NOTES ON MATH 290D: PARTICIPATING SEMINAR IN LOGIC: DESCRIPTIVE INNER MODEL THEORY UCLA WINTER 2011

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1. January 5 – Grigor Sargsyan Large Cardinals

Large Cardinals arise in 3 ways:

(I) *Logic.* The notion of weakly compact cardinals has to do with compactness of logics with uncountable cardinalities. Specifically, a cardinal κ is compact if the compactness theorem holds for languages of size κ (?). A cardinal is strongly compact if a logic allowing κ (or more?)-fold quantification is compact.

(II) *Combinatorics.* Large cardinals are those for which some combinatorial condition holds. Examples: Erdös, Jónsson, Ramsey, and Rowbottom cardinals.

(III) *Reflection properties.* Basically, a reflection principle asserts that if something is true (in the set theoretic universe V), it must have already been true at some "earlier stage," that is, in some fixed initial segment of the universe (V_κ for some cardinal κ).

Recall the definition of the **cumulative hierarchy** of sets:

$$V_0 = \emptyset$$
$$V_{\alpha+1} = \mathcal{P}(V_\alpha)$$
$$V_\lambda = \bigcup_{\alpha<\lambda} V_\alpha \text{ for limit ordinals } \lambda.$$

The axiom of foundation states that $V = \bigcup_{\alpha\in\mathrm{Ord}} V_\alpha$.

Example. $V_\kappa \prec_{\Sigma_{100}} V$ implies $V_\kappa \models \mathrm{ZFC}$. (Need to make sense of remark to the effect that this elementarity condition implies κ is inaccessible; maybe this is true under second-order elementarity?)

Recall the continuum hypothesis (CH) states that $2^{\aleph_0} = \aleph_1$; the generalized continuum hypothesis (GCH) states that $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ for all ordinals α. Gödel showed in 1940 that the constructible universe L is a model of CH and GCH, hence establishing the relative consistency of GCH with ZFC. Cohen invented forcing to show in 1963 that if ZFC is consistent, then so is ZFC $+\neg$ CH. However, one could argue that these results fail to resolve CH, for the question remains: Is CH *true*?

Gödel (a platonist) found the independence of CH from the axioms of ZFC disturbing. In a seminal paper "Cantor's Continuum Hypothesis" (? need date, check) he suggested that any universe satisfying sufficiently strong large cardinal hypotheses would be compelled to satisfy one of CH or $\neg$ CH; in brief, large cardinals could decide the continuum problem.

But some (5) years later, Levy & Solovay showed (roughly) that if φ is a large cardinal axiom and if ZFC $+\varphi$ is consistent, then so are ZFC $+\varphi + \neg$ CH *and* ZFC $+\varphi +$ CH. So (it would seem) Gödel's project failed.

Or did it? CH asks, "What is the size of the set $\mathbb{R}$ of real numbers?" or, "is there a subset of $\mathbb{R}$ with size strictly between $\aleph_0$ and 2^ω?" Things get interesting when we define the projective hierarchy.

Definition. $\mathbb{B}$ is the collection of Borel sets (of reals, or of subsets of some Polish space). The **projection of** B is $p[B] = \{x \,:\, \exists y\, (x,y) \in B\}$.

$$\begin{aligned} \boldsymbol{\Sigma}^1_1 &= \{A \,:\, \exists B \text{ Borel}, A = p[B]\}; \\ \boldsymbol{\Pi}^1_1 &= \{A \,:\, A^c \in \boldsymbol{\Sigma}^1_1\}; \\ \boldsymbol{\Sigma}^1_{n+1} &= \{A \,:\, \exists X \in \boldsymbol{\Pi}^1_n\, A = p[X]\}; \\ \boldsymbol{\Pi}^1_{n+1} &= \{A \,:\, A^c \in \boldsymbol{\Sigma}^1_{n+1}\}; \\ \boldsymbol{\Delta}^1_n &= \boldsymbol{\Sigma}^1_n \cap \boldsymbol{\Pi}^1_n. \end{aligned}$$

The **projective sets** are those that belong to the collection $\mathbb{P} = \bigcup_{n\in\omega} \boldsymbol{\Pi}^1_n$.

We then have

$$\mathbb{B} = \boldsymbol{\Delta}^1_1 \begin{array}{c} \subsetneq \\ \subsetneq \end{array} \begin{array}{c} \boldsymbol{\Sigma}^1_1 \\ \\ \boldsymbol{\Pi}^1_1 \end{array} \begin{array}{c} \subsetneq \\ \subsetneq \end{array} \boldsymbol{\Delta}^1_2 \begin{array}{c} \subsetneq \\ \subsetneq \end{array} \begin{array}{c} \boldsymbol{\Sigma}^1_2 \\ \\ \boldsymbol{\Pi}^1_2 \end{array} \begin{array}{c} \subsetneq \\ \subsetneq \end{array} \boldsymbol{\Delta}^1_3 \cdots$$

Theorem (Hausdorff/Aleksandrov). *Every Borel set either contains a perfect subset or is countable (i.e., has the* ***perfect set property****); consequently, every Borel set satisfies* CH.

So there are no Borel counterexamples to CH. But what about the projective sets?

Theorem (Solovay (I think)). *If ω_1 is inaccessible in L, then every $\boldsymbol{\Pi}^1_1$ set has the perfect set property.*

(I need to check that this is the correct result.) Remark: Forcing gives that these are equiconsistent. (Check.)

Theorem (Martin). *If there is a measurable cardinal, then the perfect set property holds for $\boldsymbol{\Pi}^1_1$ sets.*

Definition. Let $A \subseteq \mathbb{R}$. Consider the game G_A where

$$\begin{array}{c|ccccc} \text{I} & n_0 & & n_2 & & \cdots \\ \hline \text{II} & & n_1 & & n_3 & \cdots \end{array},$$

and player I wins iff $\langle n_0, n_1, \dots \rangle \in A$. The **axiom of determinacy** (AD) states that for every set $A \subseteq \mathbb{R}$, G_A is determined (i.e., some player has a winning strategy).

A problem with AD is that it implies the negation of the axiom of choice. But it also decides all sorts of questions about the reals.

Definition. For $\Gamma \subseteq \mathcal{P}(\mathbb{R})$, let AD_Γ be the statement that all games in Γ are determined.

Theorem. *Suppose* AD_Γ *and* Γ *is closed under continuous substitution. Then all sets in* Γ *have the Baire property, are Lebesgue measurable, and have the perfect set property (so satisfy* CH*).*

Theorem (Martin-Steel)**.** *If there are infinitely many Woodin cardinals, then* $\mathrm{AD}_{\mathbb{P}}$ *(the* ***axiom of projective determinacy***, PD*) holds.*

Thus it would seem one of the goals of Gödel's program has been achieved: large cardinal axioms decide whether CH holds of particular classes of sets of reals. As one further explores the consequences of large cardinals, it becomes more and more compelling to accept their existence. In fact, given virtually any set theoretic hypothesis, you can force its consistency or failure using large cardinals: "any axiom will fall somewhere in the consistency hierarchy of large cardinals."

Note that some take the previous result to imply that PD is *true* (*not* merely consistent).

Remark: Borel determinacy is a theorem (due to Martin) of ZFC. We say it's true because we believe the axioms of ZFC. But why should we believe in Woodin cardinals? It may take a lot of time to make a convincing argument, but in brief, we can build *natural canonical models* of Woodin cardinals. One of the big standing problems today is to construct such inner models for large cardinals, e.g. supercompact, measurable Woodin, etc.

