ne\tau$, let $d_0(\sigma,\tau)=2^{-\delta(\sigma,\tau)}$, where $\delta(\sigma,\tau)=$ least n such that $\sigma(n)\ne\tau(n)$. Then let

$$d(\sigma,\tau)=d_0(\sigma,\tau)+d_0(\sigma^{-1},\tau^{-1}).$$

□

Theorem. *If $A\subseteq S_\infty$ is $\mathbf{\Sigma}^1_1$, then A has the property of Baire.*

Proof. Choose $T\subseteq\omega^{<\omega}\times\omega^{<\omega}$ a tree such that

$$p[T]=\{\sigma\in S_\infty\mid \exists y\in\omega^\omega\,(\forall\ell\in\omega\,((\sigma|_\ell,y|_\ell)\in T))\}=A.$$

At each $s\in\omega^{<\omega}$, let $A_s=\{\sigma\in S_\infty\mid \exists y\supseteq s\,\forall\ell\,((\sigma|_\ell,y|_\ell)\in T)\}$.

Note. $A_\emptyset = A$, $A_s = \bigcup_{n\in\omega} A_{s^\frown n}$.

Let $B_s = S_\infty \setminus \bigcup\{\mathcal{O} \mid \mathcal{O}$ basic open, $A_s \cap \mathcal{O}$ meager$\}$. Note: each B_s is closed, $B_s \subseteq \overline{A_s}$.
Let $C_s = \bigcap_{\ell\le \mathrm{lh}(s)} B_{s|_\ell}$.

Claim. *For each $s \in \omega^{<\omega}$, $C_s \setminus \bigcup_{n\in\omega} C_{s^\frown n}$ is meager.*

Proof of claim. Suppose not. Note: Each $C_s \setminus \bigcup_{n\in\omega} C_{s^\frown n}$ is Borel. Therefore it has the property of Baire.

Suppose $C_s \setminus \bigcup_{n\in\omega} C_{s^\frown n}$ is not meager. Then we can find $\mathcal{O}$ basic open, non-empty, such that $(C_s \setminus \bigcup_{n\in\omega} C_{s^\frown n}) \cap \mathcal{O}$ is relatively comeager. So $C_s \cap \mathcal{O}$ is comeager in $\mathcal{O}$ (i.e. $\mathcal{O} \setminus C_s$ is meager) but $C_{s^\frown n} \cap \mathcal{O}$ is meager all $n \in \omega$. Hence, $A_{s^\frown n} \cap \mathcal{O}$ is meager at all $n \in \omega$. Hence $A_s = \bigcup_{n\in\omega} A_{s^\frown n}$ in $\mathcal{O}$. Therefore by definition, $B_s \cap \mathcal{O} = \emptyset$, and $C_s \cap \mathcal{O} = \emptyset$, a contradiction. □

Let $M = \bigcup_{s\in\omega^{<\omega}} (C_s \setminus \bigcup_{n\in\omega} C_{s^\frown n})$. M is meager.

Claim. *$A_\emptyset \,\Delta\, C_\emptyset$ is meager.*

Proof of claim. By definition of $C_\emptyset = B_\emptyset$, $A_\emptyset \setminus C_\emptyset = A_\emptyset \setminus B_\emptyset$ is meager.

Conversely, suppose $\sigma \in C_\emptyset$; we will show: $\sigma \notin A_\emptyset \implies \sigma \in M$. It suffices to show the contrapositive, $\sigma \notin M \implies \sigma \in A_\emptyset$.

Now, given $\sigma \notin M$, we may successively find $\emptyset = s_0 \subsetneq s_1 \subsetneq s_2 \subsetneq \dots$ such that each $s_i \in \omega^{<\omega}$, $\mathrm{lh}(s_i) = i$, and $\sigma \in C_{s_i}$ (since $\sigma \notin M$). Let $y = \bigcup_{i\in\omega} s_i$. $y \in \omega^\omega$. $C_{s_i} \subseteq \overline{A_{s_i}}$ at each i. So choose $y_i \in \omega^\omega$, $\sigma_i \in S_\infty$, such that $d(\sigma_i, \sigma) < 2^{-i}$, $(\sigma_i, y_i) \in [T]$. Then $\sigma_i \to \sigma$, $y_i \to y$, and $[T] \cap S_\infty \times \omega^\omega$ is closed in $S_\infty \times \omega^\omega$, hence $(\sigma, y) \in [T]$. □

Since $C_\emptyset$ has the property of Baire, fixed U open and N meager such that $C_\emptyset \,\Delta\, U \subseteq N$; therefore $A_\emptyset \,\Delta\, U \subseteq (A_\emptyset \,\Delta\, C_\emptyset) \cup (C_\emptyset \,\Delta\, U) \subseteq M \cup N$. □

Next time. Suppose $A \subseteq \omega^\omega \times S_\infty$ is $\mathbf{\Sigma}^1_1$, $A = p[T]$. Then for $x \in \omega^\omega$, A_x is comeager in S_∞ iff $\exists f\colon Z := \{\vec{a} \in \omega^{<\omega} \mid \vec{a} \text{ is } 1-1\} \to \omega^{<\omega}$ s.t.

