SOLUTIONS TO UCLA LOGIC QUALIFYING EXAMS

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1. Fall 2001/Winter 2002

Problem 1.1. *Let $(\varphi_e)_{e\in\mathbb{N}}$ be some canonical listing of the partial recursive functions. Call a total function f "large" if $f(e) > \varphi_e(0)$ for all e for which $\varphi_e(0)$ is defined. Show that if f is large then it has degree at least $\mathbf{0}'$.*

Solution. Let f be large. Recall $\mathbf{0}'$ is the degree of the halting problem. So it will be sufficient to show that the function

$$h(x) = \begin{cases} 1 & \text{if } \varphi_x(x) \text{ converges} \\ 0 & \text{if } \varphi_x(x) \text{ diverges} \end{cases}$$

is Turing reducible to f.

Let $\tau(x, z)$ be the partial recursive function defined by

$$\tau(x, z) = \mu y\, T_1(x, x, y).$$

Let $g(x) = S^1_1(\hat{\tau}, x)$. Then g is total recursive. Define ψ by

$$\psi(x) = \begin{cases} 1 & \text{if } (\exists y < f(g(x)))\, T_1(x, x, y) \\ 0 & \text{otherwise.} \end{cases}$$

Then $\psi(x)$ is a total function defined by recursion from f, so ψ is Turing reducible to f. We claim $\psi(x) = h(x)$ for all x.

Suppose $\psi(x) = 1$. Then for some $y < f(g(x))$, we have $T_1(x, x, y)$. It follows that $\varphi_x(x)$ converges, so $h(x) = 1$.

Suppose $h(x) = 1$; then for some y, $T_1(x, x, y)$ holds. Then $\tau(x, 0)$ converges, and we have $T_1(x, x, \tau(x, 0))$. To show $\psi(x) = 1$, it only remains to show $\tau(x, 0) < f(g(x))$: we have

$$\tau(x, 0) = \varphi_{S^1_1(\hat{\tau},x)}(0) = \varphi_{g(x)}(0) < f(g(x)),$$

since f is large. This completes the proof.

Problem 1.2. *Let Σ be a set of first-order sentences and let $\mathfrak{M}$ be a structure. Show that if every finitely generated substructure of $\mathfrak{M}$ has an extension to a model of Σ then so does $\mathfrak{M}$.*

Solution. Recall that any model of Diagram($\mathfrak{M}$) is isomorphic to an extension of $\mathfrak{M}$. So any model of Diagram($\mathfrak{M}$) $\cup\, \Sigma$ is isomorphic to an extension of $\mathfrak{M}$ that models Σ. We need to show that Diagram($\mathfrak{M}$) $\cup\, \Sigma$ is satisfiable.

Let T be a finite subset of Diagram($\mathfrak{M}$)$\cup\Sigma$. Then there is a finite $A \subset M$ so that all constants that appear in T are from A. By hypothesis, the substructure of $\mathfrak{M}$ generated by A has an extension to a model of Σ. Under the obvious interpretation of the language, this extension is a model of T.

We have shown Diagram($\mathfrak{M}$) $\cup\, \Sigma$ is finitely satisfiable; by compactness, it is satisfiable, as needed.

2. Fall 2002

Problem 2.1. *As usual, for each number $e \in \mathbb{N}$,*

$$W_e = \{x \mid \phi_e(x)\downarrow\},$$

where $\phi_0, \phi_1, \ldots$ is a standard enumeration of all recursive partial functions.

(1a) *Prove that the set*

$$A = \{e \mid W_e \text{ is finite }\}$$

is in $\Sigma_2 \setminus \Pi_2$.

(1b) *Classify in the arithmetical hierarchy the set*

$$B = \{e \mid W_e \text{ is finite and has an even number of elements}\};$$

i.e., find some n such that $B \in \Sigma_n \setminus \Pi_n$ or $B \in \Pi_n \setminus \Sigma_n$.

Solution. (1a) W_e is finite iff there exists some z so that $x > z$ implies $\phi_e(x)\uparrow$. That is,

$$e \in A \iff (\exists z)(\forall x)(\forall y)\, x > z \rightarrow \neg T_1(e, x, y).$$

So A is in Σ_2. To show it is not in Π_2, we show it is Σ_2-complete. Let C be an arbitrary Σ_2 set. Then for some recursive relation $P(x, y, z)$,

$$C = \{x \mid (\exists y)(\forall z)P(x, y, z)\}.$$

Define the partial recursive function $g(x, u)$ by

$$g(x, u) = \mu w[(\forall y < u)(\exists z < w)\neg P(x, y, z)],$$

so that $g(x, u)\uparrow$ iff $(\exists y < u)(\forall z)P(x, y, z)$. Put $f(x) = S^1_1(\hat{g}, x)$. Then f is total recursive, and

$$\begin{aligned} W_{f(x)} \text{ is finite} &\iff \{u \mid g(x, u)\downarrow\} \text{ is finite} \\ &\iff (\exists y)(\forall u > y)g(x, u)\uparrow \\ &\iff (\exists y)(\forall z)P(x, y, z) \\ &\iff x \in C. \end{aligned}$$

Hence f is a many-one reduction of C to A, so that $A \in \Sigma_2 \setminus \Pi_2$.

(1b) Note that $e \in B$ iff there is an increasing sequence of even length so that $\phi_e(x)\downarrow$ iff x appears in the sequence. This is expressed by

$$\begin{aligned} &(\exists v)(\exists n)(\exists m)(\forall x)(\forall y)\, \mathrm{seq}(v) \wedge \mathrm{lh}(v) = 2n \\ &\qquad \wedge (\forall i, j < \mathrm{lh}(v))[(i < j \rightarrow (v)_i < (v)_j) \wedge (\exists w < m)T_1(e, (v)_i, w)] \\ &\qquad \wedge [(\forall i < \mathrm{lh}(v))(x \neq (v)_i)] \rightarrow \neg T_1(e, x, y). \end{aligned}$$

So B is in Σ_2. To show B is not in Π_2, it will be sufficient to reduce A to B, since A is Σ_2-complete. Define

$$g(e, x) = \mu y\, T_1(e, \lfloor x/2 \rfloor, y).$$

Then $g(e, x)\downarrow$ iff $\phi_e(n)\downarrow$, where either $2n = x$ or $2n + 1 = x$.

Put $f(e) = S^1_1(\hat{g}, e)$. Then W_e is finite iff $W_{f(e)}$ is finite and has an even number of elements, so f is a reduction of A to B, as needed.

Problem 2.2. (2a) *Prove that for every recursive partial function* $f(x, y)$*, there is some recursive partial function* $g(x)$ *such that*

$$g(x)\downarrow \iff (\exists y)[f(x,y)\downarrow], \tag{A}$$

$$(\exists y)[f(x,y)\downarrow] \implies f(x,g(x))\downarrow. \tag{B}$$

(2b) *Show that we cannot strengthen* (2a) *by replacing* (B) *by the stronger*

$$(\exists y)[f(x,y)\downarrow] \Longrightarrow g(x) = \text{the least } y \text{ such that } f(x,y)\downarrow. \tag{B'}$$

Solution. (2a) Let e be such that $f(x, y) = \phi_e(x, y)$. Put

$$g(x) = U[\mu w\, T_2(e, x, (w)_0, (w)_1)].$$

Then

$$\begin{aligned} g(x)\downarrow &\iff (\exists w)\, T_2(e, x, (w)_0, (w)_1) \\ &\iff (\exists y)\phi_e(x, y)\downarrow \\ &\iff (\exists y) f(x, y)\downarrow, \end{aligned}$$

so (A) is satisfied. It is also clear by our definition of g that if $g(x)\downarrow$, then $f(x, g(x))\downarrow$, so (B) holds.

(2b) We define a recursive f so that (A) and (B') cannot both hold. Put

$$f(x,y) = \begin{cases} \uparrow & \text{if } y < x \\ \mu z\, T_2(x, y, z) & \text{if } y = x \\ 1 & \text{if } y > x. \end{cases}$$

Notice that $f(x, x)\downarrow$ iff $\phi_x(x)\downarrow$. Suppose $g(x)$ is a recursive function satisfying (A), (B'). Then g is total recursive, and $g(x) = x$ if $\phi_x(x)\downarrow$, $g(x) = x+1$ otherwise. Put $h(x) = 1 - (g(x) - x)$; then h is a total recursive function so that $h(x) = 1$ if $\phi_x(x)\downarrow$, $h(x) = 0$ otherwise. This contradicts the recursive unsolvability of the halting problem.

Problem 2.3. *For each sentence* θ *in the language of Peano arithmetic* PA*, let*

$$\ulcorner\theta\urcorner = \text{the (formal) numeral of the Gödel number of } \theta,$$

and let $\mathrm{Pr}(n)$ *be a formula with one free variable which expresses the relation of* provability *in Peano arithmetic, so that (in particular), for each sentence* θ*,*

$$(\mathbb{N}, 0, 1, +, \cdot) \models \mathrm{Pr}(\ulcorner\theta\urcorner) \iff \mathrm{PA} \vdash \theta.$$

Consider the following four sentences which can be constructed from an arbitrary sentence θ*:*

(a) $\theta \to \mathrm{Pr}(\ulcorner\theta\urcorner)$
(b) $\mathrm{Pr}(\ulcorner\theta\urcorner) \to \theta$
(c) $\mathrm{Pr}(\ulcorner\theta\urcorner) \to \mathrm{Pr}(\ulcorner\mathrm{Pr}(\ulcorner\theta\urcorner)\urcorner)$
(d) $\mathrm{Pr}(\ulcorner\mathrm{Pr}(\ulcorner\theta\urcorner)\urcorner) \to \mathrm{Pr}(\ulcorner\theta\urcorner)$

Determine which of these four sentences are provable in PA *(for every choice of* θ*), and* justify three of your answers *by appealing, if necessary, to standard theorems which are proved in 220. (One of the answers is more difficult to justify than the others.)*

Solution. If PA is inconsistent, then all of the above are provable. If PA is consistent, but proves its own inconsistency, then PA proves $\mathrm{Pr}(\ulcorner\theta\urcorner)$ for all sentences θ; it would then be the case that (a), (c), and (d) are all provable. We assume that PA is sound.

(a) Applying the fixed point theorem to the formula $\neg\mathrm{Pr}(x)$, there is a sentence θ so that $\mathrm{PA} \vdash \theta \leftrightarrow \neg\mathrm{Pr}(\ulcorner\theta\urcorner)$. By (a), we have $\mathrm{PA} \vdash \neg\mathrm{Pr}(\ulcorner\theta\urcorner) \to \mathrm{Pr}(\ulcorner\theta\urcorner)$, so that $\mathrm{PA} \vdash \mathrm{Pr}(\ulcorner\theta\urcorner)$. By soundness of PA, there must in fact be a proof of θ from PA, i.e. $\mathrm{PA} \vdash \theta$. But then by choice of θ, $\mathrm{PA} \vdash \neg\mathrm{Pr}(\ulcorner\theta\urcorner)$, implying inconsistency of PA. Hence (a) is not provable in PA for this θ.

(b) Recall Löb's Theorem, which states that for any sentence θ, if $\mathrm{PA} \vdash \mathrm{Pr}(\ulcorner\theta\urcorner) \to \theta$, then $\mathrm{PA} \vdash \theta$. It follows that (b) is not provable for all sentences θ, since otherwise we would have $\mathrm{PA} \vdash \theta$ for all sentences θ.

(c) For each sentence θ, $\mathrm{Pr}(\ulcorner\theta\urcorner)$ is a Σ_1 sentence in PA:

$$\mathrm{Pr}(\ulcorner\theta\urcorner) \iff (\exists y)\mathrm{Proof}_{\mathrm{PA}}(\ulcorner\theta\urcorner, y),$$

and for any Σ_1 sentence ϕ, we have $\mathrm{PA} \vdash \phi \to \mathrm{Pr}(\ulcorner\phi\urcorner)$. Hence (c) is provable for all sentences θ.

(d) As in part (b), Löb's Theorem implies that if (d) were provable for all sentences θ, we would have $\mathrm{PA} \vdash \mathrm{Pr}(\ulcorner\theta\urcorner)$ for all sentences θ, contradicting soundness of PA. Hence (d) is not provable for all sentences θ.

Problem 2.4. *Assume $V = L$, let*

$$\lambda = \aleph_\omega,$$

and prove that L_λ has the Σ_1-reflection property.

In detail, this means that if

$$\theta(x, y) \equiv (\exists x_1)(\exists x_2)\cdots(\exists x_n)\phi(x, y)$$

where $\phi(x, y)$ is a bounded *formula in which all quantifiers occur in one of the forms*

$$(\exists y \in z) \text{ or } (\forall y \in z)$$

and if for some $a \in L_\lambda$,

$$L_\lambda \models (\forall x \in a)(\exists y)\theta(x, y),$$

then there is some $b \in L_\lambda$ such that

$$L_\lambda \models (\forall x \in a)(\exists y \in b)\theta(x, y).$$

Solution. We have $a \in L_\alpha$ for some $\alpha < \lambda$. Let κ be a regular uncountable cardinal so that $\lambda < \kappa$. By Löwenheim-Skolem, let $H \prec L_\kappa$ be such that $|H| = |\alpha|$ and $L_\alpha \cup \{L_\lambda\} \subset H$. Let $\pi : H \to M$ be the Mostowski collapse embedding. Then M is a transitive model of $\mathsf{ZF} - \mathsf{Powerset} + V = L$, so by condensation, $M = L_\beta$ for some $\beta < \lambda$.

Now, $L_\lambda \models (\forall x \in a)(\exists y)\theta(x, y)$ simply means

$$(\forall x \in a)(\exists y \in L_\lambda)\theta^{L_\lambda}(x, y).$$

Since θ is Σ_1, it reflects from L_λ to L_κ; then

$$L_\kappa \models (\forall x \in a)(\exists y \in L_\lambda)\theta(x, y),$$

and by our definition of H, H satisfies the above formula as well. Applying the isomorphism π, we have

$$L_\beta \models (\forall x \in \pi(a))(\exists y \in \pi(L_\lambda))\theta(x, y).$$

But since $L_\alpha \subset H$ and L_α is transitive, π is the identity map on L_α; in particular, $\pi(a) = a$. We have

$$(\forall x \in a)(\exists y \in \pi(L_\lambda))\theta^{L_\beta}(x, y).$$

Now θ reflects from L_β to L_λ, so that

$$L_\lambda \models (\forall x \in a)(\exists y \in \pi(L_\lambda))\theta(x, y),$$

which proves the claim, with $b = \pi(L_\lambda)$.

Problem 2.5. *Suppose $V = L$.* True or false*: if α is an ordinal such that*

$$L_\alpha \models \text{ZFC},$$

then α is a strongly inaccessible cardinal. (You must prove your answer.)

Solution. It is consistent with ZFC that this is (vacuously) true, since it is consistent with ZFC that ZFC has no set model.

Assuming such an ordinal α exists, however, the claim is false. By Löwenheim-Skolem, let $H \prec L_\alpha$ be countable, and let $\pi : H \to M$ be the Mostowski collapse embedding. Then M is transitive and $M \models \text{ZFC} + V = L$, so that by condensation, $M = L_\beta$ for some countable β; but this β is not strongly inaccessible.

Problem 2.6. *A* tree *on a set A is a set $T \subseteq A^{<\omega}$ of finite sequences from A which is closed under initial segments; an* infinite branch *of a tree T is any function $\alpha : \mathbb{N} \to A$ such that for all n, $\langle \alpha(0) < \ldots, \alpha(n-1)\rangle \in T$; and T is* finitely splitting *if for each $u \in T$ there are only finitely many (perhaps 0) one-point extensions of u in T. Prove the following*

König's Lemma. Every infinite, finitely splitting tree T has an infinite branch.