Definition. An ordinal κ is said to be a **measurable cardinal** if there is some nontrivial elementary embedding $j\colon V \to M$, M transitive, such that $j \restriction \kappa = \mathrm{id}$ and $j(\kappa) > \kappa$. We say κ is the **critical point** of the embedding j, written $\mathrm{cp}(j) = \kappa$.

Note that we could require only that the class M is well-founded and set-like, since then we obtain a transitive class isomorphic to M by applying the Mostowski collapse. Also, as just formulated, this definition cannot be formalized in ZFC, since it asserts the existence of certain proper classes (the map j and the model M).

This definition may seem unnatural, but is in part guided by reflection principles:

Lemma. *Suppose* κ *is measurable. Then* κ *is inaccessible, and* $V_\kappa \models$ *"there are unboundedly many inaccessible cardinals".*

Proof. Let $j\colon V \to M$ be elementary with $\mathrm{cp}(j) = \kappa$. We wish to show κ is inaccessible; it follows that $M \models$ "κ is inaccessible" (since inaccessibility is absolute for V, M).

First we show κ is regular. Suppose not: then there exists a map $f\colon \eta \to \kappa$ cofinal in κ, with $\eta < \kappa$. Then by elementarity, $M \models$"$j(f)\colon \eta \to j(\kappa)$ is cofinal in $j(\kappa)$". But for each $\alpha < \eta$, we have $j(f)(\alpha) = j(f)(j(\alpha)) = j(f(\alpha)) = f(\alpha) < \kappa$, since j is the identity on κ. But then M sees that $j(f)$ is bounded by $\kappa < j(\kappa)$, a contradiction.

Next we show κ is strongly inaccessible. Suppose $\alpha < \kappa$. Then $\mathcal{P}(\alpha)^V = \mathcal{P}(\alpha)^M$. The inclusion $\supseteq$ is obvious, since V contains M. The inclusion $\subseteq$ follows from the observation that for any $A \subseteq \alpha$ in V, we have $j(A) = A$, since $j(\alpha) = \alpha$. So κ is strongly inaccessible.

It follows that $M \models$ "$\kappa < j(\kappa)$ and κ is inaccessible". So $M \models \exists \eta < j(\kappa)(\eta$ is inaccessible$)$. By elementarity, $V \models \exists \eta < \kappa(\eta$ is inaccessible$)$. The same argument, modified slightly, shows that $V_\kappa \models (\forall \alpha \in \mathrm{Ord})(\exists \eta > \alpha)(\eta$ is inaccessible$)$. □

Definition. Let κ be a set. A **filter** over κ is a set $\mu \subseteq \mathcal{P}(\kappa)$ such that

- $\emptyset \notin \kappa$;

- For all $A, B \subseteq \kappa$, if $A, B \in \mu$, then $A \cap B \in \mu$ (μ is closed under finite intersections);
- For all $A, B \subseteq \kappa$, if $A \in \mu$ and $A \subseteq B$, then $B \in \mu$ (μ is closed upwards).

μ is an **ultrafilter** if furthermore, for each $A \subseteq \kappa$, either $A \in \mu$ or $A^c \in \mu$. It is **nonprincipal** if it contains no singletons. It is **κ-complete** if for any $\xi < \kappa$ and collection $\{A_\alpha\}_{\alpha<\xi}$ of sets in μ, we also have $\bigcap_{\alpha<\xi} A_\xi \in \mu$ (μ is closed under $< \kappa$-intersections).

Finally, μ is **normal** if it is closed under diagonal intersections, i.e. if for every collection $\{A_\alpha\}_{\alpha<\kappa}$ of sets in μ, we have

$$\mathop{\triangle}_{\alpha<\kappa} A_\alpha := \{\beta \in \kappa \ : \ (\forall \alpha < \beta)\, \beta \in A_\alpha\} \in \mu.$$

Theorem (Scott)**.** *κ is measurable if and only if there exists a κ-complete normal nonprincipal ultrafilter over κ.*

Proof. Suppose κ is measurable. Then fix $j \colon V \to M$ with critical point κ, and define $A \in \mu$ iff $\kappa \in j(A)$. It is left as an exercise to check that μ is as desired.

Conversely, suppose μ is a κ-complete normal nonprincipal ultrafilter over κ. Define an equivalence relation $\equiv_\mu$ on the class of functions with domain κ by saying $f \equiv_\mu g$ iff $\{\alpha \in \kappa \ : \ f(\alpha) = g(\alpha)\} \in \mu$. Similarly, say $[f] \in_\mu [g]$ iff $\{\alpha \in \kappa \ : \ f(\alpha) \in g(\alpha)\} \in \mu$. Define

$$M = \mathrm{Ult}(V, \mu) = (\{[f]_\mu \ : \ f \colon \kappa \to V\}, \in_\mu);$$

this is called the **ultrapower of V by μ**. M is wellfounded because μ is countably complete. Define $j_\mu \colon V \to M$ by $j_\mu(a) = [c_a]$ (where $c_a \colon \kappa \to V$ is the constant function with value a); this is an elementary embedding (called the **ultrapower embedding**) and has critical point κ (since $[\mathrm{id}]_\mu \in_\mu j(\kappa)$). ☐

The μ of our proof is called the **ultrafilter derived from j**.

Beware! In what follows we typically identify $\mathrm{Ult}(V, \mu)$ with its transitive collapse M, and write j for the composition $\pi \circ j$ (where $\pi \colon \mathrm{Ult}(V, \mu) \to M$ is the collapse isomorphism). With this identification, we can avoid mention of the collapse embedding entirely, and statements such as "$X \in \mathrm{Ult}(\kappa, \mu)$" make sense for arbitrary sets X. By the same token, we will sometimes refer to $[f]_\mu$ as though it were its own image under the collapse on $\mathrm{Ult}(V, \mu)$. The only way this notation can give rise to multiple interpretations is if $[f]_\mu \in M$, a possibility eliminated by the following:

Fact. $\mu \notin M$.

Proof. We first show ${}^\kappa M \subseteq M$, that is, M contains all sequences of elements in M of length κ. For suppose $f \colon \kappa \to M$; for each $\alpha < \kappa$, choose $f_\alpha \colon \kappa \to V$ such that $f(\alpha) = \pi([f_\alpha]_\mu)$. Define $g \colon \kappa \to {}^\kappa V$ by letting $g(\beta) = g_\beta$ be the function satisfying $g_\beta(\alpha) = f_\alpha(\beta)$. It can be checked that $\pi([g]_\mu) = f$ (Łoś's Theorem); so $f \in M$.

Let j be the embedding of V into M. Since $\mu \in M$ and M contains all functions with domain κ, we have that for any ordinal α,

$$j(\alpha) = \sup\{\pi([f]_\mu) \ : \ f \colon \kappa \to \alpha\}$$

belongs to M. Applying replacement in M, we have $j''\kappa^+ \in M$. But $j''\kappa$ is cofinal in $j(\kappa^+)$, since if $\pi([f]_\mu) < j(\kappa^+)$, then f is bounded μ-almost everywhere by some $\alpha < \kappa^+$; whence $\pi([f]_\mu) < j(\alpha)$.

So $j''\kappa^+$ is a set of size $\kappa+$ cofinal in $j(\kappa^+) > \kappa^+$, contradicting regularity of $j(\kappa^+)$ in M. $\square$

Theorem (Scott). *If there is a measurable cardinal, then $V \neq L$.*

So is there a universe *like* L in which a measurable cardinal exists? We will see that the answer is yes.