(i) f non-decreasing
(ii) $\forall \vec{a} \in Z\, \forall \ell \in \omega\, \exists \vec{b} \in Z, \vec{a} \subseteq \vec{b}\, (f(\vec{b})$ has length $\ge \ell)$
(iii) $\forall \vec{a} \in \omega^{<\omega}\, \forall \ell \le \mathrm{lh}(\vec{a}), \mathrm{lh}(f(\vec{a}))\, (\vec{a}|_\ell, f(\vec{a})|_\ell) \in T$.

28. December 1

Theorem. *Let $A \subseteq S_\infty$, $A = p[T]$, $T \subseteq \omega^{<\omega} \times \omega^{<\omega}$ a tree. Then A is comeager in S_∞ iff $\exists (s_{\vec{a}})_{\vec{a}\in\omega^{<\omega},1-1}$ such that*

(a) *each $s_{\vec{a}} \in \omega^{<\omega}$*
(b) *$s_{\vec{a}} \subseteq s_{\vec{b}}$ for $\vec{a} \subseteq \vec{b}$*
(c) *$\forall \vec{a}\, \forall \ell \le \mathrm{lh}(\vec{a}), \mathrm{lh}(s_{\vec{a}})\, (\vec{a}|_\ell, s_{\vec{a}}|_\ell) \in T$*
(d) *for $\vec{a} \in \omega^{<\omega}$, $1-1$, all $\ell \in \omega$, $\exists \vec{b} \in \omega^{<\omega}$, $1-1$, $\vec{a} \subseteq \vec{b}$, $\mathrm{lh}(s_{\vec{b}}) \ge \ell$.*
(e) *each $W_{\vec{a}} \cap A_{s_{\vec{a}}}$ is comeager in $W_{\vec{a}}$.*

As in the proof from last time, we will let $A_s = \{x \in \omega^\omega \mid \exists y \supset s((x,y) \in [T])\}$. [Note A_s depends on T, not just on A and s.]

Proof. ($\Longleftarrow$): At each ℓ, let $\mathcal{O}_\ell = \bigcup\{W_{\vec{a}} \ : \ \mathrm{lh}(s_{\vec{a}}) \geq \ell\}$. This is open dense, by (d). Let $D = \bigcap_\ell \mathcal{O}_\ell$. This is dense G_δ, therefore comeager. Then given $\sigma \in D$, let $y = \bigcup_{n\in\omega} S_{\sigma|_n}$. By (a) & (b), and $\sigma \in D$, $y \in \omega^\omega$. By (c), $\forall \ell(\sigma_\ell, y|_\ell) \in T$.

($\Longrightarrow$): By induction on $\mathrm{lh}(\vec{a})$, we chose $s_{\vec{a}} \in \omega^{\leq \mathrm{lh}(\vec{a})}$ s.t. $s_{\vec{a}}$ is as long as possible subject to (a), (b), (c), (e). Given $s_{\vec{a}}$ with $\mathrm{lh}(s_{\vec{a}}) \leq \mathrm{lh}(\vec{a})$, (a), (b), (c), (e), and given $\vec{a}^\frown k$ $1-1$, choose $s_{\vec{b}} \supseteq s_{\vec{a}}$ with $\mathrm{lh}(s_{\vec{k}}) \leq \ell + 1$ as long as possible subject to (a), (c), (e).

We need to verify that condition (d) takes care of itself. *But*, given $s = s_{\vec{a}}$, $A_s = \bigcup_{n\in\omega} A_{s^\frown n}$; so there exists $\vec{b} \supset \vec{a}$, $n \in \omega$, $A_{s^\frown n} \cap W_{\vec{\beta}}$ is comeager in $\vec{b}$. □

Theorem. *If $B \subseteq \omega^\omega \times S_\infty$ is $\mathbf{\Sigma}^1_1$. Then $X := \{x \in \omega^\omega \mid B_x$ is comeager in $S_\infty\}$ is $\mathbf{\Sigma}^1_1$.*

Proof. Let $S \subseteq \omega^{<\omega} \times \omega^{<\omega} \times \omega^{<\omega}$ be a tree with $p[S] = B$. By the proof of the last theorem, $x \in X$ iff $\exists (s_{\vec{a}})_{\vec{a}\in\omega^{<\omega}, 1-1}$ such that

(a) each $s_{\vec{a}} \in \omega^{<\omega}$
(b) $s_{\vec{a}} \subseteq s_{\vec{b}}$ when $\vec{a} \subseteq \vec{b}$
(c) $\forall \vec{a}, 1-1, \forall \ell \leq \mathrm{lh}(\vec{a}), \mathrm{lh}(s_{\vec{a}}), (x|\ell, \vec{a}|_\ell, s_{\vec{a}}|_\ell) \in S$ iff $(\vec{a}|_\ell, s_{\vec{a}}|_\ell) \in S_x$
(d) for $\vec{a} \in \omega^{<\omega}$, $1-1$, $\ell \in \omega$, $\exists \vec{b} \supseteq \vec{a}$, $1-1$, $\mathrm{lh}(s_{\vec{b}}) \geq \ell$.