Solution. Given $u \in T$, define

$$T_u = \{v \in T \mid u \upharpoonright k = v \upharpoonright k \text{ for all } k \leq \min(\text{length}(u), \text{length}(v))\}.$$

That is, T_u is the set of elements of T that are either initial segments or extensions of u.

Now, supposing T is infinite and finitely splitting, we use choice to define an infinite branch α of T. Suppose inductively that for some n, we have defined $u = \langle \alpha(0), \ldots, \alpha(n-1)\rangle$ so that $u \in T$ and T_u is infinite (this is true automatically when $n = 0$, since $T_\emptyset = T$).

Observe that

$$T_u = \bigcup\{T_v \mid v \text{ is a one-point extension of } u \text{ in } T\}.$$

Since T is finitely splitting, there are finitely many one-point extensions of u in T. By the Pigeonhole principle, there must be a one-point extension v of u so that T_v is infinite. Let $\alpha(n)$ be defined so that $v = \langle \alpha(0), \ldots, \alpha(n)\rangle$. Then $v \in T$, and T_v is infinite, so that the induction proceeds. The function α thus obtained is an infinite branch of T.

Problem 2.7. *Recall that a model $\mathfrak{M}$ of a complete (first-order) theory T is* atomic, *if for every finite sequence $\vec{a} \in M^n$ of length n, there is a formula $\phi(\vec{v})$ with n free variables such that*

$$\mathfrak{M} \models \phi(\vec{a}),$$

and for every $\psi(\vec{v})$,

$$\textit{either } \mathfrak{M} \models (\forall \vec{v})[\phi(\vec{v}) \to \psi(\vec{v})] \textit{ or } \mathfrak{M} \models (\forall \vec{v})[\phi(\vec{v}) \to \neg\psi(\vec{v})]$$

(7a). *Does there exist a countable, complete theory with an atomic model of size $\aleph_0$ but no atomic model of size $\aleph_1$?*

(7b). *Does there exist a countable, complete theory with an atomic model of size $\aleph_1$ but no atomic model of size $\aleph_0$?*

Solution. (7a). Yes. Let $T = \mathrm{Th}(\mathbb{Z}, S)$ be the theory of the integers with successor. We claim the model $(\mathbb{Z}, S)$ is atomic. In what follows, we abbreviate the term for the m-th successor of v by $S^m(v)$, for all $m \in \omega$ (we take v to be the 0-th successor of v, $S^0(v)$).

Suppose $\vec{a} = \langle a_1, \ldots, a_n \rangle \in \mathbb{Z}^n$. Without loss of generality, we may assume $a_1 \leq a_i$ for $i = 2, \ldots, n$. Then there are naturals $m_2, \ldots, m_n$ so that $a_i = a_1 + m_i$ for $i = 2, \ldots, n$. Let $\phi(\vec{v})$ be the formula

$$\phi(\vec{v}) \equiv (v_2 = S^{m_2}(v_1)) \wedge (v_3 = S^{m_3}(v_1)) \wedge \cdots \wedge (v_n = S^{m_n}(v_1)).$$

Then $(\mathbb{Z}, S) \models \phi(\vec{a})$. Now suppose $(\mathbb{Z}, S) \models \phi(\vec{b})$ for some $\vec{b} \in \mathbb{Z}^n$. Then by our definition of ϕ, we know $\vec{b} = \langle a_1 + k, \ldots, a_n + k \rangle$ for some integer k. Then $x \mapsto x + k$ is an automorphism of $(\mathbb{Z}, S)$ that maps $\vec{a}$ to $\vec{b}$. It follows that for any formula $\psi(\vec{v})$ with $(\mathbb{Z}, S) \models \psi(\vec{a})$, we also have $(\mathbb{Z}, S) \models \psi(\vec{b})$. Since for all formulas $\psi(\vec{v})$ we have either

$$(\mathbb{Z}, S) \models \psi(\vec{a}) \text{ or } (\mathbb{Z}, S) \models \neg\psi(\vec{a}),$$

this shows $(\mathbb{Z}, S)$ is atomic.

Now suppose $\mathfrak{M} = (M, S^{\mathfrak{M}})$ is a model of T of size $\aleph_1$. Consider the 2-type

$$\Phi(v_0, v_1) = \{v_0 \neq S^m(v_1) \wedge v_1 \neq S^m(v_0) \mid m \in \omega\}.$$

Let $a_0 \in M$. Since $\mathfrak{M}$ models T, there are countably many $b \in M$ so that

$$\mathfrak{M} \models a_0 = S^m(b) \vee b = S^m(a_0)$$

for some $m \in \omega$. Hence there is $a_1 \in M$ so that $\langle a_0, a_1 \rangle$ realizes $\Phi(v_0, v_1)$. But clearly $\Phi(v_0, v_1)$ is not realized in $(\mathbb{Z}, S)$, so this type must be nonprincipal. We have that no uncountable model $\mathfrak{M}$ of T is atomic.

(7b). No. Suppose T is a theory with an atomic model $\mathfrak{M}$ of size $\aleph_1$. By downward Löwenheim-Skolem, there is a countable elementary substructure $\mathfrak{N}$ of $\mathfrak{M}$. We claim $\mathfrak{N}$ is atomic. For suppose $\vec{a} \in N^n$. Then $\vec{a} \in M^n$, and there is a formula $\phi(\vec{v})$ so that

$$\mathfrak{M} \models \phi(\vec{a})$$

and for every $\psi(\vec{v})$,

$$\text{either } \mathfrak{M} \models (\forall \vec{v})[\phi(\vec{v}) \to \psi(\vec{v})] \text{ or } \mathfrak{M} \models (\forall \vec{v})[\phi(\vec{v}) \to \neg\psi(\vec{v})].$$

Since $\mathfrak{N}$ is an elementary substructure of $\mathfrak{M}$, the same holds with $\mathfrak{N}$ substituted for $\mathfrak{M}$. Thus $\mathfrak{N}$ is atomic.

Problem 2.8. *Let $\mathfrak{M} = (\mathbb{Z}, S)$ be the model with the underlying set the integers and the successor function $S(x) = x+1$ as the only non-logical constant. Is* $\mathrm{Th}(\mathfrak{M})$ *finitely axiomatizable?*

Solution. No. Letting $S^m(x)$ abbreviate the term for the m-th successor of x, consider the theory T consisting of the following sentences:

(1) $(\forall x)(\exists! y)\, S(y) = x$
(2) For each $m \in \mathbb{N}$, $(\forall x)\, S^m(x) \neq x$

Clearly $(\mathbb{Z}, S) \models T$. We claim T is complete. Let A be the set $(\aleph_1 \times \mathbb{Z})$, and interpret S in A by $S^{\mathfrak{A}}(\alpha, n) = (\alpha, n+1)$. The structure $\mathfrak{A} = (A, S^{\mathfrak{A}})$ is a model of T, and it is straightforward to see that any model of T of size $\aleph_1$ is isomorphic to $\mathfrak{A}$. Completeness of T follows by the Loś-Vaught Test.

Suppose now for contradiction that the theory of $(\mathbb{Z}, S)$ is finitely axiomatizable. Then there is some finite subset T_0 of T so that $T_0 \models T$. Let m be the largest integer so that $(\forall x)\, S^m(x) \neq x$ is in T_0. Consider the structure with domain $\{0, 1, \ldots, m\}$, with S interpreted by $S(k) = (k+1)$ if $k \leq m$, and $S(m) = 0$. This is a model of T_0, but satisfies $S^{m+1}(0) = 0$, so that it is not a model of T, contradicting $T_0 \models T$.

Problem 2.9. *Let $T = \mathrm{Th}(\mathbb{R}, <, \mathbb{Z})$ be the theory of the real numbers, with the usual ordering and a distinguished predicate for the integers. Is T $\aleph_0$-categorical? (You must prove your answer.)*

Solution. No. Let $\mathcal{L}$ be the language expanded with constants c, d, and for each $n \in \omega$, let $\phi_n(c, d)$ be the $\mathcal{L}$-sentence that says $c < d$ and there are n distinct integers between c and d, i.e.

$$(\exists x_1) \ldots (\exists x_n)[(c < x_1 < \cdots < x_n < d) \wedge \mathbb{Z}(x_1) \wedge \cdots \wedge \mathbb{Z}(x_n)].$$

Let T' be the $\mathcal{L}$-theory $\mathrm{Th}(\mathbb{R}, <, \mathbb{Z}) \cup \{\phi_n(c, d) \mid n \in \omega\}$. Then T' is finitely satisfiable, since for any n there are elements a, b of $\mathbb{R}$ with n integers between them; so by compactness, there is a countable model of T'. Let $\mathfrak{A}$ be the reduct of this model to the language $\{<, \mathbb{Z}\}$, and by downward Löwenheim-Skolem, let $\mathfrak{B}$ be a countable elementary substructure of $(\mathbb{R}, <, \mathbb{Z})$. Then $\mathfrak{A}$ and $\mathfrak{B}$ are countable models of T, but are not isomorphic.

3. Winter 2003

Problem 3.1. *A linear order $(S, \prec)$ is* ***illfounded*** *if there is an infinite sequence $\langle s_i \mid i \in \omega \rangle$ of elements of S so that $s_{i+1} \prec s_i$ for each $i \in \omega$.*

Let φ be a sentence in the language with one binary relation symbol $\prec$. Suppose φ is true in an infinite illfounded linear order. Prove that φ is true in an uncountable illfounded linear order.

Solution. This is immediate by upward Löwenheim-Skolem. In more detail, suppose $(S, \prec)$ is an infinite illfounded linear order in which φ holds. Fix a collection $A = \{c_\alpha \mid \alpha < \omega_1\}$ of uncountably many constant symbols, and let $\mathcal{L}$ be the language $\{\prec\} \cup A$. Consider the $\mathcal{L}$-theory

$$T = \{\varphi, \text{“}\prec \text{ is a linear order”}\} \cup \{c_\alpha \neq c_\beta \mid \alpha < \beta < \omega_1\} \cup \{c_{\alpha+1} \prec c_\alpha \mid \alpha \in \omega\}$$

Note that any model of T is an uncountable ill-founded linear order that satisfies φ. Also, T is finitely satisfiable: any finite fragment is satisfied in $(S, \prec)$ under a suitable assignment of the constants in A. By the compactness theorem, there is an $\mathcal{L}$-structure satisfying T; then the reduct of this structure to $\{\prec\}$ is the desired uncountable illfounded linear order.

Problem 3.2. *For each complete theory T in the language of set theory let $A_T = \{\alpha < \omega_1 \mid L_\alpha \models T\}$. Prove that there is a complete theory T so that A_T is uncountable.*

Solution. Let T be the theory of L_{ω_1} in the language of set theory. Suppose $\delta < \omega_1$. By Löwenheim-Skolem, there is $H \subset L_{\omega_1}$ so that $\delta \subset H$, H is countable, and $H \prec L_{\omega_1}$.

Let M be the Mostowski collapse of H, and π be the collapse embedding. Since ω_1 is regular and uncountable, L_{ω_1} is a model of $\mathsf{ZF}-\mathsf{Powerset} + V = L$. Then M is a transitive set model of $\mathsf{ZF}-\mathsf{Powerset} + V = L$, so by condensation, $M = L_\alpha$ for some $\alpha < \omega_1$.

We claim that for each $\gamma \in \delta$, we have $\gamma \in L_\alpha$. For suppose inductively that $\beta \in L_\alpha$ for all $\beta < \gamma$. Then

$$\pi(\gamma) = \{\pi(\beta) \mid \beta \in \gamma \cap H\} = \{\beta \mid \beta \in \gamma\} = \gamma.$$

So $\gamma = \pi(\gamma) \in L_\alpha$ for all $\gamma \in \delta$. Then $\delta \subset L_\alpha$, and since $L_\alpha \cap \mathrm{ON} = \alpha$, we have $\delta \leq \alpha$.

We have shown that for any $\delta < \omega_1$, there is an $\alpha < \omega_1$ with $\delta \leq \alpha$ and $L_\alpha \models T$ (since $L_\alpha \cong H \prec L_{\omega_1}$). That is, we have that A_T is unbounded in ω_1; so A_T is uncountable.

Problem 3.3. *Let U be the set of all functions f so that:* $\mathrm{dom}(f) = \omega$*; and (for every $n \in \omega$) $f(n)$ belongs to ω_n. Assuming the* GCH*, prove that* $\mathrm{card}(U) = \aleph_{\omega+1}$.

Solution. Since $U \subset {\omega_\omega}^\omega$, we have

$$\mathrm{card}(U) \leq \mathrm{card}({\omega_\omega}^\omega) \leq \mathrm{card}((2^{\omega_\omega})^\omega) = \mathrm{card}(2^{\omega_\omega}) = \aleph_{\omega+1},$$

where the last equality holds by GCH.

We show there is no surjection of ω_ω onto U. Suppose for contradiction that $\{f_\alpha \mid \alpha \in \omega_\omega\}$ is an enumeration of U. Define $g : \omega \to \omega_\omega$ by putting

$$g(n) = \text{ the least element of } \omega_{n+1} \setminus \{f_\alpha(n) \mid \alpha \in \omega_n\}.$$

Note that $g(n)$ always is defined because there can be no surjection of ω_n onto ω_{n+1}. Furthermore, for any $\alpha \in \omega_\omega$, if we let n be large enough that $\alpha \in \omega_n$, we have $f_\alpha(n) \neq g(n)$; hence $g \in U$, but $f_\alpha \neq g$ for all $\alpha \in \omega_\omega$, a contradiction.

We have $\aleph_\omega = \mathrm{card}(\omega_\omega) < \mathrm{card}(U) \leq \aleph_{\omega+1}$, so we conclude $\mathrm{card}(U) = \aleph_{\omega+1}$.

Problem 3.4. (a) *For each formula φ in the language of set theory, show that* ZFC *proves* $\varphi \to \mathrm{CON}(\varphi)$.

(b) *Show that* ZFC *is not finitely axiomatizable.*

Solution. (a) Working in ZFC, the reflection principle gives us an ordinal α so that $\varphi^V \leftrightarrow \varphi^{V_\alpha}$. Now, φ^{V_α} just says that V_α is a set model of φ, which by soundness implies $\mathrm{CON}(\varphi)$. Hence $\varphi \to \mathrm{CON}(\varphi)$.

(b) Suppose ZFC is finitely axiomatizable. Then there is a single sentence φ so that $\varphi \vdash \mathsf{ZFC}$. Then by part (a), $\varphi \vdash (\varphi \to \mathrm{CON}(\varphi))$, and by the deduction theorem, $\varphi \vdash \mathrm{CON}(\varphi)$. But since φ axiomatizes ZFC, PA is interpretable in the theory φ, so this contradicts Gödel's second incompleteness theorem.

Problem 3.5. *Let $\langle \phi_e \mid e < \omega \rangle$ be a standard enumeration of all the recursive partial functions. Fix a total recursive function f. let $B = \{e \mid \phi_e = f\}$. Prove that B is Π_2 complete.*

Solution. Clearly B is Π_2, since

$$\phi_e = f \iff (\forall x)(\exists y)(T_1(e, x, y) \wedge (\forall w < y)\neg T_w(e, x, y) \wedge (y)_0 = f(x)).$$

To show it is Π_2 complete, let A be an arbitrary Π_2 set, so that for some recursive relation $P(e, x, y)$, we have

$$A = \{e \mid (\forall x)(\exists y)P(e, x, y)\}.$$

Define a partial recursive function in two variables by

$$g(e, x) = \begin{cases} f(x) & \text{if } (\exists y)P(e, x, y) \\ \uparrow & \text{otherwise.} \end{cases}$$

Let $h(e) = S^1_1(\hat{g}, e)$. Then h is total recursive, and

$$\begin{aligned} \phi_{h(e)} = f &\iff (\forall x)g(e, x) = f(x) \\ &\iff (\forall x)(\exists y)P(e, x, y) \\ &\iff e \in A. \end{aligned}$$

Hence h is a reduction of A to B. Since A was an arbitrary set in Π_2, we have that B is Π_2 complete.