Larger Cardinals:

- Strong
- Woodin
- Shelah
- Superstrong
- Supercompact

It was originally thought, in trying to solve the inner model problem, that nice inner models of large cardinals wouldn't contain sets of reals more complicated than $\boldsymbol{\Delta}^1_3$. But Magidor-Foreman-Shelah showed that you *need* messier sets of reals in an inner model of certain (very) large cardinals. For instance, Shelah-Woodin showed that an inner model of supercompact cardinals contains no $\boldsymbol{\Delta}^1_3$ wellordering of the reals.

Definition. Let $j : V \to M$ be an elementary embedding with $\text{cp}(j) = \kappa$, and let $\lambda \geq \kappa$ such that $j(\kappa) \geq \lambda$. Put

$$E = \{(a, A) \ : \ a \in [\lambda]^{<\omega}, \ A \subseteq [\kappa]^{|a|}, \ a \in j(A)\}.$$

Then E is called a (κ, λ)**-extender family derived from** j.

Lemma. *Let $E_a = \{A \ : \ (a, A) \in E\}$.*

(a) *For all a, E_a is an ultrafilter over $[\kappa]^{|a|}$.*
(b) *E_a is principal iff $a \subseteq \kappa$.*
(c) *(Coherence condition.) Let $a \subseteq b$. Define $\pi_{b,a} \colon \kappa^{|b|} \to \kappa^{|a|}$ by*

$$\pi_{b,a}(\langle c_0, c_1, \ldots, c_n \rangle) = \langle c_{i_0}, c_{i_1}, \ldots, c_{i_m} \rangle$$

where $|b| = n$, $|a| = m$, $b_{i_k} = a_k$ for all $k \leq m$.
Then if $A \in E_b$, we have $\pi''_{b,a}A \in E_a$.

(d) *(Countable completeness condition.) If $\langle (a_i, A_i) \ : \ i < \omega \rangle$ is a sequence such that $A_i \in E_{a_i}$ and $a_i \subseteq a_{i+1}$ then there is $f \colon \bigcup a_i \to \kappa$ such that $f''a_i \in A_i$ for all $i < \omega$.*

Notice that if we set $\lambda = \kappa + 1$ and $a = \{\kappa\}$, then E_a is (essentially) the same as the ultrafilter derived from j.

Proof. (a) is shown exactly as one shows the U derived from j is an ultrafilter.

(b): If $a \subseteq \kappa$ then $j(a) = a$, so E_a is the principal ultrafilter generated by a. If E_a is principle, then it is generated by some $b \in [\lambda]^{|a|}$; then $a \in j(\{b\})$, so by elementarity $a = j(b)$. Then (since $\kappa \leq \lambda \leq j(\kappa)$) we have $a = b \subseteq \kappa$.

(c) holds because $\pi_{b,a}(b) = a$.

(d): Let $(T, \leq_T)$ be the tree whose i-th level consists of the sets in A_i, with $a \leq_T b$ iff $a \subseteq b$. We will be done if we can find a branch $\langle b_i \rangle_{i \in \omega}$ through T, since then we can let f be the unique order-preserving bijection $f \colon \bigcup a_i \to \bigcup b_i$.

By elementarity $j(T)$ is the tree whose i-th level is equal to $j(A_i)$, again ordered by inclusion. By our hypotheses on the a_i, we have $a_i \in j(A_i)$ for each i, so $\langle a_i \rangle_{i\in\omega}$ furnishes a branch through the tree $j(T)$. Then $M \models$ "$\exists\langle d_i \rangle_{i\in\omega}$ a branch through $j(T)$" (the branch $\langle a_i \rangle_{i\in\omega}$ may not itself belong to M, but wellfoundedness of trees is absolute for transitive models). Again applying elementarity of j, we obtain a branch $\langle b_i \rangle_{i\in\omega}$ through T, as desired. □

Ultrapower Construction.

Suppose E is a (κ, λ)-extender derived from $j : V \to M$.

Construction 1: *Idea*: For $a \in [\kappa]^{<\omega}$, put $M_a := \mathrm{Ult}(V, E_a)$, and let $j_{E_a} : V \to M_a$ be the ultrapower embedding.

If $a \subseteq b$, then there is $\sigma_{a,b} : M_a \to M_b$ such that $j_{E_b} = \sigma_{a,b} \circ j_{E_a}$.

$$\begin{array}{ccc} V & \xrightarrow{\;j_b\;} & M_b = \{[f]_{E_b} \ : f\colon \kappa^{|b|} \to V\} \\ & \searrow^{j_a} & \uparrow \sigma_{a,b} \\ & & M_a = \{|f|_{E_a} \ : \ f\colon \kappa^{|a|} \to V\} \end{array}$$

Namely, given $f : \kappa^{|a|} \to V$, let $\sigma_{a,b}([f]_{E_a}) = [f^{a,b}]_{E_b}$; here

$$f^{a,b}(s) = f(\pi_{b,a}(s)).$$

It is easily checked that $\sigma_{a,b}$ is well-defined and elementary. Hence,

$$\mathcal{F} := \langle M_a, \sigma_{a,b} \ : \ a, b \in [\lambda]^{<\omega}, a \subseteq b\rangle$$

is a directed system. Let $\mathrm{Ult}(V, E)$ be the direct limit of $\mathcal{F}$.

Construction 2: Let $\mathrm{Ult}(V, E) = \{[a, f]_E \ : \ f\colon \kappa^{|a|} \to V, a \in [\lambda]^{<\omega}\}$; where $[a, f]_E$ is the $\equiv_E$-equivalence class of (a, f), and

$$(a, f) \equiv_E (b, g) \iff \{s \in \kappa^{|a\cup b|} \ : \ f^{a,a\cup b}(s) = g^{b,a\cup b}(s)\} \in E_{a\cup b}.$$

Note that we have an analogue of Łoś's Theorem for such things, namely

$$\mathrm{Ult}(V, E) \models \varphi([a, f]_E) \iff \{s \ : \ V \models \varphi[f(s)]\} \in E.$$

Let $j\colon V \to M$ have $\mathrm{cp}(j) = \kappa$, and let E be derived from j. Let $\sigma\colon \mathrm{Ult}(V, E) \to M$ be given by $\sigma([a, f]) = j(f)(a)$. Then σ is elementary, and $j = \sigma \circ j_E$.

Lemma. $\sigma \restriction \lambda = \mathrm{id}$*; hence if* λ *is inaccessible, then* $V_\lambda^M \subseteq \mathrm{Ult}(V, E)$*.*

I'm not sure why we need inaccessibility of λ for the conclusion. If in doubt, see Lemma 26.1 in Kanamori.

Proof. To show $\sigma \restriction \lambda = \mathrm{id}$ it is sufficient to show that every $\delta < \lambda$ belongs to the range of σ. To this end, let let a be $\{\delta\}$ and let $f\colon \{\alpha\} \mapsto \alpha$. Then by elementarity, $j(f)\colon \{\alpha\} \mapsto \alpha$ for all $\alpha \in j(\kappa)$, and

$$\sigma([a, f]) = j(f)(\{\delta\}) = \delta.$$

The conclusion $V_\lambda^M \subseteq \mathrm{Ult}(V, E)$ now follows from the usual proof (relativized to M) that if an elementary embedding is the identity on α, then it is also the identity on V_α. □

[This is why they're called "extenders" – the ultrapowers extend way beyond how far a measurable cardinal would extend.]