Note then that the set of $(x, y) \in \omega^\omega \times \omega^\omega$ such that if we define $s_{\vec{a}} = y(\ulcorner \vec{a} \urcorner)$, all $\vec{a} \in \omega^{<\omega}$, $1-1$, then we obtain Y is $\mathbf{\Sigma}^1_1$ (in fact, $\mathbf{\Delta}^1_1$), since $\mathbf{\Sigma}^1_1$ is closed under countable unions and intersections.

$X = p[Y]$, and thus it suffices to "recall" that the projection of a $\mathbf{\Sigma}^1_1$ set is $\mathbf{\Sigma}^1_1$. □

Proposition. *If $A \subseteq (\omega^\omega)^{n+1}$ is $\mathbf{\Sigma}^1_1$, then so is $\{\vec{x} \mid \exists y\, (\vec{x}, y) \in A\}$.*

Proof. Take $n = 1$ for notational simplicity. Say $A = p[T]$, $T \subseteq (\omega^{<\omega})^3$ a tree. Then let

$$S = \{(u, v) \ : \forall k \text{ with } 2^k, 3^k < \mathrm{lh}(u) = \mathrm{lh}(v)$$
$$(u|_k, (v(2), v(2^2), \ldots, v(2^k)), (v(3), v(3^2), \ldots, v(3^k))) \in T\}.$$

Then $pp[T] = p[S]$. □

Now, let's return to the game

$$G_L \quad \begin{array}{c|cc} \mathbf{I} & & x \\ \hline \mathbf{II} & & y \end{array}$$

II wins if $x \notin \mathrm{WO}$, $M_y \models \mathrm{ZFC}^* \wedge V = L$. If $x \in \mathrm{WO}$, **II** wins if $(\omega; \in_x)$ is isomorphic to an initial segment of Ord^{M_y}. Given $x \in \omega^\omega$, $\tau * x = y$, the response by **II** to x. Only **II** can win. It suffices to show, if τ is a winning strategy for **II**, then $0^\#$ exists.

For that, if $L_\alpha[z] \models \mathrm{ZFC}^*$, $\alpha < \omega_1$, then $\forall \beta < \alpha\, (\mathcal{P}(\beta)^L \subseteq L_\alpha[T])$.

First for $\beta = \omega$. Let

$$A_L = \{x \in 2^\omega \mid \exists b \in \mathrm{LO}, (\omega; \in_b) \cong \mathrm{Ord}^{\mathcal{M}}, \mathcal{M} \text{ a countable } \omega\text{-model of } \mathrm{ZFC}^*$$
$$\forall^* \sigma \in S_\infty\, \tau * (\sigma \cdot b) = y, \text{ has } a \in \mathcal{P}(\omega)^{M_y}, M_y \text{ an } \omega\text{-model}\}.$$

By the last theorem, A_L is $\mathbf{\Sigma}^1_1$.

29. December 3

Theorem. *If $A \subset \omega^\omega \times S_\infty$, $A \in \mathbf{\Sigma}^1_1$, then $\{x \in \omega^\omega \mid \forall^* \sigma \in S_\infty\, ((x,\sigma) \in A)\}$ is $\mathbf{\Sigma}^1_1$.*
Similarly $\{x \in \omega^\omega \mid \exists^* \sigma \in S_\infty\, ((x,\sigma) \in A)\}$ is $\mathbf{\Sigma}^1_1$.

In fact, the proof showed that is $A \in \Sigma^1_1(z)$ (some $z \in 2^\omega$), then these two sets are also $\Sigma^1_1(z)$.

$$G_L \quad \begin{array}{c|cc} \mathbf{I} & & x \\ \hline \mathbf{II} & & y \end{array}$$

II wins if either (a) $x \notin \mathrm{WO}$, or (b) M_y is an ω-model of ZFC*, $V = L$, $(\omega; \in_x) \cong$ an initial segment of Ord^{M_y}.

We have previously seen **I** cannot have a winning strategy (boundedness). So let $\tau : \omega^{<\omega} \to \omega$ be a winning strategy for **II**. Want: If $\alpha < \delta < \omega_1$, and $L_\delta[\tau] \models$ ZFC*, then $\mathcal{P}(\alpha)^L \subseteq L_\delta[\tau]$.