Problem 3.6. *Let $\varphi(x_1, \ldots, x_n)$ be a Σ_1 formula in the language of set theory. Suppose $V = L$. Let $a_1, \ldots, a_n \in L_{\omega_1}$. Prove that*

$$\varphi(a_1, \ldots, a_n) \longleftrightarrow \varphi^{L_{\omega_1}}(a_1, \ldots, a_n).$$

Solution. $\varphi(x_1, \ldots, x_n)$ is Σ_1; so for some Δ_0 formula θ we have

$$\varphi(x_1, \ldots, x_n) \equiv (\exists y_1) \ldots (\exists y_m)\theta(x_1, \ldots, x_n, y_1, \ldots, y_m).$$

Then θ is absolute for L_{ω_1}, L, and by $V = L$, we have

$$\begin{aligned} \varphi^{L_{\omega_1}}(a_1, \ldots, a_n) &\longleftrightarrow (\exists y_1 \in L_{\omega_1}) \ldots (\exists y_m \in L_{\omega_1})\theta^{L_{\omega_1}}(a_1, \ldots, a_n, y_1, \ldots, y_m) \\ &\longrightarrow (\exists y_1) \ldots (\exists y_m)\theta(a_1, \ldots, a_n, y_1, \ldots, y_m) \\ &\longleftrightarrow \varphi(a_1, \ldots, a_n). \end{aligned}$$

Now suppose $\varphi(a_1, \ldots, a_n)$ holds. Then there are $b_1, \ldots, b_m$ so that

$$\theta(a_1, \ldots, a_n, b_1, \ldots, b_m)$$

holds. By $V = L$, there is a regular uncountable cardinal κ large enough that $b_1, \ldots, b_m \in L_\kappa$. Let α be countable and large enough that $a_1, \ldots, a_n \in L_\alpha$. By Löwenheim-Skolem, let $H \prec L$ be countable so that $L_\alpha \cup \{b_1, \ldots, b_n\} \subset H$. Let $\pi : H \to M$ be the collapse embedding. Then M is a transitive countable set model of $\mathsf{ZF} - \mathsf{Powerset} + V = L$. By condensation, $M = L_\delta$ for some countable δ.

Notice that $\pi \restriction L_\alpha = \mathrm{Id}$, since L_α is a transitive subset of H. In particular, $\pi(a_i) = a_i$ for $i = 1, \dots, n$. We have

$$\begin{aligned}\theta(a_1,\dots,a_n,b_1,\dots,b_m) &\longleftrightarrow \theta^{L_\kappa}(a_1,\dots,a_n,b_1,\dots,b_m)\\ &\longleftrightarrow \theta^{H}(a_1,\dots,a_n,b_1,\dots,b_m)\\ &\longleftrightarrow \theta^{L_\delta}(a_1,\dots,a_n,\pi(b_1),\dots,\pi(b_m))\\ &\longrightarrow (\exists y_1 \in L_\delta)\dots(\exists y_m \in L_\delta)\theta^{L_\delta}(a_1,\dots,a_n,y_1,\dots,y_m)\\ &\longrightarrow (\exists y_1 \in L_{\omega_1})\dots(\exists y_m \in L_{\omega_1})\theta^{L_{\omega_1}}(a_1,\dots,a_n,y_1,\dots,y_m)\\ &\longleftrightarrow \varphi^{L_{\omega_1}}(a_1,\dots,a_n).\end{aligned}$$

Here we have several times used absoluteness of θ for transitive models of $\mathsf{ZF} - \mathsf{Powerset}$, as well as the facts that $H \prec L_\kappa$, $\pi : H \to L_\delta$ is an isomorphism, and $L_\delta \subset L_{\omega_1}$. This finishes the proof.

Problem 3.7. *Let A and B be r.e. sets so that $A \cap B = \emptyset$.*
(a) *Show that there is a formula φ in the language of arithmetic so that:* (i) *If $n \in A$ then $\mathsf{PA} \vdash \varphi(\tilde{n})$; and* (ii) *If $n \in B$ then $\mathsf{PA} \vdash \neg\varphi(\tilde{n})$. ($\tilde{n}$ here is the formal term in the language of arithmetic for the nth successor to* 0.*)*
(b) *Is it possible to strengthen the above to require also:* (iii) *For every $n \in \omega$, ($\mathsf{PA} \vdash \varphi(\tilde{n})$ or $\mathsf{PA} \vdash \neg\varphi(\tilde{n})$)?*

Solution. (a) Let a, b be natural numbers so that $A = W_a$ and $B = W_b$. Recall that for any index e, $n \in W_e$ iff $(\exists y)\, T_1(e, n, y)$. Let $\varphi(x)$ be the formula

$$(\exists y)(\forall w < y)\, T_1(a, x, y) \wedge \neg T_1(b, x, w).$$

Suppose $n \in A$. Then $\varphi(\tilde{n})$ is a true Σ_1 sentence, so $\mathsf{PA} \vdash \varphi(\tilde{n})$.

Next, suppose $n \in B$. Then there is some fixed p so that $T_1(b, n, p)$, and for each $k \le p$, $\neg T_1(a, n, k)$. Then

$$\mathsf{PA} \vdash (\forall y)\, T_1(\tilde{b}, \tilde{n}, \tilde{p}) \wedge (y \le \tilde{p} \to \neg T_1(\tilde{a}, \tilde{n}, y)),$$

which clearly implies $\neg\varphi(\tilde{n})$ in PA.

(b) This is not possible in general. Note that for any formula θ, $\{n \mid \mathsf{PA} \vdash \theta(\tilde{n})\}$ is a r.e. set. Any r.e. set with r.e. complement is recursive; so (iii) implies that for any r.e. sets A, B such that $A \cap B = \emptyset$, there is a recursive set C so that $A \subset C$ and $C \cap B = \emptyset$. But this contradicts the existence of recursively inseparable r.e. sets.

4. Fall 2003

Problem 4.1. *For this problem, work in* ZF *(*ZFC *minus the Axiom of Choice). If κ is a cardinal number and X is a set, then $\mathcal{P}_\kappa(X)$ is the set of all subsets of X of size $< \kappa$. Suppose that $f : \mathcal{P}_{\omega_1}(\mathbb{R}) \to \mathbb{R}$ is one-to-one. Prove that there exists a sequence of ω_1 distinct reals.*

Solution. Define the sequence $\{x_\alpha\}_{\alpha\in\omega_1}$ by transfinite recursion so that

$$x_\alpha = f(\{x_\beta \mid \beta \in \alpha\}).$$

Note that since f is injective, $x_\alpha \ne x_\beta$ for all $\beta < \alpha$. It follows that $\{x_\alpha\}_{\alpha\in\omega_1}$ is a sequence of distinct reals.

Problem 4.2. *A subset X of a limit ordinal α is* stationary *in α if X meets every closed, unbounded subset of α. Let κ be a regular cardinal and let $X \subseteq \kappa$ be stationary in κ. Let M be a transitive class model of* ZFC *such that $X \in M$. Prove that X is stationary in κ in M.*

Solution. Note (X is stationary in κ) $\leftrightarrow (\forall C)(C$ club in $\kappa \to (\exists a \in X)\, a \in C)$. Now,

$$\begin{aligned} C \text{ is club in } \kappa &\iff C \text{ is closed in } \kappa \text{ and } C \text{ is unbounded in } \kappa \\ &\iff (\forall\lambda \in \kappa)[(\forall\alpha \in \lambda)(\exists\beta \in C)(\alpha \in \beta \in \lambda) \to \lambda \in C] \\ &\qquad \wedge [(\forall\alpha \in \kappa)(\exists\beta \in C)\, \alpha \in \beta \in \kappa]. \end{aligned}$$

This is Δ_0, so absolute for M.

We wish to show (X is stationary in $\kappa)^M$ (note $X \in M$, and since M is a transitive model of ZFC, $\bigcup X = \kappa \in M$, so this makes sense). This is just

$$(\forall C \in M)(C \text{ is club in } \kappa)^M \to (\exists a \in X \cap M)\, a \in C.$$

By the above, and since M is transitive, this is equivalent to

$$(\forall C \in M)(C \text{ is club in } \kappa) \to (\exists a \in X)\, a \in C$$

which is clearly implied by our stronger assumption that X is stationary in κ.

Problem 4.3. *Assume $V = L$. Define $\langle A_\alpha \mid \alpha < \omega_1\rangle$ as follows. Let A_α be the $<_L$-least $A \subseteq \alpha$ such that $(\forall\beta < \alpha) A \cap \beta \neq A_\beta$ if such an A exists and let $A_\alpha = \emptyset$ otherwise. Prove that for all $A \subseteq \omega_1$ there exists an $\alpha < \omega_1$ such that $A \cap \alpha = A_\alpha$.*

The problem as stated is trivial: take $\alpha = 0$. Even requiring $\alpha \neq 0$, it is easily seen that if $A \cap n \neq A_n$ for all nonzero $n < \omega$, then $A \cap \omega = A_\omega$. To correct this, α and β should be assumed to be infinite throughout.

Solution. Suppose there exists $A \subseteq \omega_1$ so that for all infinite $\alpha < \omega_1$, $A \cap \alpha \neq A_\alpha$. Let A be the $<_L$-least such. By $V = L$, there is a regular uncountable cardinal $\theta > \omega_1$ such that $A, \langle A_\alpha \mid \alpha < \omega_1\rangle \in L_\theta$. By Löwenheim-Skolem, let $H \subset L_\theta$ be such that

- $(H; \in\restriction H) \prec (L_\theta; \in\restriction L_\theta)$
- $|H| = \omega$
- $H \supset \{A, \omega_1, \langle A_\alpha \mid \alpha < \omega_1\rangle\}$
- $H \cap \omega_1$ is an ordinal.

Put $\lambda = \omega_1 \cap H$. Then λ is an ordinal with $\lambda < \omega_1$. Let $\pi : H \to M$ be the Mostowski collapse. Then $(M; \in\restriction M)$ is a countable transitive set model of $\mathsf{ZF}{-}\mathsf{Powerset}{+}V = L$, so by condensation, $M = L_\delta$ for some δ.

Since $\lambda \subset H$, we have $\pi(\alpha) = \alpha$ for all $\alpha < \lambda$. Then by definition of π,

$$\begin{aligned} \pi(\omega_1) &= \{\pi(\alpha) \mid \alpha \in \omega_1 \cap H\} \\ &= \{\pi(\alpha) \mid \alpha \in \lambda\} \\ &= \{\alpha \mid \alpha \in \lambda\} = \lambda, \end{aligned}$$

and

$$\begin{aligned} \pi(A) &= \{\pi(\alpha) \mid \alpha \in A \cap H\} \\ &= \{\pi(\alpha) \mid \alpha \in A \cap \lambda\} \\ &= \{\alpha \mid \alpha \in A \cap \lambda\} = A \cap \lambda. \end{aligned}$$

We claim $\pi(\langle A_\alpha \mid \alpha < \omega_1\rangle) = \langle A_\alpha \mid \alpha < \lambda\rangle$. Since π is an isomorphism, and

$$(H; \in\restriction H) \models \langle A_\alpha \mid \alpha < \omega_1\rangle \text{ is a sequence of length } \omega_1,$$

we have

$$(L_\delta; \in\restriction L_\delta) \models \pi(\langle A_\alpha \mid \alpha < \omega_1\rangle) \text{ is a sequence of length } \lambda.$$

By absoluteness, $\pi(\langle A_\alpha \mid \alpha < \omega_1\rangle)$ is a sequence of length λ. By similar arguments, we can show that for each $\alpha < \lambda$, the α-th element of $\pi(\langle A_\alpha \mid \alpha < \omega_1\rangle)$ is a subset of α that is equal to the α-th element of $\langle A_\alpha \mid \alpha < \omega_1\rangle$. This proves our claim.

We have that $A \cap \lambda$ is a subset of λ such that $(A \cap \lambda) \cap \alpha \neq A_\alpha$ for all infinite $\alpha < \lambda$. We claim it is the $<_L$-least such. For suppose there is some $A' <_L A \cap \lambda$ with these properties. By definition of $<_L$, L_δ is downward closed under $<_L$, so $A' \in L_\delta$. By absoluteness of $<_L$, we have

$$(L_\delta; \in\restriction L_\delta) \models (A' \subset \lambda) \wedge (A' <_L A \cap \lambda) \wedge (A' \cap \alpha \neq A_\alpha \text{ all infinite } \alpha < \lambda),$$

and inverting the collapse embedding π,

$$\begin{aligned}(H; \in\restriction H) \models &(\pi^{-1}(A') \subset \omega_1) \wedge (\pi^{-1}(A') <_L A) \\ &\wedge (\pi^{-1}(A') \cap \alpha \neq A_\alpha \text{ all infinite } \alpha < \omega_1).\end{aligned}$$

By elementarity, this holds in L_θ, and by absoluteness, then holds in V; but this contradicts our choice of A as the $<_L$-least set with these properties.

We have that $A \cap \lambda$ is the $<_L$-least subset of λ such that $(A \cap \lambda) \cap \alpha \neq A_\alpha$ for all infinite $\alpha < \lambda$. Hence by definition, $A_\lambda = A \cap \lambda$; but this contradicts our initial assumption about A. This completes the proof.

Problem 4.4. *Work in* ZF. *Let* $\mathrm{AC}^{\mathrm{fin}}$ *be the restriction of the Axiom of Choice to collections of* finite *sets. Prove that the Compactness Theorem of model theory implies* $\mathrm{AC}^{\mathrm{fin}}$.

Solution. Let $\mathcal{F}$ be a family of finite sets. Let $\mathcal{L}$ be a language with a single unary relation symbol f, constants C_X for each $X \in \mathcal{F}$, and constants c_a for each $a \in \bigcup \mathcal{F}$. Let T be the $\mathcal{L}$-theory

$$\begin{aligned}T =&\{f(C_X) = c_{a_1} \vee \cdots \vee f(C_X) = c_{a_n} \mid X \in \mathcal{F},\ X = \{a_1, \dots, a_n\}\} \\ &\cup \{c_a \neq c_b \mid a, b \in \textstyle\bigcup \mathcal{F}, a \neq b\}\end{aligned}$$

It should be noted that we are not choosing orderings for the elements X of $\mathcal{F}$ in order to write down the formulas in T; we simply include one sentence for every ordering of X.

T is finitely satisfiable, since a choice function for any finite subset of $\mathcal{F}$ can be constructed by finite choice in ZF. By compactness, let $\mathfrak{A}$ be an $\mathcal{L}$-structure modeling T. Let g be the function on $\mathcal{F}$ defined by

$$g(X) = a \iff \mathfrak{A} \models f(C_X) = c_a.$$

Then $g(X)$ is uniquely defined, and $g(X) \in X$ for all $X \in \mathcal{F}$ by our definition of T. So g is a choice function for F, and $\mathrm{AC}^{\mathrm{fin}}$ holds.

Problem 4.5. *Let* $S(n) = n + 1$ *for* $n \in \omega$. *Prove that the theory of* (ω, S) *is not finitely axiomatizable.*

Solution. The solution is essentially the same as that to Fall 2002 Problem 8. For $m \geq 1$, let $S^m(n)$ denote the term $S(S(\cdots S(n) \cdots))$ (m times). Let T be the set of sentences

(1) $(\exists! x)(\forall y)\, S(y) \neq x$
(2) $(\forall x)(\forall y)\, x \neq y \to S(x) \neq S(y)$
(3) For each $m \in \mathbb{N}$, $(\forall x)\, S^m(x) \neq x$

Then (1) says that there is a unique element with no predecessor, so that (2) implies that every other element has a unique predecessor. The axioms (3) ensure that there are no "loops".