Definition. κ is a **λ-strong cardinal** if there is a (κ, λ)-extender E such that $V_\lambda \subseteq \mathrm{Ult}(V, E)$; equivalently, if there exists an elementary $j : V \to M$ with $\mathrm{cp}(j) = \kappa$, $j(\kappa) > \lambda$, and $V_\lambda \subseteq M$.

Note that we have here the *actual* V_λ contained in $\mathrm{Ult}(V, E)$, not merely V_λ^M. (Why is it the case the the first definition implies the *strict* inequality $j(\kappa) > \lambda$ in the second?)

Definition. κ is **strong** if it's λ-strong for every λ.

Exercise. If κ is $(2^\kappa)^+$-strong, then there exists a normal κ-complete ultrafilter U over κ such that

$$\{\xi < \kappa \ : \xi \text{ is a measurable cardinal}\} \in U.$$

Definition. κ is a **Woodin cardinal** if κ is regular and for every $f : \kappa \to \kappa$, there exists $\eta < \kappa$ such that for some (η, λ)?-extender E with $\mathrm{cp}(j_E) = \eta$, we have

$$V_{j_E(f)(\eta)} \subseteq \mathrm{Ult}(V, E).$$

Note that Woodins needn't be measurable.

Exercise. κ is Woodin iff

$$(\forall A \subseteq \kappa)(\exists \eta < \kappa)(\forall \lambda \in (\eta, \kappa))(\exists j \colon V \to M)[\mathrm{cp}(j) = \eta,\ j(\eta) \geq \lambda, \text{ and } j(A) \cap \lambda = A \cap \lambda].$$

Example. κ is **Shelah** if for all $f \colon \kappa \to \kappa$ there exists $j : V \to M$ with $\mathrm{cp}(j) = \kappa$ and $V_{j(f)(\kappa)} \subseteq M$.

Definition. κ is **superstrong** if there exist λ and a (κ, λ)-extender E such that $V_\lambda \subseteq \mathrm{Ult}(V, E)$, and $j_E(\kappa) = \lambda$.

Exercise. Suppose there exists $j : V \to M$ such that $\mathrm{cp}(j) = \kappa$ and for all $f : \kappa \to \kappa$, $V_{j(f)(\kappa)} \subseteq M$. Then κ is superstrong.

Definition. κ is **λ-supercompact** if there exists $j : V \to M$ with $\mathrm{cp}(j) = \kappa$, $j(\kappa) > \lambda$, and $M^\lambda \subseteq M$.

κ is **supercompact** if κ is λ-supercompact for all λ.

Exercise.
(a) If κ is Woodin, then $V_\kappa \models$"there are unboundedly many strong cardinals".
(b) If κ is superstrong, then κ is Woodin, and $V_\kappa \models$"there are unboundedly many Woodin cardinals".
(c) If κ is superstrong, then κ is Shelah.
(d) If κ is supercompact, then κ is strong and Woodin. In addition, if there exists $j : V \to M$ witnessing $(2^\kappa)^+$-supercompactness, then κ is superstrong in M.

The latter conclusion in part (b) tells us that V thinks $j(\kappa)$ is a limit of Woodins (where j witnesses superstrength of κ?). I'm not sure if the second part of exercise (d) is stated correctly.

Exercise. Suppose $j : V \to M$ has $\mathrm{cp}(j) = \kappa$ and that for some $\lambda > \kappa$, $j''\lambda \in M$. Then κ is λ-supercompact.

Proof sketch. Define $\mu \subseteq \mathcal{P}_\kappa(\lambda) = \{A \subseteq \lambda \ : \ |A| < \kappa\}$ by letting $A \in \mu$ iff $j''\lambda \in j(A)$. Then show that $j_\mu \colon V \to \mathrm{Ult}(V, \mu)$ witnesses that κ is λ-supercompact. □

2. January 10 – Anush Tserunyan
Measurable Cardinals and $\mathbf{\Pi}^1_1$-Determinacy

Definition. A set $A \subseteq X^\omega$ (X a Polish space) is called **κ-Souslin** (κ a cardinal) if there exists a tree on $X \times \kappa$ such that $A = p[T]$.

For example, analytic sets are ω-Souslin. Many proofs of facts about analytic sets can be adapted to prove analogous facts for κ-Souslin sets (e.g. Kunen's theorem that an analytic wellfounded relation has countable rank can be adapted to show that a wellfounded κ-Souslin relation has rank below κ^+).

We might naïvely attempt to show that κ-Souslin sets are determined as follows: Let $A \subseteq X$ be κ-Souslin with $A = p[T]$, T a tree on $X \times \kappa$. The game $G(A)$ is played like so:

$$G(A) = \begin{array}{c|cccc} \text{I} & x_0 & & x_2 & \cdots \\ \hline \text{II} & & x_1 & & \cdots \end{array}$$

I wins iff $x = (x_0, x_1, \dots) \in A$. This game has its unraveled version,

$$G^* = \begin{array}{c|cccc} \text{I} & x_0, y_0 & & x_2, y_1 & \cdots \\ \hline \text{II} & & x_1 & & \cdots \end{array}$$

where I wins iff $(x, y) \in [T]$; that is, to win, I needs to simultaneously play a run x and a y witnessing $x \in A$. This is a closed game, and therefore is determined. Clearly a winning strategy for I in G^* yields a winning strategy for I in $G(A)$. So suppose II has a winning strategy in G^*. We might be inclined simulate the game $G(A)$ in G^*, by letting I play a random sequence $y_0, y_1, \dots$. But there is nothing to ensure we pick a witness in T that $x \in A$.

We try to recover the desired strategy by demanding player I play a witness y "generically".

For $s \in X^{<\omega}$, let $|s|$ denote the length of s, and put

$$T_s := \{t \in \kappa^{|s|} \ : \ (s,t) \in T\}.$$

For $s \subseteq t$ and $B \subseteq T_s$, define the set of descendants of B in T_t by

$$\text{proj}_s^{-1}(B) = \{u \in T_t \ : \ u \restriction |s| \in B\}.$$

Definition. A tree on $X \times Y$ is called **homogeneous** if there exists a family $\langle \mu_s \ : \ s \in X^{<\omega}\rangle$ such that

(i) μ_s is a countably additive two-valued measure on T_s.
(ii) if $s \subseteq t$ then for $B \subseteq T_s$, $\mu_s(B) = \mu_t(\text{proj}_s^{-1}(B))$.
(iii) if $x \in p[T]$, then the ultrapower $\text{Ult}(V, \langle \mu_{x\restriction n} \ : n \in \omega\rangle)$ is wellfounded.
(iii)' if $x \in p[T]$, then for all $X_n \subseteq T_{x\restriction n}$ with $\mu_{x\restriction n}(X_n) = 1$ for all $n \in \omega$, there exists $f \in Y^\omega$ such that $f \restriction n+1 \in X_n$ for all n.

Conditions (iii) and (iii)' are equivalent under (i) and (ii).

Definition. A homogenous tree is called **κ-homogeneous** if each μ_s is κ-complete.

Definition. A set $A \subseteq X^\omega$ is called **κ-homogeneously Souslin** if there exists $T \subseteq X \times Y$ κ-homogeneous such that $A = p[T]$.

(should Y be κ?)

Theorem (Martin, Steel). *If $A \subseteq X^\omega$ is κ-homogeneously Souslin, $\kappa > |X|$, then A is determined.*

Proof. If I has a winning strategy in G^*, it also has one in $G(A)$. So assume II has a winning strategy σ^* in G^*. We define the derived strategy σ for II in G.