Notation. For $x \in \omega^\omega$,

$$\begin{aligned} \tau * x &= \mathbf{II}\text{'s response to } x \text{ according to } \tau \\ &= (\tau(\langle x(0)\rangle), \tau(\langle x(0), x(1)\rangle), \tau(\langle x(0), x(1), x(2)\rangle), \dots) \end{aligned}$$

Notation.

$$\begin{aligned} \mathrm{LO} = \{x \in 2^\omega\ &: x(k) = 0 \text{ unless } k \text{ has the form } k = 2^n 3^m \\ &\text{and } \in_x = \{(n,m)\ :\ x(2^n 3^m) = 1\} \text{ linearly orders } \omega\}. \end{aligned}$$

For $x \in \mathrm{LO}$, $\sigma \in S_\infty$, let $\sigma \cdot x \in \mathrm{LO}$ be defined by

$$\sigma \cdot x(2^n 3^m) = x(2^{\sigma^{-1}(n)} 3^{\sigma^{-1}(m)}).$$

Fact. For $x_1, x_2 \in \mathrm{LO}$, $(\omega; \in_{x_1}) \cong (\omega; \in_{x_2})$ iff $\exists \sigma \in S_\infty\, (\sigma \cdot x_1 = x_2)$.

Lemma. *Let M be an illfounded countable ω-model of* ZFC**. Then $\mathrm{Ord}^M \cong (1+\eta)\alpha$ for some countable ordinal α.*

(here $\eta =$order type of $(\mathbb{Q}, <)$.)

Proof. Fix M. Let $\alpha =$(order type) of the wellfounded part of the ordinals of M. For $a, b \in \mathrm{Ord}^M$, set $a \sim b$ if $\exists \beta < \alpha$ such that $M \models$"$a + \beta = b \vee b + \beta = a$".

Suppose $a \not\sim b$, $M \models a < b$. Let $A = \{d \in \mathrm{Ord}^M\ :\ M \models a < a + d < a + d + d < b\}$. $A \supseteq \alpha$. But α is *not* definable in M. Therefore $\exists d \in A$, d in the illfounded part of M ($\Longrightarrow \forall \beta < \alpha\, M \models$ "$\beta < d$"). Then let $c = a + d$. □

Definition. Let

$$\begin{aligned} A_L = \{a \in \mathcal{P}(\omega)\ &: \exists \text{ a countable illfounded } \omega\text{-model of ZFC}^*, M, \\ &\exists x \in \mathrm{LO}, \mathrm{Ord}^M \cong (\omega; \in_x), \text{ and } \forall^* \sigma \in S_\infty\, (a \in \mathcal{P}(\omega)^{M_{\tau * (\sigma \cdot x)}})\}. \end{aligned}$$

Note. The first condition is $\Sigma^1_1(a, \tau)$; the second is $\Sigma^1_1(a, \tau, M)$; and the third is $\Delta^1_1(a, \tau, x)$ or $\Delta^1_1(a, \tau, \sigma, x)$ (?) so A_L is $\Sigma^1_1(\tau)$.

Note also. $\mathcal{P}(\omega)^L \subseteq A_L$ (Given $a \in \mathcal{P}(\omega)^L$, choose $\alpha < \omega_1$, $L_\alpha \models$ ZFC*, $a \in L_\alpha$. By the boundedness theorem, we can find $w \in 2^\omega$, $M_w = (\omega; \in_w) \models$ ZFC*, $\alpha \cong$ an initial segment of Ord^{M_w}, M_w illfounded. *Then* for any x $(\sigma \cdot x!)$ in LO with $(\omega; \in_x) \cong \mathrm{Ord}^{M_w}$, we have $\alpha \in$ wellfounded part of $M_{\tau * x}$ s.t. $a \in M_{\tau * x}$.)

Theorem. $|A_L| \leq \aleph_0$.

Proof. For $w \in 2^\omega$ as above ($M_w \models \mathrm{ZFC}^*$, illfounded ω-model), $x \in \mathrm{LO}$, $(\omega; \in_x) \cong \mathrm{Ord}^{M_w}$, let $A_{w,x} = \{a \in \mathcal{P}(\omega) \ : \ \forall^* \sigma \in S_\infty \, (a \in \mathcal{P}(\omega)^{M_{\tau * (\sigma \cdot x)}})\}$.

Note. $(\omega; \in_{x_1}) \cong (\omega; \in_{x_2})$ implies $A_{w,x_1} = A_{w,x_2}$. Then let $A_w = A_{w,x}$ for any (all!) $x \in \mathrm{LO}$ as above.

Note also. A_ω *only* depends on Ord^{M_ω} up to isomorphism.

Claim. *Each such A_ω is countable.*

30. December 6

□