Clearly $(\omega, S) \models T$. Let A be the set $\mathbb{N} \cup (\aleph_1 \times \mathbb{Z})$, and interpret S by $S(n) = n+1$ and $S(\alpha, n) = (\alpha, n+1)$. The structure $\mathfrak{A}$ thus obtained is a model of T, and it is easily seen that any model of T of size $\aleph_1$ is isomorphic to $\mathfrak{A}$. Then T is $\aleph_1$-categorical, so by the Loś-Vaught Test, T is complete.

Suppose now for contradiction that the theory of (ω, S) is finitely axiomatizable. Then there is some finite subset T_0 of T so that $T_0 \models T$. Let m be the largest integer so that $(\forall x)\, S^m(x) \neq x$ is in T_0. Consider the structure with domain $(\{0\} \times \mathbb{N}) \cup (\{1\} \times \{1, 2, \ldots, m+1\})$, with S interpreted by $S(0, n) = (0, n+1)$, $S(1, k) = (1, k+1)$ if $k \leq m$, and $S(1, m+1) = (1, 1)$. This is a model of T_0, but $S^{m+1}(1, 1) = (1, 1)$, so that it is not a model of T, contradicting $T_0 \models T$.

Problem 4.6. *Let $\kappa = \omega_1$ and let $T = \mathrm{Th}(V_\kappa, \in)$. Prove that there is no saturated countable model of T.*

Solution. For each subset A of ω, let $p_A(x)$ be the partial 1-type

$$p_A(x) = \{n \in x \mid n \in A\} \cup \{n \notin x \mid n \notin A\}$$

(here "$n \in x$" abbreviates a formula in the language of set theory with single free variable x that expresses this relationship, for each $n \in \omega$). Then $\{p_A(x) \mid A \subset \omega\}$ is an uncountable collection of pairwise incompatible partial 1-types, and each $p_A(x)$ is realized in $(V_\kappa, \in)$ because $A \in \mathcal{P}(\omega) \subset V_\kappa$. So T has uncountably many distinct types, and therefore has no countable saturated model.

Problem 4.7. *Let A be an infinite recursively enumerable set. Show that $\{e \mid W_e = A\}$ is many-one complete for Π_2. (Here W_e is the eth r.e. set in some standard enumeration.*

Solution. Any r.e. set is the domain of some partial recursive function, so let z be an index with $A = W_z$. Then $W_e = A$ iff

$$(\forall x)(\forall y)(\exists w)\, (T_1(z, x, y) \to T_1(e, x, w)) \wedge (T_1(e, x, y) \to T_1(z, x, w)),$$

so $\{e \mid W_e = A\}$ is Π_2.

To show many-one completeness for Π_2, let B be an arbitrary Π_2 set, so that

$$B = \{x \mid (\forall y)(\exists z)\, P(x, y, z)\}$$

for some recursive relation P. Let α be a partial recursive function so that

- $n \in A \iff \alpha(n)\downarrow$, and
- α is onto $\mathbb{N}$.

Such an α exists because A is an infinite r.e. set.

Now put

$$g(x, n) = \mu z\, P(x, \alpha(n), z).$$

Then g is partial recursive. Set $f(x) = S^1_1(\hat{g}, x)$. Notice that for any n, if $g(x, n)\downarrow$, then $\alpha(n)\downarrow$, whence $n \in A$. So we automatically have $A \supset \{n \mid g(x, n)\downarrow\}$ for all x, and

$$\begin{aligned} A = W_{f(x)} &\iff A = W_{S^1_1(\hat{g},x)} \\ &\iff (\forall n)[n \in A \longleftrightarrow g(x,n)\downarrow] \\ &\iff (\forall n)[n \in A \to (\exists z)\, P(x,\alpha(n),z)] \\ &\iff (\forall y)(\exists z)\, P(x,y,z) \\ &\iff x \in B. \end{aligned}$$

The second-to-last equivalence holds because α is onto $\mathbb{N}$ with domain A. Hence f is a many-one reduction of B to A, so that A is seen to be many-one complete for Π_2.

Problem 4.8. *Let* $\mathrm{Prov}(v_1, v_2)$ *represent in Peano Arithmetic (*PA*) the set of all pairs* (a, b) *such that* a *is the Gödel number of a sentence* τ *and* b *is the Gödel number of a proof of* τ *from the axioms of* PA*. Let* σ *be gotten from the Fixed Point Lemma applied to* $\forall v_2 \neg\mathrm{Prov}(v_1, v_2)$*. In other words, let* σ *be a sentence such that* $\mathrm{PA} \vdash (\sigma \leftrightarrow \forall v_2 \neg\mathrm{Prov}(\mathbf{k}, v_2))$*, where* k *is the Gödel number of* σ*. Let* T *be the theory gotten from* PA *by adding* $\neg\sigma$ *as an axiom. Show that* T *is* ω-in*consistent: that there is a formula* $\psi(v_1)$ *such that* $T \vdash \exists v_1 \psi(v_1)$ *and* $T \vdash \neg\psi(\mathbf{n})$ *for each numeral* n*.*

Solution. We have $\mathrm{PA} \vdash (\neg\sigma \leftrightarrow \exists v_2\, \mathrm{Prov}(\mathbf{k}, v_2))$, and since T strengthens PA with $T \vdash \neg\sigma$, we have $T \vdash \exists v_1\, \mathrm{Prov}(\mathbf{k}, v_1)$. We claim $T \vdash \neg\mathrm{Prov}(\mathbf{k}, \mathbf{n})$ for each numeral n.

Note first that $\mathrm{PA} \nvdash \sigma$. For suppose otherwise; then some n is the code of a proof of σ from PA. Then $\mathrm{PA} \vdash \mathrm{Prov}(\mathbf{k}, \mathbf{n})$, since PA is sound and $\mathrm{Prov}(v_1, v_2)$ represents a primitive recursive relation in PA. But then $\mathrm{PA} \vdash \neg\sigma$ by choice of σ, so that $\mathrm{PA} \vdash \sigma \wedge \neg\sigma$. This contradicts consistency of PA.

Now, for each n, we have that n is not the code of a proof of σ from PA. Then again by soundness of PA, we have $\mathrm{PA} \vdash \neg\mathrm{Prov}(\mathbf{k}, \mathbf{n})$ for all numerals n. Since T strengthens PA, this proves the claim, with $\psi(v_1) \equiv \mathrm{Prov}(\mathbf{k}, v_1)$.

5. Winter 2004

Problem 5.1. *Let* T *be the theory of the model* $(\mathbb{N}; +, \cdot)$*. Show that* T *has uncountably many non-isomorphic countable models.*

Solution. First note that we can express, for any $n \in \mathbb{N}$, the statement "$x = n$" in the language $\{+, \cdot\}$. "$x = 0$" is just $(\forall u) x \cdot u = x$; "$x = 1$" is $(\forall u) x \cdot u = u$; and for each $n > 1$, we can then express "$x = n$" by $(\exists z)(z = 1) \wedge (x = z + z + \cdots + z)$ (n times). We may then also express "n divides x" by $(\exists u)(\exists v)(v = n) \wedge (v \cdot u = x)$.

Now, let $\mathbf{P}$ denote the set of prime numbers. For each $Q \subset \mathbf{P}$, define the partial 1-type

$$T_Q(x) = T \cup \{n \text{ divides } x \mid n \in Q\} \cup \{\neg(n \text{ divides } x) \mid n \in \mathbf{P} \setminus Q\},$$

where "n divides x" is the formula introduced above.

We claim $T_Q(x)$ is a consistent type. If Q is empty, $T_Q(x)$ is realized by $x = 1$; if Q is nonempty, suppose $q(x)$ is a finite fragment of $T_Q(x)$. Let m be an element of Q so that $n \leq m$ for all n for which "n divides x" $\in q(x)$. Then if p is the product of all elements of $\{n \in Q \mid n \leq m\}$, we have that $q(x)$ is realized in $\mathbb{N}$ by $x = p$. We have shown that $T_Q(x)$ is finitely realized in $(\mathbb{N}; +, \cdot)$, so it is a consistent type.

Now the collection $\{T_Q(x) \mid Q \subset \mathbf{P}\}$ is an uncountable set of pairwise incompatible types. Each type $T_Q(x)$ is realized in some countable model of T; but since a countable model can realize only countably many types, there must be uncountably many non-isomorphic countable models of T.

Problem 5.2. *Let $\langle \phi_e \mid e < \omega \rangle$ be a standard enumeration of the recursive partial functions. Show that $\{e \mid \phi_e \text{ is bounded}\}$ is Σ_2-complete. (A partial recursive function is **bounded** if its range is bounded in ω.)*

Solution. We have ϕ_e is bounded iff there exists some m so that for all x, if $\phi_e(x)$ converges, then $\phi_e(x) < m$. That is, iff

$$(\exists m)(\forall x)(\forall y)\,[T_1(e, x, y) \wedge (\forall z < y)\neg T_1(e, x, z)] \rightarrow (y)_0 < m.$$

Hence $\{e \mid \phi_e \text{ is bounded}\}$ is Σ_2. To show it is Σ_2-complete, suppose A is a Σ_2 set; we will define a total recursive function f so that $x \in A$ iff $\phi_{f(x)}$ is bounded.

Since A is Σ_2, there is some recursive relation P so that

$$x \in A \iff (\exists y)(\forall z)P(x, y, z).$$

Define a partial recursive function $g(x, y)$ by

$$g(x, u) = \begin{cases} u & \text{if } (\forall y < u)(\exists z)\neg P(x, y, z) \\ \uparrow & \text{otherwise.} \end{cases}$$

Put $f(x) = S^1_1(\hat{g}, x)$. Then f is total recursive, and for all x,

$$\begin{aligned} \phi_{f(x)} \text{ is bounded} &\iff g(x, u) \text{ is a bounded function of } u \\ &\iff (\exists m)(\forall u)\, u > m \rightarrow g(x, u) \uparrow \\ &\iff (\exists m)(\forall u)\, u > m \rightarrow (\exists y < u)(\forall z)P(x, y, z) \\ &\iff (\exists y)(\forall z)P(x, y, z) \\ &\iff x \in A. \end{aligned}$$

So $\{e \mid \phi_e \text{ is bounded}\}$ is Σ_2-complete.

Problem 5.3. *Let φ be Goldblach's conjecture: Any even number ≥ 4 is the sum of two primes. Let T be some system of axioms stronger than ZFC, and suppose that $\mathsf{T} \vdash \varphi$. Prove that $\mathrm{CON}(\mathsf{T}) \rightarrow \varphi$.*

Solution. We prove the contrapositive. Suppose $\neg\varphi$. Notice that $\neg\varphi$ is a Σ_1 sentence: it states

$$(\exists n \in \omega)(\forall p, q < n)\; p \text{ and } q \text{ prime} \rightarrow p + q \neq n.$$

If $\neg\varphi$ holds, there is in fact some fixed natural number n witnessing this; we have

$$(\forall p, q < n)\; p \text{ and } q \text{ prime} \rightarrow p + q \neq n.$$

Then this is a true Δ_0 sentence in arithmetic, so is certainly provable in ZFC (in fact, it is provable in Robinson's arithmetic Q), hence provable in T. But then $\mathsf{T} \vdash \varphi \wedge \neg\varphi$, whence $\neg\mathrm{CON}(\mathsf{T})$.

Problem 5.4. *Work with the language of arithmetic and the standard Gödel numbering of sentences. For each sentence σ let $\mathrm{ConSeq}(\sigma)$ be the set of consequences of σ, namely $\{\tau \mid \sigma \vdash \tau\}$. Let $A = \{\ulcorner\sigma\urcorner \mid \mathrm{ConSeq}(\sigma) \text{ is decidable}\}$. Is A recursive? (Prove your answer.)*

Solution. No. We give a reduction of the halting problem to A, that is, a total recursive function f so that $\phi_e(e)\downarrow$ iff $f(e) \in A$.

Let τ be the conjunction of the axioms of Robinson's arithmetic Q. Let $\theta(v)$ be the formula

$$\tau \wedge \neg(\exists y)\mathbf{T}_1(v, v, y),$$

where $\mathbf{T}_1$ is the numeralwise representation in Q of the Kleene T-predicate. Put $f(e) = \ulcorner\theta(\Delta e)\urcorner$.

Suppose $\phi_e(e)\downarrow$. Then $(\exists y)T_1(e, e, y)$; so $\mathsf{Q} \vdash (\exists y)\mathbf{T}_1(\Delta e, \Delta e, y)$. Then $\theta(\Delta e)$ is an inconsistent sentence, so $\mathrm{ConSeq}(\theta(\Delta e))$ is decidable: it contains the codes of all sentences in the language of arithmetic. Hence $f(e) \in A$.

Now suppose $\phi_e(e)\uparrow$. Then $\neg(\exists y)T_1(e, e, y)$, so that $\theta(\Delta e)$ is a sound extension of Q. It follows that $\mathrm{ConSeq}(\theta(\Delta e))$ is undecidable, so $f(e) \notin A$.

Problem 5.5. *Let $\mathcal{L}$ be the language consisting of the logical symbols and a binary relation symbol R. We view models of $\mathcal{L}$ as graphs, with the universe of the model determining the nodes, and the interpretation of R determining the edges. Is there a theory T in $\mathcal{L}$ such that (for every model $\mathfrak{A}$ of $\mathcal{L}$) $\mathfrak{A} \models T$ iff $\mathfrak{A}$ is* connected *as a graph? (Prove your answer.)*

Solution. No. Suppose T is a theory so that if $\mathfrak{A}$ is an $\mathcal{L}$-structure that is connected as a graph, then $\mathfrak{A} \models T$. We show the existence of an $\mathcal{L}$-structure $\mathfrak{B}$ so that $\mathfrak{B}$ is not connected as a graph, but $\mathfrak{B} \models T$.

Let $\mathcal{L}'$ be the language $\mathcal{L}$ with fresh constants c, d adjoined. Let Σ be the $\mathcal{L}'$-theory

$$\{c \neq d \wedge \neg cRd\} \cup \{\neg(\exists v_1)(\exists v_2)\dots(\exists v_n)(cRv_1 \wedge v_1Rv_2 \wedge \dots \wedge v_nRd) \mid n \in \mathbb{N}\}.$$

So Σ is a collection of $\mathcal{L}'$-sentences expressing "there are distinct points c and d that are not connected by any path of length n, for all natural numbers n". The theory $T \cup \Sigma$ is finitely satisfiable, since for each n, there is a connected graph containing two vertices not joined by a path of length $\leq n$. By compactness, there is a model $\mathfrak{B}$ of $T \cup \Sigma$. Then the reduct $\mathfrak{B} \restriction \mathcal{L}$ models T, but is not connected as a graph, since $c^{\mathfrak{B}}$ and $d^{\mathfrak{B}}$ are not joined by any path.

Problem 5.6. *Assume the CH (but not the GCH). Prove that ${\omega_n}^{\omega} = \omega_n$ for every natural number $n \geq 1$.*

Solution. By CH, $\omega_1 = \omega^\omega$. Then

$${\omega_1}^{\omega} = (\omega^\omega)^\omega = \omega^{\omega\cdot\omega} = \omega^\omega = \omega_1.$$

Now assume the result holds for ω_{n-1} for some $n > 1$. Every function $f : \omega \to \omega_n$ is bounded in ω_n, since successor cardinals are regular. Then any $f \in {}^\omega\omega_n$ is in ${}^\omega\alpha$ for some $\alpha < \omega_n$. It follows that

$${\omega_n}^{\omega} = \mathrm{card}\left(\bigcup_{\alpha<\omega_n} {}^\omega\alpha\right) \leq \omega_n \cdot \sup_{\alpha<\omega_n} \mathrm{card}({}^\omega\alpha) = \omega_n \cdot {\omega_{n-1}}^{\omega} = \omega_n \cdot \omega_{n-1} = \omega_n.$$

Note that we used the inductive hypothesis in the second to last equality. Since clearly $\omega_n \leq {\omega_n}^{\omega}$, we have ${\omega_n}^{\omega} = \omega_n$. By induction, the claim follows.