Suppose $s = (x_i)_{i \le 2n}$ is the sequence played in G up to the n-th move:

$$G(A) : \begin{array}{c|ccccc} \text{I} & x_0 & & x_2 & \dots & x_{2n} \\ \hline \text{II} & & x_1 & & \dots & \end{array}$$

$$G^* : \begin{array}{c|cccccc} \text{I} & x_0, y_0 & & x_2, y_1 & & \dots & x_{2n}, y_n \\ \hline \text{II} & & x_1 & & x_3 & \dots & \end{array}$$

Let $s' = s \restriction n+1$. By $|X|^+$-completeness of $\mu_{s'}$, there exists (a unique) $z \in X$ such that

$$X_s := \{t \in T_{s'} \ : \ \sigma^*(s,t) = z\} \in \mu_{s'}.$$

Put $\sigma(s) = z = x_{2n+1}$.

Assume for contradiction that $x = (x_i)_{i\in\omega} \in A$. Let $X_n = X_{x\restriction n}$, $\mu_{x\restriction n}(X_n) = 1$. Since $x \in A = p[T]$, by (iii)', there exists $y \in Y^\omega$ such that $y \restriction n+1 \in X_n \subseteq T_{x\restriction n}$. This implies $(x,y) \in [T]$. Hence II loses in G^*, contradicting the fact that (x,y) was a play according to σ^*. □

Theorem (Martin, Steel). *If κ is a measurable cardinal and $A \subseteq \omega^\omega$ is $\mathbf{\Pi}^1_1$, then A is κ-homogeneously Souslin.*

[Soon we will see how this result is translated for $\mathbf{\Pi}^1_n$ sets using Woodin cardinals.]

For an infinite sequence x, T_x denotes the union $\bigcup_{n\in\omega} T_{x\restriction n}$. As is standard, we fix an enumeration $\{s_i\}_{i\in\omega}$ of $\omega^{<\omega}$ such that $|s_i| \le i$ for all i.

Proof. Let $\neg A = p[\tilde{T}]$. By definition, $x \in A$ iff $\tilde{T}_x$ is wellfounded iff the Kleene-Brouwer ordering $<_{\mathrm{KB}}$ on $\tilde{T}_x$ is a wellordering.

For $s \in \omega^{<\omega}$, let $<_s$ be a linear order for $|s|$ such that for $i, j < |s|$,

$$\begin{aligned} i <_s j \iff & s_i, s_j \notin T_{<s} = \bigcup_{\ell \le |s|} T_{s\restriction \ell} \text{ and } i < j, \\ & \text{or } s_i \notin T_{<s} \text{ and } s_j \in T_{<s}, \\ & \text{or } s_i, s_j \in T_{<s} \text{ and } s_i <_{\mathrm{KB}} s_j. \end{aligned}$$

Claim. *If $s \subseteq s'$, then $<_{s'}$ extends $<_s$.*

Proof of claim. For $i < |s|$ we have $|s_i| \le i < |s|$, so that $s_i \in T_{<s}$ iff $s_i \in T_{<s'}$. Hence for $i, j < |s|$, the ordering $<_{s'}$ on i, j is already determined by $<_s$. □

So for $x \in \omega^\omega$, we may define $<_x = \bigcup_{n\in\omega} <_{x\restriction n}$. Observe that $<_x$ is the usual order on the naturals for those i such that $s_i \notin T_x$; for those i with $s_i \in T_x$, $<_x$ is the ordering induced by the Kleene-Brower order on these s_i. Then $x \in A$ iff $<_x$ is a wellordering (since $<_{\mathrm{KB}}$ is a wellordering of T_x iff $x \in A$).

For $\kappa \ge \omega_1$, we can define a tree T on $\omega \times \kappa$:

$$T = \{(s,t) \in (\omega \times \kappa)^{|s|} \ : \ \forall i, j < |s|, (i <_s j \iff t(i) < t(j))\}.$$

Claim. $A = p[T]$.

Proof of claim. Suppose $(x, f) \in [T]$; then $f\colon \omega \to \kappa$ is order-preserving for $<_x$. Hence T_x is wellordered, so that $x \in A$.

Conversely, if $x \in A$ then $<_x$ is a wellordering of T_x; so there exists some order preserving map $f\colon \omega \to \kappa$. By our definition of T, we have $(x, f) \in [T]$. □

So far, we haven't used measurability of κ. We use that now, to show that the tree just defined is κ-homogeneous, as desired.

For each $s \in \omega^{<\omega}$, T_s may be regarded as a subset of $[\kappa]^{|s|}$, since for any t, t' with $\mathrm{range}(t) = \mathrm{range}(t')$ and $(s, t), (s, t') \in T$, we have $t = t'$ and $|\mathrm{range}(t)| = |t| = |s|$.

(there's a remark here about $<_{\mathrm{KB}}$ giving a corr. of nodes of the tree with κ)

Fix a normal κ-complete measure μ on κ. Let $B \subseteq T_s$. Define measures μ_s on T_s by

$$\mu_s(B) = 1 \iff \exists Y \subseteq \kappa \text{ such that } \mu(Y) = 1 \text{ and } \forall t \in T_s, \mathrm{range}(t) \subseteq Y \implies t \in B.$$

This defines a measure for each s, by

Rowbottom's Theorem. *Let μ be a normal measure on κ. Then for any $F\colon [\kappa]^n \to \{0,1\}$, there exists $Y \subseteq \kappa$ with $\mu(Y) = 1$ such that $F \restriction Y$ is constant.*

Specifically, we apply Rowbottom's Theorem to the characteristic function of B, that is, $F_B\colon T_s \to \{0,1\}$ so that $F_B(t) = 1$ iff $t \in B$.

μ_s is κ-complete since μ is: for suppose $\lambda < \kappa$ and $B_\alpha \in \mu_s$ for all $\alpha < \lambda$; for each α, let Y_α witness $B_\alpha \in \mu_s$. Then $Y := \bigcap_{\alpha<\lambda} Y_\alpha \in \mu$. If $t \in T_s$ and $\mathrm{range}(t) \subseteq Y \subseteq Y_\alpha$, then $t \in B_\alpha$ for all $\alpha < \lambda$, so that $t \in B := \bigcap_{\alpha<\lambda} B_\alpha$ and $\mu_s(B) = 1$.

For (ii), suppose $s \subseteq s'$ and $B \subseteq T_s$. We need to show that $\mu_s(B) = 1$ iff $\mu_{s'}(\mathrm{proj}_s^{-1}(B)) = 1$. Suppose $\mu_s(B) = 1$. Then there is $Y \subseteq \kappa$ with $\mu(Y) = 1$ witnessing $\mu_s(B) = 1$. For all $t \in T_{s'}$, if $\mathrm{range}(t) \subseteq Y$ then $\mathrm{range}(t \restriction |s|) \subseteq Y$, so that $t \restriction |s| \in B$ and $t \in \mathrm{proj}_s^{-1}(B)$; then $\mu_{s'}(\mathrm{proj}_s^{-1}(B)) = 1$. Conversely, if $\mu_s(B) = 0$, then $\mu_s(\neg B) = 1$, so that by the above, $\mu_{s'}(\mathrm{proj}_s^{-1}(\neg B)) = 1$, and $\mu_{s'}(\neg\,\mathrm{proj}_s^{-1}(\neg B)) = \mu_{s'}(\mathrm{proj}_s^{-1}(B) = 0$.

For (iii)', Let $x \in A = p[T]$, and for each $n \in \omega$, $X_n \subseteq T_{x\restriction n}$ such that $\mu_{x\restriction n}(X_n) = 1$. Let $Y_n \subseteq \kappa$ be μ-measure 1 sets witnessing $\mu_{x\restriction n}(X_n) = 1$. Let $Y = \bigcap_n Y_n$; then still $\mu(Y) = 1$, so Y is uncountable, since μ is nonprincipal. Since $x \in A$, $<_x$ is a wellordering, and there exists $f : \omega \to Y$ preserving the order $<_x$.

We claim that for all $n \in \omega$, we have $f \restriction n+1 \in X_n$. It's clear that $(x, f) \in [T]$; so for all n, $f \restriction n+1 \in T_{x\restriction n+1}$. And $\mathrm{range}(f \restriction n+1) \subseteq \mathrm{range}(f) \subseteq Y \subseteq Y_n$, so that $f \restriction n+1 \in X_n$.

Thus T is κ-homogeneous, which completes the proof of the theorem. □

3. January 17 – Grigor Sargsyan
Mice and Comparison

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4. January 24 – Sherwood Hachtman
Ultrapowers and Iterability

All of what follows is cribbed from Kanamori, section 19.

We work in the language of set theory plus a unary relation symbol A. M is a transitive model of ZFC$^-$ (set theory minus Powerset) and $M \models$ "κ is a cardinal".

Definition. U is an **M-ultrafilter over κ** iff

(i) $\langle M, \in U\rangle \models$ "U is a normal ultrafilter over κ"
(ii) (weak amenability) $(\forall F \in M^\kappa \cap M)\{\xi < \kappa \mid F(\xi) \in U\} \in M$.

Condition (ii) is *not* superfluous, since instances of Replacement that hold in M are not expected to contain instances of the symbol A.

The ultrapower of M by U is taken in the usual fashion, considering only functions $f : \kappa \to M$ inside M; A is interpreted in the ultrapower structure by setting

$$[f]_U \in_U A_U \iff \{\xi < \kappa \;:\; f(\xi) \in U\} \in U.$$

Note we needed to use weak amenability to guarantee definability of A_U in M (since U is only an M-ultrafilter). We have Łos's theorem for $\langle M^\kappa/U, \in_U\rangle$:

$$M^\kappa/U \models \varphi([f_1], \ldots, [f_n]) \iff \{\xi < \kappa \mid M \models \varphi(f_1(\xi), \ldots, f_n(\xi))\} \in U.$$

Here it was crucial that $M \models$ ZFC$^-$: the set on the right belongs to M because of Comprehension; and we need Choice in M to prove the existential quantifier case. (Note however that we do *not* necessarily have Łos's theorem for the full structure with A_U, since instances of Comprehension are expected not to contain the symbol A.)

If the ultrapower is wellfounded, let the collapse be $\langle N, \in, W\rangle$, with $j : \langle M, \in\rangle \to \langle N, \in\rangle$ the usual embedding. Then we have the following:

(i) $\mathrm{cp}(j) = \kappa$;
(ii) $|M| = |N|$ when M is a set;
(iii) $U \notin N$;
(iv) W is an N-ultrafilter over $j(\kappa)$;
(v) For all $x \in N$, there is some $f : \kappa \to M$ in M so that $x = j(f)(\kappa)$.

Definition. The **iteration** $\langle M_\alpha, U_\alpha, \kappa_\alpha, i_{\alpha,\beta}\rangle_{\alpha\leq\beta\in\tau}$ of $\langle M, \in, U\rangle$ is defined recursively by

$\langle M_0, U_0, \kappa_0\rangle = \langle M, U, \kappa\rangle$;

$\langle M_{\alpha+1}, U_{\alpha+1}, \kappa_{\alpha+1}\rangle$ is the transitive collapse of the ultrapower of M_α by U;

$i_{\alpha,\alpha+1}$ is the ultrapower embedding composed with the collapse, j;

$i_{\gamma,\alpha+1} = i_{\alpha,\alpha+1} \circ i_{\gamma,\alpha}$ for $\gamma < \alpha$; and $\kappa_{\alpha+1} = j(\kappa_\alpha)$;

$\langle M_\lambda, U_\lambda, \kappa_\lambda\rangle$ is the transitive collapse of the direct limit of $\langle M_\alpha, U_\alpha, i_{\alpha,\beta}\rangle_{\alpha\leq\beta<\lambda}$,

with $i_{\alpha,\lambda}$ the induced embedding for $\alpha < \lambda; \kappa_\lambda = i_{0,\lambda}(\kappa_0)$.

The **length** τ of the iteration is the least stage at which this construction halts due to illfoundedness. $\langle M_\alpha, \in, U_\alpha\rangle$ is the **α-th iterate of** $\langle M, \in, U\rangle$.

Some facts:

(i) $\mathrm{cp}(i_{\alpha,\beta}) = \kappa_\alpha$, $i_{\alpha,\beta}(\kappa_\alpha) = \kappa_\beta$;

(ii) $|M_\alpha| = |M||\alpha|$ for M a set;
(iii) $\beta \mapsto \kappa_\beta$ is continuous.

5. January 31 – Sherwood Hachtman
Inner Models of Measurability

Fix a set X. Recall the definition of the constructible hierarchy relative to X:

$$\begin{aligned} L_0[X] &= \emptyset, \\ L_{\alpha+1}[X] &= \{B \subseteq L_\alpha[X] \ : \ B \text{ is definable over } \langle L_\alpha[X], \in, L_\alpha[X] \cap X\rangle\}, \\ L_\lambda[X] &= \bigcup_{\alpha<\lambda} L_\alpha[X]. \end{aligned}$$

Then $L[X] = \bigcup_{\alpha \in \mathrm{Ord}} L_\alpha[X]$.

Notice that it needn't be the case that $X \in L[X]$; however, if we put $\overline{X} = L[X] \cap X$, then $L[\overline{X}] = L[X]$, so we can assume without loss of generality that $X = \overline{X}$ and $X \in L[X]$.

Much as in the case of L, there is a sentence in the language $\{\in, A\}$ (call it "$\sigma_{L[A]}$") such that for any transitive class N and set X,

$$\langle N, \in, X \cap N\rangle \models \sigma_{L[A]} \iff N = L_\delta[X] \text{ for some } \delta \text{ limit, } \delta > \omega.$$

(So in arguments using the condensation lemma, we will need to verify that the object interpreting A in the Mostowski collapse M is really $M \cap X$ to conclude $M = L_\delta[X]$.) Similarly, there is a single two-variable formula $\varphi(v_1, v_2)$ that defines a wellorder $<_{L[A]}$ of $L_\delta[A]$ in $L_\delta[A]$ for all limit $\delta > \omega$.

Definition. $\langle L[U], \in, U\rangle$ is a κ-**model** iff $\langle L[U], \in, U\rangle \models$"$U$ is a normal ultrafilter over κ".