Problem 5.7. *Prove that there is a Π_2 sentence φ which is true in L_{ω_1} but false in L.*

It is unlikely that such a sentence can be exhibited to have the desired properties in ZFC alone. In particular, a result of Silver and others implies that if κ is a measurable cardinal, then $L_{\omega_1} \prec L_\kappa$. It follows that $L_{\omega_1} \models \mathsf{ZFC}+V=L$. Now, the formulas φ such that $\mathsf{ZFC} \vdash \varphi^L$ are exactly the consequences of $\mathsf{ZFC}+V=L$; hence if we show φ^L in ZFC, the existence of a measurable cardinal implies $L_{\omega_1} \models \varphi$.

To avoid this difficulty, we need to assume that $V = L$.

Solution. Let φ be the sentence expressing "all sets are countable", i.e.

$$\varphi \equiv (\forall x)(\exists f) f : x \to \omega \text{ is an injection.}$$

Clearly φ is false in L since one can prove in ZFC the existence of uncountable sets. To show φ is true in $\mathcal{L}_{\omega_1}$, it is sufficient to show

$$(\forall x \in L_{\omega_1})(\exists f \in L_{\omega_1}) f : x \to \omega \text{ is an injection,}$$

by absoluteness of the notions ω and injective function for transitive models of $\mathsf{ZF} - \mathsf{Powerset}$.

So suppose $x \in L_{\omega_1}$; then $x \in L_\xi$ for some infinite $\xi < \omega_1$, and x is countable. Let $f : x \to \omega$ be an injection. By $V = L$, $f \in L_\kappa$ for some regular uncountable κ. By Löwenheim-Skolem, there exists $H \prec L_\kappa$ so that $|H| = \omega$, $f \in H$, and $L_\xi \subset H$. Then by elementarity,

$$H \models f : x \to \omega \text{ is injective.}$$

Let $\pi : H \to M$ be the Mostowski collapse embedding. Then M is a countable transitive model of $\mathsf{ZF} - \mathsf{Powerset} + V = L$, so by condensation, $M = L_\alpha$ for some countable α. Applying the isomorphism, we have

$$L_\alpha \models \pi(f) : \pi(x) \to \pi(\omega) \text{ is injective.}$$

But since $L_\xi \subset H$, we have that π is the identity on L_ξ; in particular, $\pi(x) = x$ and $\pi(\omega) = \omega$, so that

$$L_\alpha \models \pi(f) : x \to \omega \text{ is injective.}$$

Then by absoluteness, $\pi(f)$ is an injection from x to ω, and since $\pi(f) \in \mathcal{L}_\alpha \subset L_{\omega_1}$, this proves $L_{\omega_1} \models \varphi$.

It remains to show φ is Π_2. Indeed, as stated, it may not be. However, the sentence

$$(\forall x)(\forall z)(\exists e)(\exists f)[(\forall u \in e)(u \neq u) \wedge (f \text{ is a function from } x \text{ to } z)$$
$$\wedge\, [(e \in z) \wedge (\forall u \in z)(S(u) \in z)] \to (f \text{ is injective})]$$

is Π_2; the notions of successor $S(u)$, injective function, etc. are all Δ_0. This sentence says "for all sets x and z, there exists a function $f : x \to z$ so that if z is inductive (that is, z contains the empty set and is closed under the successor operation) then f is an injection". This is clearly equivalent in $\mathsf{ZF} - \mathsf{Powerset}$ to our φ, which completes the proof.

Problem 5.8. *Let $\mathcal{L}$ be the language of arithmetic, and let $\mathfrak{N}$ be the standard model of arithmetic. Let $\langle \theta_n \mid n < \omega \rangle$ be some standard enumeration of the sentences of $\mathcal{L}$.*

(a) *Recall that the fixed point lemma states that for every formula $\varphi(v)$ of $\mathcal{L}$ there exists a number n so that $\mathfrak{N} \models (\theta_n \leftrightarrow \varphi(\underset{\sim}{\mathbf{n}}))$. Sketch a proof of this lemma.*

(b) *Prove that truth is not definable in $\mathfrak{N}$. More precisely prove that there is no formula $\tau(v)$ in $\mathcal{L}$ with the property that $\mathfrak{N} \models \theta_n$ iff $\mathfrak{N} \models \tau(\underset{\sim}{\mathbf{n}})$.*

Solution. (a) Let $\mathbf{Sub}(x, m, n)$ numeralwise express in $\mathcal{L}$ the relation that x is the code of a formula $\chi(v)$ with one free variable v, such that $\chi(\underset{\sim}{\mathbf{m}})$ is the sentence θ_n in our standard enumeration. Put

$$\psi(v) \equiv (\exists n)[\mathbf{Sub}(v, v, n) \; \& \; \varphi(n)].$$

Let m be the code of the formula $\psi(v)$, and fix n so that θ_n is the sentence $\psi(\underset{\sim}{\mathbf{m}})$. We have

$$\begin{aligned}
\mathfrak{N} \models \theta_n &\iff \mathfrak{N} \models \psi(\tilde{\mathbf{m}}) \\
&\iff \mathfrak{N} \models (\exists n)[\mathbf{Sub}(\tilde{\mathbf{m}}, \tilde{\mathbf{m}}, n) \; \& \; \varphi(n)] \\
&\iff m \text{ codes a formula } \chi(v) \text{ so that } \theta_n \text{ is } \chi(\tilde{\mathbf{m}}), \text{ and } \mathfrak{N} \models \varphi(\tilde{\mathbf{n}}) \\
&\iff \mathfrak{N} \models \varphi(\tilde{\mathbf{n}}).
\end{aligned}$$

Hence $\mathfrak{N} \models (\theta_n \leftrightarrow \varphi(\underset{\sim}{\mathbf{n}}))$.

(b) Suppose $\tau(v)$ was such a sentence. Then by the fixed point lemma applied to $\neg\tau(v)$, there is a number n so that $\mathfrak{N} \models (\theta_n \leftrightarrow \neg\tau(\underset{\sim}{\mathbf{n}}))$. But then $\mathfrak{N} \models \theta_n$ iff $\mathfrak{N} \models \tau(\underset{\sim}{\mathbf{n}})$ iff $\mathfrak{N} \models \neg\theta_n$, a contradiction.

6. Fall 2004

Problem 6.1. *Let $\mathcal{L}$ contain a two-place relation symbol P. Let $\mathfrak{A}$ be an infinite model for $\mathcal{L}$ such that $P_{\mathfrak{A}}$ is an equivalence relation. Prove that at least one of the following holds:*

(i) *There is some positive integer n such that there are infinitely many n-element equivalence classes of $P_{\mathfrak{A}}$.*
(ii) *There is an elementary extension $\mathfrak{B}$ of $\mathfrak{A}$ such that $P_{\mathfrak{B}}$ is an equivalence relation with an infinite equivalence class.*

Solution. Suppose $\mathfrak{A}$ doesn't satisfy (i). We show (ii) holds. If $\mathfrak{A}$ has an infinite $P_{\mathfrak{A}}$-equivalence class, (ii) holds with $\mathfrak{B} = \mathfrak{A}$. So suppose all $P_{\mathfrak{A}}$-equivalence classes are finite; then since (i) fails, there are $P_{\mathfrak{A}}$-equivalence classes of arbitrarily large finite size.

Now, let $\mathcal{L}'$ be the language $\mathcal{L}(A) = \{c_a \mid a \in A\}$ expanded further by a fresh constant d. Define for each $n \in \omega$ the $\mathcal{L}'$-sentence

$$\phi_n \equiv \exists u_1 \dots u_n \left(\bigwedge_{1 \le i < j \le n} u_i \neq u_j \right) \wedge \left(\bigwedge_{1 \le k \le n} P(u_k, d) \right).$$

That is, ϕ_n asserts that d belongs to a P-equivalence class of size at least n. Put

$$T = \text{EDiagram}(\mathfrak{A}) \cup \{\phi_n\}_{n \in \omega}.$$

Then T is finitely satisfiable: for suppose T_0 is a finite subset of T, and that n is the largest index so that $\phi_n \in T_0$. Then the expansion of $\mathfrak{A}$ to $\mathcal{L}'$ which interprets each c_a by a, and d by a member of A that belongs to a $P_{\mathfrak{A}}$-equivalence class of size at least n, is a model of T_0.

By compactness, T has a model, $\mathfrak{B}$. Since $\mathfrak{B} \models \phi_n$ for all n, $d_{\mathfrak{B}}$ belongs to an infinite $P_{\mathfrak{B}}$-equivalence class. Since $\mathfrak{B} \models \text{EDiagram}(\mathfrak{A})$, there is an $\mathcal{L}$-structure $\mathfrak{B}' \succeq \mathfrak{A}$ isomorphic to the reduct $\mathfrak{B} \restriction \mathcal{L}$, so that $\mathfrak{B}'$ has an infinite $P_{\mathfrak{B}'}$-equivalence class. This proves (ii).

Problem 6.2. *Let T be a consistent theory in a countable language containing a two place predicate symbol $<$. Assume that $T \vdash$ "$<$ is a linear ordering of the universe." Assume also that T has no model of size $\aleph_1$ in which each point has only countably many $<$-predecessors. Prove that there is a countable model $\mathfrak{A}$ of T such that, for all $\mathfrak{B} \succ \mathfrak{A}$,*

$$(\exists a \in A)(\exists b \in B \setminus A)\, b <_{\mathfrak{B}} a.$$

Solution. Suppose not. Then for each countable model $\mathfrak{A}$ of T, there exists a proper elementary extension $\mathfrak{B} \succ \mathfrak{A}$ so that

$$(\forall a \in A)(\forall b \in B \setminus A)\, a <_{\mathfrak{A}} b.$$

That is, every countable model of T has a proper elementary end extension; applying downward Löwenheim-Skolem, we can furthermore assume this end extension is countable.

We define an elementary chain $\{\mathfrak{A}_\alpha\}_{\alpha<\omega_1}$ as follows. Let $\mathfrak{A}_0$ be a countable model of T. For each $\mathfrak{A}_\alpha$, let $\mathfrak{A}_{\alpha+1}$ be some countable elementary end extension of $\mathfrak{A}_\alpha$ as guaranteed above. For limit $\lambda < \omega_1$, let $\mathfrak{A}_\lambda = \bigcup_{\alpha<\lambda} \mathfrak{A}_\alpha$; note that $\mathfrak{A}_\lambda$ is then the limit structure of a countable elementary chain of countable models, so is countable.

Now, let $\mathfrak{A} = \bigcup_{\alpha<\omega_1} \mathfrak{A}_\alpha$. Then $\mathfrak{A}$ is a model of T of size $\aleph_1$. For each $a \in A$, there is some $\alpha < \omega_1$ so that $a \in A_\alpha$; by construction, $\mathfrak{A}$ is an end extension of $\mathfrak{A}_\alpha$, so a has only countably many $<$-predecessors. We have shown the existence of a model $\mathfrak{A}$ of T of size $\aleph_1$ in which each point has only countably many $<$-predecessors, contrary to our assumption about T. It follows that some model of T exists that has no proper elementary end extension, as claimed.

Problem 6.3. *A* sound interpretation *of Peano arithmetic into a theory T (in any language) is a recursive function $\theta \mapsto \theta^*$ on the sentences of PA to the sentences of T which satisfies the following properties, for every sentence θ in the language of PA:*

1. *If $PA \vdash \theta$, then $T \vdash \theta^*$.*
2. *If $T \vdash \theta^*$, then θ is true.*
3. *$(\neg\theta)^* \equiv \neg\theta^*$.*

Prove that if T is axiomatizable and there exists a sound interpretation of PA into T, then T is incomplete.

Hint. *Use the Fixed Point Lemma in Peano Arithmetic.*

Solution. Let $\mathbf{Form}(x)$ represent in PA the relation that says x is the code of a formula in PA; let $\mathbf{I}(x, y)$ represent the sound interpretation of PA into T (so that in particular $PA \vdash \mathbf{I}(\ulcorner\theta\urcorner, \ulcorner\theta^*\urcorner)$ for all sentences θ in PA); and let $\mathbf{Prov}_T(y)$ be the formula in PA representing the provability predicate for T (this exists because T is axiomatizable). Let $\varphi(x)$ be the formula

$$\mathbf{Form}(x) \wedge (\exists y)(\mathbf{I}(x, y) \wedge \mathbf{Prov}_T(y)).$$

Then $PA \vdash \varphi(\ulcorner\theta\urcorner)$ iff $T \vdash \theta^*$, for all formulas θ in PA.

By the Fixed Point Lemma, there is a sentence σ in the language of PA so that

$$PA \vdash \sigma \leftrightarrow \neg\varphi(\ulcorner\sigma\urcorner).$$

We claim that neither σ^* nor $\neg\sigma^*$ is a theorem of T. For if $T \vdash \sigma^*$, then $PA \vdash \varphi(\ulcorner\sigma\urcorner)$ by definition of $\varphi(x)$. By choice of σ, $PA \vdash \neg\sigma$, so that by (1), $T \vdash (\neg\sigma)^*$. Then by (2), both σ and $\neg\sigma$ are true, a contradiction.

Now suppose $T \vdash \neg\sigma^*$. Then by (3), $T \vdash (\neg\sigma)^*$, so that $\neg\sigma$ is true by (2). Since $\neg\sigma$ is equivalent in PA to a true Σ_1 sentence (namely $\varphi(\ulcorner\sigma\urcorner)$), we have $PA \vdash \neg\sigma$. Then by choice of σ, $PA \vdash \varphi(\ulcorner\sigma\urcorner)$, so that $T \vdash \sigma^*$. But then by (2), σ and $\neg\sigma$ are both true, a contradiction.

Problem 6.4. *Prove that the relation*

$$P(e,m) \iff W_e = W_m$$

is Π_2 *but not* Σ_2*.*

Solution. We have $W_e = W_m$ iff for all x, if $\varphi_e(x)$ converges, then so does $\varphi_m(x)$, and vice versa. This is expressed by

$$(\forall x)(\forall y)(\exists w)[(T_1(e,x,y) \to T_1(m,x,w))\&(T_1(m,x,y) \to T_1(e,x,w))],$$

so $P(e,m)$ is a Π_2 relation.

To show $P(e,m)$ is not Σ_2, we show it is Π_2-complete. Suppose A is Π_2, so that there is some recursive relation $Q(u,v,w)$ with

$$A = \{w \mid (\forall u)(\exists v)Q(u,v,w)\}.$$

Let k be an index so that $W_k = \mathbb{N}$. Let g be the partial recursive function defined by

$$g(w,u) = \mu v\, Q(u,v,w).$$

We claim the map $w \mapsto \langle S^1_1(\hat{g},w),k\rangle$ is a many-one reduction of A to the relation $P(e,m)$. For

$$\begin{aligned} P(S^1_1(\hat{g},w),k) &\iff W_{S^1_1(\hat{g},w)} = W_k = \mathbb{N} \\ &\iff (\forall u)\varphi_{S^1_1(\hat{g},w)}(u)\downarrow \\ &\iff (\forall u)g(w,u)\downarrow \\ &\iff (\forall u)(\exists v)Q(u,v,w) \\ &\iff w \in A. \end{aligned}$$

Thus the relation $P(e,m)$ is Σ_2-complete, hence not Π_2.

Problem 6.5. (a) *Prove that if* $f : \mathbb{N} \to \mathbb{N}$ *is a total, recursive function, then there is a number* z *such that* $W_z = \{f(z)\}$*.*

(b) *Prove that there is a number* z *such that* $\varphi_z(z)\downarrow$ *and* $W_z = \{\varphi_z(z)\}$*.*

(c) *Suppose that* $f : \mathbb{N} \to \mathbb{N}$ *is a (total) recursive function. Prove that there is a number* z *such that either* $f(z) = z$ *and* $W_z = \{0\}$*, or* $f(z) \neq z$ *and* $W_z = \{1\}$*.*

Solution. (a) Define

$$g(x,y) = \begin{cases} 1 & \text{if } f(x) = y \\ \uparrow & \text{otherwise.} \end{cases}$$

Then by the second recursion theorem, there is a number z so that for all y, $g(z,y) = \varphi_z(y)$. We have

$$y \in W_z \iff \varphi_z(y)\downarrow \iff g(z,y)\downarrow \iff f(z) = y,$$

so that $W_z = \{f(z)\}$.