Lemma. *Let $\langle L[U], \in, U\rangle$ be a κ-model. Then in $L[U]$, for all $\lambda \geq \kappa$, we have $2^\lambda = \lambda^+$.*

Proof. The argument is the same as that for GCH in L. We show $\mathcal{P}(\lambda) \cap L[U] \subseteq L_{\lambda^+}[U]$; the result follows, since $|L_\alpha[U]| = |\alpha|$ for $\alpha \geq \omega$. Suppose $x \in \mathcal{P}(\lambda) \cap L[U]$. Fix $\gamma > \lambda$ limit with $x, U \in L_\gamma[U]$. By Löwenheim-Skolem, fix $H \prec L_\gamma[U]$ such that $\lambda \cup \{x, U\} \subseteq H$ and $|H| = \lambda$. Let $\langle N, \in, W\rangle$ be the Mostowski collapse of $\langle H, \in, U \cap H\rangle$. Since $\lambda \subseteq H$, we have $\pi(y) = y$ for all $y \in \mathcal{P}(\lambda) \cap H$; so $\pi(x) = x$ and $\pi(U \cap H) = \pi(U \cap N)$. It follows by elementarity and condensation that $N = L_\beta[U]$ for some $\beta < \lambda^+$; then $x \in L_\beta[U] \subseteq L_{\lambda^+}[U]$. □

Definition. $\langle M, \in, U\rangle$ is a ZFC^- **premouse for** κ iff U is an M-ultrafilter over κ and $M = L_\zeta[U]$ for some ζ (possibly $\zeta = \mathrm{Ord}$).

Note that an iterate of a premouse is a premouse.

Lemma. *Suppose $\langle L_\zeta[U], \in, U\rangle$ is a premouse, $L_\zeta[U] \models \mathrm{ZFC}$, and $\omega_1 \cup \{U\} \subseteq L_\zeta[U]$. Then $\langle L_\zeta[U], \in, U\rangle$ is iterable.*

Proof. Since $L_\zeta[U] \models$"U is a normal ultrafilter over κ", $L_\zeta[U] \models$"$L_\zeta[U]$ is iterable". Since the iteration is defined internally, we have for any $\alpha < \zeta$ that the α-th iterate $\langle M_\alpha, \in, U_\alpha\rangle$ is absolute for $L_\zeta[U], V$; and since $L_\zeta[U]$ models ZFC, this iterate is really wellfounded. Since $\omega_1 \subseteq L_\zeta[U]$, the (real) iteration of $\langle L_\zeta[U], \in, U\rangle$ has length at least ω_1; hence $\langle L_\zeta[U], \in, U\rangle$ is iterable. □

Definition. For regular cardinals ν, let C_ν denote the club filter on ν.

Lemma. *Suppose $\langle M, \in, U\rangle$ is a premouse at κ with iteration $\langle M_\alpha, U_\alpha, \kappa_\alpha, i_{\alpha,\beta}\rangle_{\alpha\leq\beta\in\tau}$ and $\nu \in \tau$ is regular, with $\nu > |\kappa^\kappa \cap M|$. Then for some ζ,*

$$M_\nu = L_\zeta[C_\nu];\ \textit{hence}\ U_\nu = C_\nu \cap L_\zeta[C_\nu].$$

Proof. By two lemmas from last time, we have $\kappa_\nu = \nu$, and $X \in U_\nu$ iff there exists $\alpha < \nu$ such that $\{\kappa_\gamma\ :\ \alpha \leq \gamma < \nu\} \subseteq X$. Since each such set is club, $U_\nu \subseteq C_\nu \cap M_\nu$; the opposite inclusion holds since U_ν is a normal ultrafilter in M_ν. Furthermore, by elementarity,

$$\langle M_\nu, \in, C_\nu \cap M_\nu\rangle = \langle M_\nu, \in, U_\nu\rangle \models \sigma_{L[A]},$$

so that $M_\nu = L_\zeta[C_\nu]$ for some ζ. □

Definition. Two premice $\langle M, \in, U\rangle$ and $\langle N, \in, W\rangle$ are **comparable** if there exists an F so that $U = F \cap M$, $W = F \cap N$, and $M = L_\zeta[F]$, $N = L_\eta[F]$ for some ζ, η.

Thus comparable premice are initial segments of the same relative constructible hierarchy. In particular, if M and N contain all ordinals, then $M = N$.

Lemma. *If $\langle M, \in, U\rangle$, $\langle N, \in, W\rangle$ are iterable premice, then they have some comparable iterates.*

Proof. By the previous lemma: let $\nu > |\kappa^\kappa \cap M|, |\lambda^\lambda \cap N|$ (where U is an M-ultrafilter over κ, W is an N-ultrafilter over λ). □

Lemma. *Suppose $\langle L[U], \in, U\rangle$ is a κ-model, $B \subseteq \mathrm{Ord}$ with $|B| \geq \kappa^+$, and δ is a limit ordinal with $B \subseteq L_\delta[U]$. Then*

$$\mathcal{P}(\kappa) \cap L[U] \subseteq \mathrm{SkolemHull}_{\langle L_\delta[U],\in,U\rangle}(\kappa \cup B).$$

Here we are referring to the definable Skolem hull of $\kappa \cup B$ in $\langle L_\delta[U], \in, U\rangle$, as obtained using the definable wellordering of $L_\delta[U]$. The lemma asserts that for any $x \in \mathcal{P}(\kappa \cap L[U])$, there exists a formula φ and $\xi_1, \ldots, \xi_m \in \kappa$, $\eta_1, \ldots, \eta_n \in B$ such that x is unique satisfying

$$\langle L_\delta[U], \in, U\rangle \models \varphi(x, \xi_1, \ldots, \xi_m, \eta_1, \ldots, \eta_n)$$

Proof. Let $\langle H, \in, U \cap H\rangle$ be the Skolem Hull, $\langle N, \in, W\rangle$ its transitive collapse. Since $\kappa \subseteq H$, we have $\pi(x) = x$ for all $x \in \mathcal{P}(\lambda) \cap H$, so that $W = U \cap N$. Then $N = L_\zeta[U]$ for some $\zeta \geq \kappa^+$ (since $B \subseteq H$), so that by the proof of GCH above κ in $L[U]$, $\mathcal{P}(\kappa) \cap L[U] \subseteq N$. So $\mathcal{P}(\kappa) \cap L[U] \subseteq H$, as needed. □

Theorem. *Suppose $\langle L[U], \in, U\rangle$ and $\langle L[W], \in, W\rangle$ are κ-models. Then $U = W$ and so $L[U] = L[W]$.*

Proof. They have some comparable iterates, so they share a common iterate, say $\langle L[F], \in, F\rangle$. Let $i : L[U] \prec L[F]$, $j : L[W] \prec L[F]$ be the corresponding elementary embeddings.

Since $L[U]$ and $L[W]$ both satisfy GCH above κ, we have that all sufficiently large regular cardinals are strong limit with cofinality greater than κ in both $L[U]$ and $L[W]$. By a previous lemma,

$$\{\theta \in \mathrm{Ord}\ :\ \theta = i(\theta) = j(\theta)\}$$

for all such θ; so this is a proper class. Let B be a subset of this class with $|B| > \kappa^+$, and fix $\delta > \sup B$ with $i(\delta) = j(\delta) = \delta$.

Now if $X \in U$, then by the last lemma,

$$X \in \text{SkolemHull}_{\langle L_\delta[U], \in, U\rangle}(\kappa \cup B);$$

so for some formula φ, some $\xi_1, \ldots, x_n \in \kappa$, and some $\eta_1, \ldots, \eta_m \in B$, we have that X is the unique set satisfying $\varphi(X, \xi_1, \ldots, \xi_m, \eta_1, \ldots, \eta_n)$ in $L_\delta[U]$. Since i is elementary, and since i fixes all elements of κ, B, $i(X)$ is the unique set in $L[F]$ satisfying $\varphi(i(X), \xi_1, \ldots, \xi_m, \eta_1, \ldots, \eta_n)$ in $L_\delta[F]$. Then by elementarity of j, we can let Y be the unique set satisfying φ with these parameters in $L_\delta[W]$. But by the same argument, we must have that $j(Y)$ satisfies this formula in $L_\delta[F]$; so $i(X) = j(Y)$. But since X and Y are both subsets of κ, we have

$$X = i(X) \cap \kappa = j(Y) \cap \kappa = Y,$$

so that $X = Y \in W$. It follows that $U \subseteq W$; by symmetry, $W \subseteq U$. □

Corollary. *If $\langle L[U], \in, U\rangle$ is the κ-model, then U is the only normal ultrafilter over κ in $L[U]$.*

Theorem. *Let $\langle L[U], \in, U\rangle$ be the κ-model, and $\langle L[W], \in, W\rangle$ be the λ-model where $\kappa < \lambda$. Then the latter is an iterate of the former.*

Proof. Let $\langle L[U_\alpha], U_\alpha, \kappa_\alpha, i_{\alpha,\beta}\rangle_{\alpha\leq\beta\in\text{Ord}}$ be the iteration of $\langle L[U], \in, U\rangle$. Since the κ_β's form a club class, there is a unique β such that

$$\kappa_\beta \leq \lambda < \kappa_{\beta+1}.$$

If $\kappa_\beta = \lambda$, we're done by the previous corollary. So assume for a contradiction that $\kappa_\beta < \lambda < \kappa_{\beta+1}$. Let $L[F]$ be a common iterate, and let $i_{\beta,\gamma}, j$ be the corresponding embeddings for $L[U_\beta], L[W]$, respectively.

Let $B \cup \{\delta\}$ consist of cardinals fixed by $i_{\beta,\gamma}, j$ so that $|B| = \kappa_\beta^+$ and $\delta > \sup B$.

Now, since $\lambda \in L[U_{\beta+1}]$, we have some $f \in \kappa_\beta^{\kappa_\beta} \cap L[U_\beta]$ so that $\lambda = i_{\beta,\beta+1}(f)(\kappa_\beta)$. By the Skolem Hull lemma, there is some formula ψ and elements $\xi_1, \ldots, \xi_m \in \kappa_\beta$, $\eta_1, \ldots, \eta_n \in B$ so that f is the unique element satisfying $\varphi(f, \xi_1, \ldots, \xi_m, \eta_1, \ldots, \eta_n)$ in $L_\delta[U_\beta]$. Now since

$$\lambda = i_{\beta,\beta+1}(f)(\kappa_\beta),$$

it follows that λ is the unique element satisfying $\psi(\lambda, \xi_1, \ldots, \xi_m, \kappa_\beta, \eta_1, \ldots, \eta_n)$ in $L_\delta[U_{\beta+1}]$ for some formula ψ. Since $i_{\beta+1,\gamma}$ fixes all of κ_β, λ, and B, we apply $i_{\beta+1,\gamma}$ and see that λ is the unique element in $L[F]$ satisfying $\psi(\lambda, \xi_1, \ldots, \xi_m, \kappa_\beta, \eta_1, \ldots, \eta_n)$ in $L_\delta[F]$.

Now, applying elementarity of j, there is some $\mu \in L[W]$ satisfying this formula ψ with these parameters in $L_\delta[W]$. But then we have $j(\mu) = \lambda$; this contradicts $\text{cp}(j) = \lambda$. □

Lemma. *Let U be a normal measure on a cardinal κ, and let $\lambda < \kappa$. Suppose $(\mathcal{A}, \ldots)$ is an $\mathcal{L}$-structure with $|\mathcal{L}| \leq \lambda$, and $A \supset \kappa$. If $P \subseteq A$ has $|P| < \kappa$, and $X \subseteq A$ has $|X| \leq \lambda$, then there exists an elementary substructure $\mathcal{B} = (B, \ldots) \prec \mathcal{A}$ with $|B| = \kappa$, $|P \cap B| \leq \lambda$, $X \subseteq B$, and $B \cap \kappa \in U$.*

Proof. Adjoin to $\mathcal{L}$ a unary predicate with interpretation P and constants c_x for each $x \in X$. For each formula φ in this expanded language, let h_ϕ be a Skolem function for $(A, \ldots, P, x)_{x\in X}$. Adjoin symbols for these, and let $\mathcal{L}'$ be this language, with $\mathcal{A}'$ the expanded structure.

Since κ is normal, it is Ramsey, so there exists a set of indiscernibles $I \subseteq \kappa$ for $\mathcal{A}'$ with $I \in U$, by the usual argument.

Let B be the closure of I under the Skolem functions h_φ; then $\mathcal{B} \prec \mathcal{A}$, B has size κ, and $|P \cap B| < \kappa$; by indiscernibility, we get $|P \cap B| \leq \lambda$. $\square$

Theorem. *If $\langle L[U], \in, U\rangle$ is a κ-model, then $L[U] \models \mathrm{GCH}$.*

Proof. We already have GCH in $L[U]$ above κ. Arguing in $L[U]$, suppose for contradiction that $|\mathcal{P}(\lambda)| \geq \lambda^+$. Let X be the λ^+-th element in the wellorder $<_{L[U]}$, and let α be the least ordinal with $X \in L_\alpha$. Since each $L_\zeta[U]$ is a $<_{L[U]}$-initial segment of $L[U]$, we have $|\mathcal{P}(\lambda) \cap L_\alpha[U]| \geq \lambda^+$.

Let $\eta > \alpha$ be a cardinal so that $U \in L_\eta[U]$, and let $\mathcal{A} = (A, \in)$ with $A = L_\eta[D]$. Let $P = \mathcal{P}(\lambda) \cap A$. Since κ is measurable, we have $2^\lambda < \kappa$, so $|P| < \kappa$. By the previous lemma, there is an elementary $\mathcal{B} \prec \mathcal{A}$ so that $|P \cap B| \leq \lambda$, $\lambda \cup \{U, X, \alpha\} \subseteq B$, and $\kappa \cap B \in U$.

Let $\pi : B \to M$ be the collapse embedding. Then $M = L_\gamma[\pi(U)]$ for some γ. We claim $\pi(U) = M \cap U$. Since $\kappa \cap B \in U$, we have $|\kappa \cap B| = \kappa$, so $\pi(\kappa) = \kappa$. Since $\pi(\xi) \leq \xi$ for all $\xi < \kappa$, we have by normality that there exists $Z \in U$ with $\pi(\xi) = \xi$ for all $\xi \in Z$. If $Y \in U \cap B$, then $\pi(Y) \supseteq \pi(Y \cap Z) = Y \cap Z \in U$, so $\pi(Y) \in U$. Similarly, if $\pi(Y) \in U$ for $Y \in B$, then $\pi(Y) \cap Z \in U$, so that $\pi^{-1}(\pi(Y) \cap Z) = Y \cap Z \in U$; hence $Y \in U$. So $\pi(U) = U \cap M$.

We have that $M = L_\gamma[U]$, and since $\lambda \subseteq B$, every subset of λ is mapped to itself by π, so that $\mathcal{P}(\lambda) \cap B = \mathcal{P}(\lambda) \cap M$. Also, $\pi(X) = X \in L_\gamma[U]$, so $\alpha \leq \gamma$. But then

$$\lambda \geq |\mathcal{P}(\lambda) \cap B| = |\mathcal{P}(\lambda) \cap L_\gamma[U]| \geq |\mathcal{P}(\lambda) \cap L_\alpha[U]| \geq \lambda^+,$$

a contradiction. $\square$