(b) Define

$$h(x,y) = \begin{cases} x & \text{if } x = y \\ \uparrow & \text{otherwise.} \end{cases}$$

By the second recursion theorem, there is a z so that for all y, $h(z,y) = \varphi_z(y)$. Then

$$y \in W_z \iff \varphi_z(y)\downarrow \iff h(z,y)\downarrow \iff h(z,y) = z = y.$$

Hence $W_z = \{z\} = \{h(z,z)\} = \{\varphi_z(z)\}$.

(c) Define

$$g(x,y) = \begin{cases} 1 & \text{if } y = 0 \text{ and } x = f(x) \\ 1 & \text{if } y = 1 \text{ and } x \neq f(x) \\ \uparrow & \text{otherwise.} \end{cases}$$

Then g is recursive, so that by the second recursion theorem, there exists some z with $\varphi_z(y) = g(z,y)$ for all y. If $f(z) = z$, then $W_z = \{y \mid g(z,y)\downarrow\} = \{0\}$; if $f(z) \neq z$, then $W_z = \{1\}$.

7. Winter 2005

Problem 7.1. *Work in* ZFC. *Prove that there is a limit ordinal α so that* (a) *α is countable; and* (b) *there is no bijection $f : \omega \to \alpha$ with $f \in L_{\alpha+1}$.*

Solution. Let $\kappa > \omega_1$ be a regular cardinal. By Löwenheim-Skolem, there is a countable $H \prec L_\kappa$ so that $\omega_1 \in H$. Let M be the Mostowski collapse of H, and π the collapse embedding. Since $M \cong H \prec L_\kappa$ with κ regular uncountable, we have that M is a countable transitive model of ZF$-$Powerset$+V = L$. By condensation, $M = L_\delta$ for some countable δ. Put $\alpha = \pi(\omega_1)$. Then $\alpha < \delta$. Furthermore,

$$L_\kappa \models \omega_1 \text{ is a limit ordinal } \wedge \neg(\exists f)(f : \omega \to \omega_1 \text{ is a bijection}),$$

and since $H \prec L_\kappa$ and $\pi : H \to L_\delta$ is an isomorphism, we have

$$L_\delta \models \alpha \text{ is a limit ordinal } \wedge \neg(\exists f)(f : \omega \to \alpha \text{ is a bijection}).$$

By absoluteness of the notions of limit ordinal and bijective function, we have that there is no bijection $f : \omega \to \alpha$ with $f \in L_\delta$; since $L_{\alpha+1} \subset L_\delta$, this proves the claim.

Problem 7.2. *Let $M \subseteq N$ be transitive models of* ZFC. *Let $r \in M$ be a relation and suppose that* (r is wellfounded)^M. *Prove that* (r is wellfounded)^N.

Solution. Let dom(r) denote the domain of the relation r, i.e. $\mathrm{dom}(r) = \bigcup\bigcup r$. We first prove in ZFC that r is a wellfounded relation iff there exists an ordinal α and a bijective function $f : \alpha \to \mathrm{dom}(r)$ such that for all $\beta, \gamma \in \alpha$, $\beta \in \gamma$ implies $f(\gamma) \not r\, f(\beta)$.

Suppose the latter condition holds. If $x \subset \mathrm{dom}(r)$ is nonempty, fix $w \in x$ so that $f^{-1}(w)$ is $\in$-minimal in $\{f^{-1}(u) \mid u \in x\}$. Then w is r-minimal in x, so r is wellfounded.

Now suppose r is wellfounded. Let z be some set with $z \notin \mathrm{dom}(r)$. Using choice, define by transfinite induction a class function F on ordinals so that for all ordinals β,

- $F(\beta)$ is r-minimal in $\mathrm{dom}(r) \setminus \{F(\delta) \mid \delta \in \beta\}$ if this is nonempty;
- $F(\beta) = z$ otherwise.

Let $\alpha = \{\beta \mid F(\beta) \in \mathrm{dom}(r)\}$, and let f be the restriction of f to α. Then if $\beta \in \gamma \in \alpha$, we have that $f(\beta)$ is r-minimal in $\mathrm{dom}(r) \setminus \{f(\delta) \mid \delta \in \beta\}$; in particular, $f(\gamma) \not r\, f(\beta)$, so this f and α have the desired properties.

Now we have

$$\begin{aligned}
(r \text{ is wellfounded})^M &\iff (\exists \alpha \in M)(\exists f \in M)(f : \alpha \to \mathrm{dom}(r) \text{ is a bijection,}\\
&\qquad \alpha \text{ is an ordinal, and } \beta \in \gamma \in \alpha \to f(\gamma) \not r\, f(\beta))^M\\
&\implies (\exists \alpha \in N)(\exists f \in N)(f : \alpha \to \mathrm{dom}(r) \text{ is a bijection,}\\
&\qquad \alpha \text{ is an ordinal, and } \beta \in \gamma \in \alpha \to f(\gamma) \not r\, f(\beta))^N\\
&\iff (r \text{ is wellfounded})^N.
\end{aligned}$$

Here the forward implication holds because $M \subseteq N$, and because of the absoluteness of the notions of ordinal, bijective function, etc. for transitive models of ZFC.

8. Fall 2005

Problem 8.1. *A group $\mathfrak{G} = (G; \cdot, e)$ is* torsion*, if for every $g \in G$ there exists some $n \in \mathbb{N}$ such that $g^n = e$, where e is the identity and $g^n = g \cdot g \dots g$ (n-times); $\mathfrak{G}$ is of* unbounded rank *if there is no single n such that $g^n = e$ for all $g \in G$.*

1a. *Prove that every countable torsion group $\mathfrak{G}$ of unbounded rank has a countable, elementary extension which is not torsion.*

1b. *Prove that there is no first order theory in the language of groups whose models are exactly the torsion groups.*

Solution. 1a. Let $\mathfrak{G}$ be a countable torsion group of unbounded rank. Let $\mathcal{L}$ be the language of groups expanded by a set $\{a_g \mid g \in G\}$ of constant symbols for each element of G, with an additional fresh constant c. Define the $\mathcal{L}$-theory

$$T = \mathrm{EDiagram}(\mathfrak{G}) \cup \{c^n \neq e \mid n \in \mathbb{N}\}.$$

Notice that T is finitely satisfiable: if T_0 is a finite subset, let m be the largest integer so that $c^m \neq e$ appears in T_0; then T_0 is satisfied by expanding $\mathfrak{G}$ to an $\mathcal{L}$-structure interpreting c as any $g \in G$ so that for all positive $n \leq m$, $g^n \neq e$; such g exists because $\mathfrak{G}$ is of unbounded rank. By compactness, T is satisfiable; let $\mathfrak{H}$ be a countable model of T. Then the reduct of $\mathfrak{H}$ to the language of groups is isomorphic to an elementary extension of $\mathfrak{G}$ which is not torsion, since $(c^{\mathfrak{H}})^n \neq e$ for all $n \in \mathbb{N}$.

1b. Suppose Σ was such a theory. Let $\mathfrak{G}$ be a countable torsion group of unbounded rank (for example, $\mathfrak{G}$ could be the group S_∞ consisting of all finitely supported permutations of $\mathbb{N}$). Then $\mathfrak{G} \models \Sigma$. By 1a, there is an elementary extension of $\mathfrak{G}$ that is not torsion; but then this is a model of Σ that is not torsion, a contradiction.

Problem 8.2. *Recall that a model is* atomic *if the only [complete] types realized in the model are principal.*

Show that if a countable complete theory T has only countably many non-isomorphic countable models, then it has an atomic model.

Solution. This is immediate by the theorem for omitting countably many types; the hypothesis implies that T has at most countably many types, so we simply let $\mathfrak{A}$ be a model of T omitting all of those types that are nonprincipal.

In more detail, let $\{\mathfrak{M}_i \mid i \in \omega\}$ be an enumeration of countable models of T, so that every countable model of T is isomorphic to $\mathfrak{M}_i$ for some i. A countable model can realize only countably many types: so for each i, let $\mathcal{F}_i = \{\Phi_{i,j}(\vec{v}) \mid j \in \omega\}$ be an enumeration of all nonprincipal types realized in $\mathfrak{M}_i$.

By the theorem for omitting countably many types, there is a model $\mathfrak{A}$ omitting all types in $\bigcup_{i\in\omega}\mathcal{F}_i$. Now, any nonprincipal n-type $\Phi(\vec{v})$ of T is realized in some countable model $\mathfrak{M}$ of T. There is some $i \in \omega$ so that M_i is isomorphic to M; so $\Phi(\vec{v})$ is realized in M_i, hence is $\Phi_{i,j}(\vec{v})$ for some $j \in \omega$, hence is omitted by $\mathfrak{A}$. Thus $\mathfrak{A}$ is atomic.

Problem 8.3. *For each natural number n, $\Delta n = S^n(0)$ is the* numeral *which denotes n in the standard (and every) model of Peano Arithmetic,* PA.

Prove that there is a formula $\phi(v)$ in the language of PA *with just one free variable v, such that*

(1) *For every $n \in \mathbb{N}$,* $\mathrm{PA} \vdash \phi(\Delta n)$.
(2) $\mathrm{PA} \nvdash (\forall v)\phi(v)$.

(Here $\phi(\Delta n)$ is the sentence resulting from $\phi(v)$ by replacing v in all its free occurrences by the numeral Δn.)

Solution. Let $\mathbf{Proof}_{\mathrm{PA}}(x,y)$ be the formula in the language of PA expressing "x is the code of a proof in PA of the sentence coded by y". Let

$$\phi(v) \equiv \neg\mathbf{Proof}_{\mathrm{PA}}(v, \ulcorner 0=1 \urcorner).$$

By consistency of PA, no n codes a proof of 0=1 from PA, so that $\phi(\Delta n)$ is a true Δ_0 sentence. It follows that $\mathrm{PA} \vdash \phi(\Delta n)$ for all $n \in \mathbb{N}$. On the other hand, $(\forall v)\phi(v)$ expresses consistency of PA, so that by Gödel's second incompleteness theorem, $\mathrm{PA} \nvdash (\forall v)\phi(v)$.

Problem 8.4. *For each of the following statements, determine whether it is true or false and prove your answer.* Caution*: Two of the statements are reformulations of standard results and one of them is completely trivial.*

4a. *If $A \subseteq B \subseteq \mathbb{N}$, A is recursively enumerable and B is Π^0_1, then there exists a recursive set C such that $A \subseteq C \subseteq B$.*

4b. *If $A \subseteq B \subseteq \mathbb{N}$, A is Π^0_1 and B is recursively enumerable, then there exists a recursive set C such that $A \subseteq C \subseteq B$.*

4c. *If $A \subsetneq B \subseteq \mathbb{N}$, and both A and B are recursively enumerable, then there exists a recursive set C such that $A \subseteq C \subseteq B$.*

Solution. 4a. False. Let A and B be sets so that A and B^c are recursively inseparable r.e. sets; then B is Π^0_1, and there is no recursive set C with $A \subseteq C$ and $B^c \subseteq C^c$ (equivalently, $C \subseteq B$).

For an example of such recursively inseparable r.e. sets, put

$$A = \{x \mid \phi_x(x) = 0\}, \qquad D = \{x \mid \phi_x(x) = 1\}.$$

Then A and D are disjoint r.e. sets. Suppose C is recursive and that $A \subseteq C$, $D \subseteq C^c$. Then χ_C is ϕ_e for some e. If $e \in C$, then $\phi_e(e) = 1$, so $e \in D$, contradicting $D \subseteq C^c$; and if $e \notin C$ then $\phi_e(e) = 0$ and $e \in A$, contradicting $A \subseteq C$. Hence no recursive C can separate A and D.

4b. True, by the separation property for r.e. complements. We may prove it directly as follows: to decide if $x \in C$, examine successive y to see if y witnesses a proof that $x \in A^c$ or $x \in B$. If the least such y is a witness to $x \in B$, designate $x \in C$; otherwise, let $x \in C^c$. Formally, we may fix indices a and b so that $W_a = A^c$ and $W_b = B$, and define χ_C by

$$\chi_C(x) = \begin{cases} 1 & \text{if } T_1(b, x, \mu y[T_1(a,x,y) \vee T_1(b,x,y)]) \\ 0 & \text{otherwise.} \end{cases}$$

Since A^c and B are r.e. sets with $A^c \cup B = \mathbb{N}$, we have that C is a recursive set, and clearly it is the case that $A \subseteq C \subseteq B$.

4c. False. Let B be a nonrecursive r.e. set (such as $\{x \mid \phi_x(x)\downarrow\}$). Let A be B with a single point removed. Then A is nonrecursive r.e., $A \subsetneq B$, but if $A \subseteq C \subseteq B$, then C is one of A and B, so is nonrecursive.

Problem 8.5. *Assume that the Kleene (computation) predicate $T_1(e,x,y)$ is defined in some natural way, so that it has the standard properties:*

(a) *$T_1(e,x,y)$ is (primitive) recursive.*
(b) *If we set $\varphi_e(x) = U(\mu y T_1(e,x,y))$, then $\varphi_0, \varphi_1, \ldots$ enumerates all the unary, recursive partial functions.*
(c) *For all e, x,*

$$\textit{if } T_1(e,x,y), \textit{ then } x < y.$$

(The last property (c) *holds because the computation which establishes that $\varphi_e(x) = w$ for some w has larger code than the input x.) As usual, we set*

$$W_e = \{n \mid \varphi(x)\downarrow\}.$$

A number e is self-verifying *if for all x, y,*

$$\textit{if } T_1(e,x,y), \textit{ then } y \in W_e.$$

Classify in the arithmetical hierarchy the set

$$A = \{e \mid e \text{ is self-verifying}\}.$$

Solution. We have that e is self-verifying iff

$$(\forall x)(\forall y)(\exists z)[T_1(e,x,y) \to T_1(e,y,z)],$$

so A is Π_2. To show it is not Σ_2, we show it is Π_2-complete, by providing a reduction of a known Π_2-complete set to A.

Put

$$B = \{e \mid \varphi_e \text{ is total}\} = \{e \mid (\forall x)(\exists y)T_1(e,x,y)\}.$$

So B is Π_2-complete. Define

$$g(e,x) = \mu y\,(\forall u < x)(\exists v < y)T_1(e,u,v).$$

Then g is partial recursive. Let $f(e) = S^1_1(\hat{g}, e)$. Then f is total recursive, and we claim it reduces B to A.

Suppose $e \in B$. Then φ_e is total, so that $g(e,x) = \varphi_{f(e)}(x)$ converges for all x. Since $\varphi_{f(e)}$ is total, it follows that $f(e)$ is self-verifying.

Suppose $e \notin B$. Then φ_e is not total; let x_0 be the least x so that $\varphi_e(x)$ diverges. Then by our definition of g, $g(e,x_0) = \varphi_{f(e)}(x_0)$ converges, and $\varphi_{f(e)}(x)$ diverges for all $x > x_0$. Then there is some y so that $T_1(f(e),x_0,y)$ holds; but by (c) above, $y > x_0$, so that $y \notin W_{f(e)}$. It follows that $f(e)$ is not self-verifying.

Problem 8.6. *Prove that for each ordinal number α, the structure $(\alpha, \in)$ is* rigid*, i.e., there is no bijection $\pi : \alpha \to \alpha$ (other than the identity) such that*

$$\beta \in \gamma \iff \pi(\beta) \in \pi(\gamma) \qquad (\beta, \gamma \in \alpha).$$

Solution. Suppose $\pi : \alpha \to \alpha$ was such a bijection. Let β be the $\in$-least ordinal in α so that $\pi(\beta) \neq \beta$. If $\pi(\beta) \in \beta$, then $\pi(\pi(\beta)) \in \pi(\beta)$, contradicting minimality of β. Then $\beta \in \pi(\beta)$. But in this case, $\pi^{-1}(\beta) \in \pi^{-1}(\pi(\beta)) = \beta$, which again contradicts minimality of β, since $\pi(\pi^{-1}(\beta)) \neq \pi^{-1}(\beta)$.

Problem 8.7. *For this problem, assume that every set is constructible, $V = L$.*

Recall that a set of ordinals $\alpha \subseteq \omega_1$ below the first uncountable ordinal is stationary *if it intersects every closed, unbounded subset of ω_1. Let*

$$A = \{\alpha < \omega_1 \mid L_\alpha \models \text{“there exists an uncountable cardinal”}\}.$$

7a. *Prove that A is non-empty, and, in fact, unbounded in ω_1.*

7b. *Prove that A is not stationary.*

Solution. 7a. Fix $\beta < \omega_1$. Let κ be a regular uncountable cardinal so that $L_\kappa \models$ “there exists an uncountable cardinal”. By downward Löwenheim-Skolem, let $H \prec L_\kappa$ be a countable elementary submodel so that $\beta \subset H$. Let $\pi : H \to M$ be the Mostowski collapse embedding. Then M is a countable transitive model of ZFC$-$Powerset$+V = L$, so by condensation, $M = L_\alpha$ for some countable α. Furthermore, $\beta \subset L_\alpha$ since the Mostowski collapsing function is the identity map on transitive sets; so $\beta < \alpha$. Now, since $L_\alpha \cong H \prec L_\kappa$, we have that $L_\alpha \models$ “there exists an uncountable cardinal”, so that $\alpha \in A$. Since $\beta < \omega_1$ was arbitrary, we have that A is unbounded in ω_1.

7b. Recall that $L_{\omega_1} \models$ “all ordinals are countable” (for a proof of this fact, see the solution to Winter 2004 Problem 7). Inductively define an elementary chain $\{H_\alpha \mid \alpha < \omega_1\}$ so that for each α, $\alpha \subset H_\alpha$ and $H_\alpha \prec H_{\alpha+1} \prec L_{\omega_1}$; and put $H_\lambda = \bigcup_{\alpha<\lambda} H_\alpha$, for limit $\lambda < \omega_1$. Finally, let $\pi : H_\alpha \to M_\alpha$ be the collapse embedding. Then by condensation, each M_α is $L_{\psi(\alpha)}$ for some countable $\psi(\alpha) > \alpha$.

Clearly for each α, $L_{\psi(\alpha)} \models$ “all ordinals are countable”; we claim

$$C = \{\psi(\alpha) \mid \alpha < \omega_1\}$$

is closed unbounded in ω_1. Clearly it is unbounded. Suppose $\{\psi(\alpha_n) \mid n \in \omega\}$ is an increasing sequence in C. Then if we let λ be $\sup_{n\in\omega} \alpha_n$, we have

$$H_\lambda = \bigcup_{\alpha<\lambda} H_\alpha = \bigcup_{n\in\omega} H_{\alpha_n},$$

by our definition of the H_α. Then by definition of the Mostowski collapse embedding,

$$L_{\psi(\lambda)} = \bigcup_{\alpha<\lambda} L_{\psi(\alpha)} = \bigcup_{n\in\omega} L_{\psi(\alpha_n)}.$$

It follows that $\sup_{n\in\omega} \psi(\alpha_n) = \psi(\lambda) \in C$, so that C is closed. Since A has empty intersection with C, A is not stationary.

Problem 8.8. Caution: *This problem is very easy if you use some basic theorems from Recursion Theory and Proof Theory (in part 8a) and from Proof Theory (in part 8b), but practically impossible to do from scratch.*

We assume that the sentences of ZFC *and* PA *have been coded in some standard way, and so that (for convenience), every natural number n is the code of some sentence θ_n^T, with $T =$* ZFC *or $T =$* PA. *(You can do this starting from any coding and setting $\theta_n \equiv (\forall x)[x = x]$ if n is not the code of a sentence.) Set*

$$T_{\text{ZFC}} = \{n \mid \text{ZFC} \vdash \theta_n^{\text{ZFC}}\}, \qquad T_{\text{PA}} = \{n \mid \text{PA} \vdash \theta_n^{\text{PA}}\}.$$

8a. *Prove that there is a recursive permutation $\pi : \mathbb{N} \to \mathbb{N}$ of the natural numbers such that*

$$n \in T_{\text{ZFC}} \iff \pi(n) \in T_{\text{PA}}.$$

8b. *Show that (the formal version of)* **8a** *cannot be proved in* ZFC.

Solution. 8a. Let $\mathbf{Prov}_{\mathrm{ZFC}}(v)$ express the proof predicate for ZFC in the language of PA with respect to the coding given above, so that PA $\vdash \mathbf{Prov}_{\mathrm{ZFC}}(\Delta n)$ iff ZFC $\vdash \theta_n^{\mathrm{ZFC}}$. Let f be the mapping

$$n \mapsto \#\mathbf{Prov}_{\mathrm{ZFC}}(\Delta n),$$

so that $\theta^{\mathrm{PA}}_{f(n)}$ is $\mathbf{Prov}_{\mathrm{ZFC}}(\Delta n)$ for all n. Then f is total recursive and injective. Now $n \in T_{\mathrm{ZFC}}$ if and only if θ_n^{ZFC} is a theorem of ZFC, which is true if and only if PA $\vdash \mathbf{Prov}_{\mathrm{ZFC}}(\Delta n)$; so $f(n) \in T_{\mathrm{PA}}$. It follows that f is a one-one reduction of T_{ZFC} to T_{PA}.

A one-one reduction of T_{PA} to T_{ZFC} is obtained in an identical fashion: simply replace every instance of ZFC in the above paragraph with PA, and vice versa.

By Myhill's isomorphism theorem, there is a recursive isomorphism $\pi : \mathbb{N} \to \mathbb{N}$ between T_{ZFC} and T_{PA}. This proves the claim.

8b. ZFC proves that $(\mathbb{N}; 0, S, +, \cdot)$ is a set model of PA, and by the completeness theorem, ZFC proves consistency of PA. Hence ZFC proves there is some sentence not provable in PA, that is, $(\exists n)\, n \notin T_{\mathrm{PA}}$. But if ZFC proves the result above, then ZFC $\vdash (\exists n)\, n \notin T_{\mathrm{ZFC}}$. But this sentence is equivalent to consistency of ZFC, so that ZFC proves its own consistency, contradicting Gödel's second incompleteness theorem.

9. Fall 2008

Problem 9.1. *An* ordering *is a structure* $(A, \leq)$ *with one binary relation* $\leq$ *which is a linear ordering of the universe* A*, and it is a* wellordering *if* $\leq$ *is well founded, i.e., if there are no infinite,* $>$*-descending chains. Put*

$$\mathcal{W} = \{(A, \leq) \mid (A, \leq) \text{ is a wellordering } \}$$
$$\mathcal{W}^c = \{(A, \leq) \mid \text{ is an ordering but not a wellordering}\}.$$

(1a) *Determine whether* $\mathcal{W}$ *is an elementary class or not and prove your answer.*
(1b) *Determine whether* $\mathcal{W}^c$ *is an elementary class or not and prove your answer.*

Solution. Neither class is elementary. Clearly the set of natural numbers with the usual order, $(\mathbb{N}, \leq)$, is in $\mathcal{W}$. We show the existence of an ordering $(A, \leq)$ that satisfies the same sentences as $(\mathbb{N}, \leq)$, but is in $\mathcal{W}^c$.

Let $\{a_n\}_{n\in\omega}$ be fresh constants. Consider the theory T in the expanded language,

$$T = \mathrm{Th}(\mathbb{N}, \leq) \cup \{(a_{n+1} \leq a_n) \wedge (a_{n+1} \neq a_n) \mid n \in \omega\}.$$

Then T is finitely satisfiable, since $\mathbb{N}$ has arbitrarily large $>$-descending chains of finite length; so by compactness, T has a model. Let $(A, \leq)$ be the reduct of this model to the language of orderings. Then A is linearly ordered by $\leq$, and contains an infinite, $>$-descending chain; so $(A, \leq)$ is in $\mathcal{W}^c$, even though $(\mathbb{N}, \leq) \equiv (A, \leq)$.

10. Fall 2009

Problem 10.1. *Let* T *be an axiomatizable extension of* PA *(in the language of* PA*), suppose* $\mathbf{Proof}_T(v, u)$ *numeralwise expresses in* PA *the relation*

$$\mathrm{Proof}_T(x, y) \iff x \text{ is the code of } \mathsf{PA}\text{-sentence } \theta \text{ and } y \text{ is the code of a proof of } \theta \text{ in } T,$$

and set

$$\mathbf{Provable}_T(v) :\equiv (\exists u)\mathbf{Proof}_T(v, u).$$

Write $\ulcorner\sigma\urcorner$ *for the numeral of the Gödel number of the sentence* σ*, in some standard Gödel number; let* $\Delta(n)$ *be the numeral of* n*.*

(a) *True or false: for any* PA*-sentence* σ*,*

$$\mathsf{PA} \vdash \mathbf{Provable}_{T\cup\{\neg\sigma\}}(\ulcorner\sigma\urcorner) \to \mathbf{Provable}_T(\ulcorner\sigma\urcorner).$$

For a fixed formula $\theta(v)$ *with just* v *free, let*

$$R_\theta(x,y) \iff y \text{ is the code of the term } \ulcorner\theta(\Delta x)\urcorner,$$

and let $\mathbf{R}_\theta(v,w)$ *numeralwise express* R_θ *in* PA*.*

(b) *True or false, for any formula* $\theta(v)$ *with just* v *free:*

$$\mathsf{PA} \vdash \mathbf{Provable}_T(\ulcorner(\forall v)\theta(v)\urcorner) \to (\forall v)(\exists w)[\mathbf{R}_\theta(v,w)\&\mathbf{Provable}_T(w)].$$

(c) *True or false, for* θ *and* $\mathbf{R}_\theta$ *as above:*

$$\mathsf{PA} \vdash (\forall v)(\exists w)[\mathbf{R}_\theta(v,w)\,\&\,\mathbf{Provable}_T(w)] \to (\forall v)\theta(v).$$

Solution. (a) True. By the deduction theorem, $\mathbf{Provable}_{T\cup\{\neg\sigma\}}(\ulcorner\sigma\urcorner)$ implies $\mathbf{Provable}_T(\ulcorner\neg\sigma \to \sigma\urcorner)$. By the contraction rule, we obtain $\mathbf{Provable}_T(\ulcorner\sigma\urcorner)$. Since the proof of the deduction theorem and the rules of the predicate calculus are formalizable within PA, this shows $\mathbf{Provable}_{T\cup\{\neg\sigma\}} \to \mathbf{Provable}_T(\ulcorner\sigma\urcorner)$ is a theorem of PA.

(b) This may be false, if one isn't careful about the formula $\mathbf{R}_\theta(v,w)$. As an example, let $\theta(v)$ be $v = v$, and let $\mathbf{P}_\theta(v,w)$ numeralwise express R_θ in PA. Let $\mathbf{R}_\theta(v,w)$ be the formula

$$\mathbf{R}_\theta(v,w) \equiv \neg\mathbf{Proof}_{\mathsf{PA}}(v,\ulcorner 0=1\urcorner)\,\&\,\mathbf{P}_\theta(v,w).$$

Then since PA is consistent, $\mathbf{R}_\theta(v,w)$ also numeralwise expresses R_θ in PA. Now, clearly $\mathsf{PA} \vdash \mathbf{Provable}_T(\ulcorner(\forall v)\theta(v)\urcorner)$, but by Gödel's second incompleteness theorem, PA does *not* prove $(\forall v)(\exists w)\mathbf{R}_\theta(v,w)$, since this sentence implies consistency of PA.

(c) False; indeed, the given sentence need not even be true, let alone provable in PA. As an example, suppose T is inconsistent, and let $\theta(v)$ be $v \neq v$. Then $(\forall v)(\exists w)[\mathbf{R}_\theta(v,w)\,\&\,\mathbf{Provable}_T(w)]$ is true, but $(\forall v)(v \neq v)$ is not.

Problem 10.2. *True or false? For every two structures* $\mathcal{A}$ *and* $\mathcal{B}$ *for the same language, there is a sentence* σ *which is valid in both* $\mathcal{A}$ *and* $\mathcal{B}$ *but not logically valid.*

Solution. True. Let ψ_n be the sentence expressing "there do not exist exactly n elements", i.e.

$$\psi_n \equiv \neg(\exists v_1)(\exists v_2)\dots(\exists v_n)(\forall u)\left[\bigwedge_{1\le i<j\le n}(v_i \neq v_j) \ \wedge\ \bigvee_{i=1}^{n}(u = v_i)\right].$$

Clearly if n is any number so that $\mathcal{A}$ and $\mathcal{B}$ don't both have n elements, then ψ_n is valid in both $\mathcal{A}$ and $\mathcal{B}$. However, no ψ_n is logically valid, since for every language $\mathcal{L}$ and all n there are structures for $\mathcal{L}$ of size n.

Problem 10.3. *Let* $\mathcal{L}$ *be a finite language and* T *a finite consistent* $\mathcal{L}$*-theory. Suppose that any two countable models of* T *are isomorphic. Show that* T *is decidable.*

Solution. We claim T is complete. For otherwise, there is some sentence σ so that $T \cup \{\sigma\}$ and $T \cup \{\neg\sigma\}$ are both consistent. By Gödel's completeness theorem, there exist countable models of both theories; but these cannot be isomorphic, contrary to assumption.

Now, since T is a finite theory in a finite language, we can outline a procedure to enumerate all proofs from T. For any sentence σ, we have either $T \vdash \sigma$ or $T \vdash \neg\sigma$. So if we follow our proof enumeration procedure, we will eventually find a proof of either σ if σ is a consequence of T, or $\neg\sigma$ if it is not. Hence T is decidable.

Problem 10.4. *Let $\mathcal{L} = \{<\}$ with a binary relation symbol $<$.*

(a) *Let $\mathcal{M} = (M, <^{\mathcal{M}})$ be an ordered set. We say that $\mathcal{M}$ can be extended to a total ordering if there is a total ordering $\prec$ of M with $<^{\mathcal{M}} \subseteq \prec$. Show that $\mathcal{M}$ can be extended to a total ordering if every finite $\mathcal{L}$-substructure of $\mathcal{M}$ can be extended to a total ordering.*

(b) *Let $\mathcal{L}^*$ be a language extending $\mathcal{L}$, and let T be a consistent $\mathcal{L}^*$-theory. Suppose that for every $\mathcal{M} \models T$, the interpretation $<^{\mathcal{M}}$ of $<$ in $\mathcal{M}$ is a well-ordering of M, that is, there are no infinite sequences $a_0 >^{\mathcal{M}} a_1 >^{\mathcal{M}} > \cdots$. Show that every model of T is finite.*

Solution. (a) Let TO be the sentence in $\mathcal{L}$ expressing "$<$ is a total ordering". Let Σ be the $\mathcal{L}(\mathcal{M})$-theory

$$\Sigma = \text{Diagram}(\mathcal{M}) \cup \{\mathsf{TO}\}.$$

Any model of Σ is (isomorphic to) an extension of $\mathcal{M}$ that satisfies TO, and so induces a total ordering $\prec$ of M extending $<^{\mathcal{M}}$. Hence we need only show Σ is satisfiable.

Let Σ_0 be a finite subset of Σ. Then the set A of elements of M whose constants appear in Σ_0 is finite. Now $\mathcal{A} = (A, <^{\mathcal{M}} \restriction A)$ is a finite $\mathcal{L}$-substructure of $\mathcal{M}$. By hypothesis, $\mathcal{A}$ can be extended to a total ordering, $(A, \prec)$. Under the obvious interpretation of the constants in M, this structure is a model of Σ_0, so Σ_0 is satisfiable. It follows that Σ is finitely satisfiable, hence satisfiable by compactness.

(b) Suppose not, and that there is an infinite model $\mathcal{M}$ of T. Adjoin to the language $\mathcal{L}^*$ countably many fresh constants $c_0, c_1, \ldots$. Let T^* be the theory in the expanded language

$$T^* = T \cup \{c_n > c_{n+1} \mid n \in \omega\}.$$

Since $\mathcal{M}$ is an infinite set well-ordered by $<^{\mathcal{M}}$, we have that for any n there exists a descending sequence $a_0 >^{\mathcal{M}} a_1 >^{\mathcal{M}} \cdots >^{\mathcal{M}} a_n$ in $\mathcal{M}$. It follows that T^* is finitely satisfiable, hence satisfiable by compactness. But the reduct of any model of T^* to $\mathcal{L}^*$ is a model of T that contains an infinite descending sequence, a contradiction. Hence every model of T must be finite.

Problem 10.5. *A topology on the class of ordinal numbers is defined by stipulating that*

$$\{\{0\}\} \cup \{(\alpha, \beta) \; : \; \alpha + 1 < \beta\}$$

is the class of basic open sets, where

$$(\alpha, \beta) = \{\gamma \; : \; \alpha < \gamma < \beta\}.$$

The topology of any ordinal is the relative topology. Determine (with proof) the ordinals which are compact in this topology.

Solution. The compact ordinals are precisely 0 and the successor ordinals. Clearly 0 is compact. If α is compact, let V be an open cover of $\alpha+1$. Then there is a finite subset V_0 of V so that V_0 covers α. Let U be an element of V with $\alpha \in V$. Then $V_0 \cup \{U\}$ is a finite subset of V that covers $\alpha+1$, so $\alpha+1$ is compact.

Now suppose λ is limit, and that $\alpha+1$ is compact for all $\alpha<\lambda$. We show $\lambda+1$ is compact. Let V be an open cover of $\lambda+1$. Then there is an open set U so that $\lambda \in U$. For some $\alpha<\lambda$, we have $(\alpha, \lambda+1) \subset U$. Now there is some finite $V_0 \subset V$ so that V_0 covers $\alpha+1$, by inductive hypothesis. Then $V_0 \cup \{U\}$ is the desired finite subcover of $\lambda+1$.

By transfinite induction, it follows that for all α, $\alpha+1$ is compact. It remains to show that no limit ordinal λ is compact. But this is obvious: for if λ is limit, then

$$\{\{0\}\} \cup \{(0, \alpha) \;:\; \alpha<\lambda\}$$

is an infinite open cover of λ that admits no finite subcover.

Problem 10.6. *Is there an ordinal number $\alpha<\omega_\omega$ such that $L_\alpha \models$ "ω_ω exists"?*

Solution. Yes. "ω_ω exists" is equivalent in $\mathsf{ZF}-\mathsf{Powerset}$ to the sentence

$$(\exists \lambda)[\lambda \text{ is an uncountable cardinal with countable cofinality}].$$

Let $f: \omega \rightarrow \omega_\omega$ be cofinal. Then if κ is any regular cardinal so that $\omega, \omega_\omega, f \in L_\kappa$, we have that $L_\kappa \models$ "ω_ω exists".

Let H be a countable elementary substructure of L_κ, and let $\pi: H \rightarrow M$ be the Mostowksi collapse embedding. Then M is a countable transitive model of $\mathsf{ZF}-\mathsf{Powerset}+V=L$, so by condensation, $M=L_\alpha$ for some countable α. Furthermore, since $L_\alpha \cong H \prec L_\kappa$, $L_\alpha \models$"ω_ω exists", as needed.

Problem 10.7. *Let κ be inaccessible. Let M be an elementary submodel of V_κ such that $\kappa \subseteq M$. Prove that $M=V_\kappa$.*

Solution. Trivially $M \subseteq V_\kappa$. Suppose for contradiction that $V_\kappa \nsubseteq M$. Then there is a $\beta<\kappa$ so that $V_\beta \nsubseteq M$. Now $V_\beta \in V_\kappa$, and if we let $V(u, v)$ define the class function $\alpha \mapsto V_\alpha$ in V_κ, then by elementarity we must have

$$M \models (\exists! X)\, V(\beta, X).$$

Again by elementarity, it must then be the case that this X is V_β, so $V_\beta \in M$.

Now, since κ is inaccessible, $V_\kappa \models \mathsf{ZFC}$; so there is an ordinal $\gamma<\kappa$ and a function $f: \gamma \rightarrow V_\beta$ in V_κ so that f is a bijection. We have

$$V_\kappa \models (\exists f)\ f: \gamma \rightarrow V_\beta \text{ is a bijection,}$$

so by elementarity, there is a function g in M so that

$$M \models g: \gamma \rightarrow V_\beta \text{ is a bijection.}$$

Then g is also a bijection in the sense of V_κ, so by absoluteness, g is really a bijection from γ to V_κ.

Suppose $x \in V_\beta$. Then for some $\delta \in \gamma$, we have $g(\delta)=x$. Now $\delta \in M$, and by elementarity,

$$M \models (\exists! y) g(\delta)=y.$$

But again by elementarity, this implies that $x \in M$. It follows that $V_\beta \subset M$, contrary to our initial assumption about β. Thus $V_\kappa \subseteq M$.

Problem 10.8. *Prove that*

$$K = \{e \ : \ e \in W_e\}$$

is one-one reducible to

$$\text{Fin} = \{e \ : \ W_e \text{ is finite}\}.$$

Solution. Define

$$g(x, y) = \begin{cases} \uparrow & \text{if } (\exists z \leq y)\, T_1(x, x, z) \\ 1 & \text{otherwise.} \end{cases}$$

Let $f(e) = S^1_1(\hat{g}, e)$. Then f is total recursive and injective. If $e \in W_e$, then for some z, we have $T_1(e, e, z)$. Then for all $y > z$, we have that $g(e, y)$ diverges; so $W_{f(e)} = W_{S^1_1(\hat{g},e)} = \{y \ : \ g(e, y)\downarrow\}$ is finite, and $f(e) \in \text{Fin}$.

Conversely, if $e \notin W_e$, then for all z, it is not the case that $T_1(x, x, z)$, so that $g(e, y) = 1$ for all y. Hence $W_{f(e)}$ is infinite, so that $f(e) \notin \text{Fin}$. Thus f is a one-one reduction of K to Fin.

11. Spring 2010

Problem 11.1. *Let* $\text{Fin} = \{e \mid W_e \text{ is finite}\}$ *be the set of (standard) codes of all recursively enumerable sets which are finite.*
(1a) *Classify* Fin *in the arithmetical hierarchy.*
(1b) *Prove that there is no recursive partial function* $f(e)$ *which gives an upper bound for the members of every finite, r.e. set, i.e., such that*

$$\text{if } W_e \text{ is finite, then } f(e)\downarrow \& (\forall x)[x \in W_e \Longrightarrow x \leq f(e)].$$

Solution. (1a) Fin is in $\Sigma_2 \setminus \Pi_2$. See Fall 2002 Problem (1a).

(1b) Suppose for contradiction that such a recursive f exists. Put

$$g(x, z) = \begin{cases} 1 & \text{if } z = \mu y\, T_1(x, x, y) \\ \uparrow & \text{otherwise.} \end{cases}$$

Then $W_{S^1_1(\hat{g},x)} = \{z \mid g(x, z)\downarrow\}$ is finite for all x, so the map $x \mapsto f(S^1_1(\hat{g}, x))$ is total recursive.

Define

$$h(x) = \begin{cases} 1 & \text{if } (\exists z \leq f(S^1_1(\hat{g}, x)))\, T_1(x, x, z) \\ 0 & \text{otherwise.} \end{cases}$$

Then h is total recursive. If $\varphi_x(x)\uparrow$, then for all z, $\neg T_1(x, x, z)$, so $h(x) = 0$.

If $\varphi_x(x)\downarrow$, then let $z = \mu y\, T_1(x, x, y)$. We have $g(x, z)\downarrow$, so $z \in W_{S^1_1(\hat{g},x)}$, whence $z \leq f(S^1_1(\hat{g}, x))$ by assumption. So $h(x) = 1$.

Thus $h(x)$ is the characteristic function of $\{x \mid \phi_x(x)\downarrow\}$; but this contradicts the recursive unsolvability of the halting problem.

Problem 11.2. *A collection* F *of subsets of* ω *has the* reduction *property if for any* $A, B \in F$*, there exist* $A^*, B^* \in F$ *so that*

$$A^* \subseteq A, \ B^* \subseteq B, \ A^* \cup B^* = A \cup B, \ \textit{and } A^* \cap B^* = \emptyset.$$

For each total recursive function f*, let*

$$A_f = \{n \mid W_{f(n)} \text{ is finite}\},$$

and let F *be the collection of sets* A_f*. Prove that* f *has the reduction property.*

Solution. First, we observe that F is the collection of Σ_2 sets. For suppose f is total recursive. Then

$$n \in A_f \iff W_{f(n)} \text{ is finite } \iff f(n) \in \text{Fin}.$$

Since Fin is Σ_2, each set A_f is Σ_2. Conversely, if A is Σ_2, then since Fin is Σ_2-complete, there is a many-one reduction f of A to Fin. Then

$$n \in A \iff f(n) \in \text{Fin} \iff W_{f(n)} \text{ is finite } \iff n \in A_f.$$

So $A = A_f \in F$.

We only need to show that the family of Σ_2 sets has the reduction property. Suppose A, B are Σ_2. Then there are recursive relations P, Q so that

$$A = \{x \mid (\exists y)(\forall z)P(x,y,z)\},$$
$$B = \{x \mid (\exists y)(\forall z)Q(x,y,z)\}.$$

Then put

$$A^* = \{x \mid (\exists y)(\exists w)(\forall z)[P(x,y,z)\&(\forall u < y)(\exists v < w)\neg Q(x,u,v)]\},$$
$$B^* = \{x \mid (\exists y)(\exists w)(\forall z)[Q(x,y,z)\&(\forall u \leq y)(\exists v < w)\neg P(x,u,v)]\}.$$

These are Σ_2, and clearly $A^* \subseteq A$ and $B^* \subseteq B$. Suppose $x \in A \cup B$. Then let y be the least number so that $(\forall z)P(x,y,z) \vee (\forall z)Q(x,y,z)$ holds. If $(\forall z)P(x,y,z)$, then $x \in A^* \setminus B^*$. Otherwise, $x \in B^* \setminus A^*$. This shows both that $A \cup B = A^* \cup B^*$, and that $A^* \cap B^* = \emptyset$. Thus A^* and B^* have the desired properties, proving that F has the reduction property.

Problem 11.3. *True or false (and you must prove your answer): there is a nonstandard model*

$$\mathfrak{N}^* = (\mathbb{N}^*, 0, 1, +^*, \cdot^*)$$

of Peano arithmetic PA such that the set of its true sentences

$$\text{True}(\mathfrak{N}^*) = \{\theta \mid \mathfrak{N}^* \models \theta\}$$

is arithmetical.

Solution. True. It is sufficient to show that there is some consistent complete theory T extending PA and a formula $\varphi(x)$ so that $\varphi(\Delta n)$ is true in the standard model of arithmetic iff n is the code of a sentence in T. Then any model $\mathfrak{N}^*$ of PA that satisfies T is necessarily nonstandard, since its truth relation is arithmetical.

Such a theory may be constructed as follows: fix a coding of the formulas of PA, and let a_0 be the least number that codes a sentence σ_0 consistent with PA. Having defined sentences $\sigma_0, \ldots, \sigma_n$ consistent with PA, let a_{n+1} be the least number greater than a_n that codes a sentence σ_{n+1} consistent with $\text{PA} \cup \{\sigma_0, \ldots, \sigma_n\}$. Then the collection $T = \{\sigma_n \mid n \in \omega\}$ is a complete consistent extension of PA.

To show the theory T is arithmetical, we essentially formalize the above construction in PA. Let $\text{CON}_{\text{PA}}(a)$ express in PA the relation that a codes a sequence of sentences $\sigma_0, \ldots, \sigma_n$ in the language of PA, so that the theory $\text{PA} \cup \{\sigma_0, \ldots, \sigma_n\}$ is consistent. Then let $\varphi(x)$ be the sentence

$$(\exists a)[\text{Seq}(a) \wedge (x = a_{\text{length}(a)-1}) \wedge$$
$$(\forall i < \text{length}(a))(a_i = \mu n\, [(\forall j < i)(a_i \neq a_j) \wedge \text{CON}_{\text{PA}}(a \restriction i^\frown\langle n\rangle)])].$$

Then $\varphi(x)$ simply expresses that x is the code of a sentence σ appearing in the construction outlined above. Thus $\sigma \in T$ iff $\varphi(\ulcorner\sigma\urcorner)$ holds in the standard model of arithmetic.

Problem 11.4. *Let*

$$\Box\theta :\equiv (\exists y)\mathbf{Proof}_{\mathrm{PA}}(\ulcorner\theta\urcorner, y), \mathbf{Proof}_{\mathrm{PA}}(\ulcorner\theta\urcorner, y),$$

where $\ulcorner\theta\urcorner$ *is the numeral of the code (Gödel number) of* θ *and* $\mathbf{Proof}_{\mathrm{PA}}(\mathsf{v}_1, \mathsf{v}_2)$ *numeralwise expresses the relation*

$$\begin{aligned}\mathrm{Proof}_{\mathrm{PA}}(e, y) \iff\ & e \text{ is the code of a sentence } \theta \\ & \text{and } y \text{ is the code of a proof of } \theta \text{ in PA.}\end{aligned}$$

We also let $\mathsf{Con}_{\mathrm{PA}}$ *be the formal sentence which expresses the consistency of* PA*,*

$$\mathsf{Con}_{\mathrm{PA}} :\equiv \neg(\exists y)\mathbf{Proof}_{\mathrm{PA}}(\ulcorner 0 = 1\urcorner, y).$$

For each of the following four sentences, determine whether or it is provable in PA*. You must prove your answers by reference to standard results).*

(i) $\mathsf{Con}_{\mathrm{PA}} \to \Box\mathsf{Con}_{\mathrm{PA}}$.
(ii) $\mathsf{Con}_{\mathrm{PA}} \to \neg\Box\mathsf{Con}_{\mathrm{PA}}$.
(iii) $\Box\mathsf{Con}_{\mathrm{PA}} \to \mathsf{Con}_{\mathrm{PA}}$.
(iv) $\neg\Box\mathsf{Con}_{\mathrm{PA}} \to \mathsf{Con}_{\mathrm{PA}}$.

Solution. (i) Not provable. Indeed, (i) is not even true: Gödel's second incompleteness theorem states that $\mathsf{Con}_{\mathrm{PA}}$ implies $\neg\Box\mathsf{Con}_{\mathrm{PA}}$.

(ii) Provable – this is just the formal version of Gödel's second incompleteness theorem, the proof of which can be formalized in PA.

(iii) Not Provable. Löb's theorem states that if PA $\vdash \Box\mathsf{Con}_{\mathrm{PA}} \to \mathsf{Con}_{\mathrm{PA}}$, then PA $\vdash \mathsf{Con}_{\mathrm{PA}}$, which would contradict Gödel's second incompleteness theorem.

(iv) Provable. In fact, $\neg\Box\phi \to \mathsf{Con}_{\mathrm{PA}}$ is provable for any sentence ϕ, since PA can prove the basic fact of predicate logic that a theory T is consistent iff some sentence is not provable from T.

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