

**NOTES ON MATH 290D: PARTICIPATING SEMINAR IN LOGIC:
COUNTABLE BOREL EQUIVALENCE RELATIONS
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1. SEPTEMBER 30 - GREG HJORTH

Definition. Let X be a Polish space. An equivalence relation E on X is *Borel* if it is Borel as a subset of $X \times X$. It is *countable* if every equivalence class is countable.

E.g. 1. $\text{id}(C) = \{(x, x) \mid x \in X\}$

E.g. 2. E on $2^{\mathbb{N}}$; $\vec{x} E_0 \vec{y}$ iff $\exists N \forall m > N (x_m = y_m)$

E.g. 3. For Γ a countable group acting in a Borel manner on X ,

$$E_{\Gamma} = \{(x, \gamma \cdot x) \mid \gamma \in \Gamma, x \in X\}$$

E.g. 4. $\Gamma \curvearrowright 2^{\Gamma}$ by $(\gamma \cdot f)(\sigma) = f(\gamma^{-1}\sigma)$. Let μ be the product measure; then this action is measure preserving and a.e. free ($\gamma \neq 1, x \in X \Rightarrow \gamma \cdot x \neq x$ [a.e.]).

E.g. 5. For $n \in \mathbb{N}$, $S(\mathbb{Q}^n) =$ subgroups of \mathbb{Q}^n ; $\cong_n =$ isomorphism on $S(\mathbb{Q}^n)$.

E.g. 6. (\mathbb{F}_2 the free group on 2 generators,) $\mathbb{F}_2 = \langle a, b \rangle$. $E_{\infty} = E_{\mathbb{F}_2}$ for $\mathbb{F} \curvearrowright 2^{\mathbb{F}_2}$.

Fact. If E is a countable Borel equivalence relation, then $E \leq_b E_{\infty}$.

Definition. E, F on X, Y , $E \leq_B F$ iff there is a Borel $f : X \rightarrow Y$ such that $\forall x_1, x_2 \in X$,

$$x_1 E x_2 \iff f(x_1) F f(x_2),$$

$X/E \hookrightarrow Y/F$.

Fact. ($\text{AD}^{L(\mathbb{R})}$) For E, F Borel (or even Δ_1^2) $E \leq_{L(\mathbb{R})} F$ iff $|X/E|_{L(\mathbb{R})} \leq |Y/F|_{L(\mathbb{R})}$.

E.g. 7. For $A, B \in 2^{\mathbb{N}} \sim \mathcal{P}(\mathbb{N})$, $A \equiv_T B$ if $A \leq_T B$ and $B \leq_T A$.

Question 1. Do there exist infinitely many countable Borel equivalence relations up to \leq_B ?

YES! But all known proofs use ideas outside logic.

Question 2. $\cong_1 \equiv_B E_0$, $E_0 <_B \cong_2$. Is $\cong_n <_B \cong_{n+1}$?

YES! Thomas uses deep facts from the theory of elementary groups.

Theorem (Harrington-Kechris-Louveau). *If E is Borel, then exactly one of*

- (I) $E \leq_B \text{id}(\mathbb{R})$, or
- (II) $E_0 \leq_B E$.

Question 3. Is $E_{\infty} \leq_B \equiv_T$?

Definition. Let E, F be equivalence relations on Polish spaces X, Y . A **reduction** of E to F is a map $f : X \rightarrow Y$ such that

$$x E y \iff f(x) F f(y).$$

If there's a Borel such f , write $E \leq_B F$; if there's a continuous such f , write $E \leq_c F$.

Definition. Let E be an equivalence relation on X . We say E has **perfectly many equivalence classes** if there's a perfect $A \subseteq X$ such that $x \neq y \in A \implies x \not E y$.

Proposition. *The following are equivalent:*

- (i) E has perfectly many classes.
- (ii) $\text{id}_Y \leq_c E$ for some uncountable Y .
- (iii) $\text{id}_Y \leq_B E$ for some uncountable Y .
- (iv) $\text{id}_Z \leq_B E$ for all uncountable Y .

Proof. (i) \implies (ii) (any perfect subset of X is Polish in its own right. Use inclusion map)

(ii) \implies (iii) \checkmark

(iii) \implies (iv) Because $\text{id}_Y \equiv_B \text{id}_Z$ since any two uncountable Polish spaces are Borel isomorphic.

(iv) \implies (iii) \checkmark

(iii) \implies (ii) By increasing the topology we can make the Borel reduction continuous.

(ii) \implies (i) Take a copy of Cantor space in Y , and look at $f[C]$. \square

Proposition. *The following are equivalent (for E a Borel equivalence relation):*

- (i) E has countably many equivalence classes.
- (ii) $E \leq_B \text{id}_\omega$.

Proof. (ii) \implies (i) \checkmark

(i) \implies (ii) Let $\theta_0, \dots, \theta_n, \dots$ be the equivalence classes: $f : X \rightarrow \omega$ by $f(x) = n \iff x \in \theta_n$. \checkmark \square

Theorem (Silver). *Let E be a Borel equivalence relation. Either:*

- (i) $E \leq_B \text{id}_\omega$.
- (ii) $\text{id}_{2^\omega} \leq_B E$.

[We will try to prove this without the technical machinery of Effective Descriptive Set Theory (ala Ben Miller).]

Theorem (Mycielski). *Let E be a meager equivalence relation on X . Then $\text{id}_{2^\omega} \leq_c E$.*

Proof. X must be perfect, otherwise $\langle x, x \rangle$ would be open for some X and then E couldn't be meager. Let $F_0 \subseteq F_1 \subseteq F_2 \subseteq \dots$ nowhere dense with $E \subseteq \bigcup_{n < \omega} F_n$.

Inductively construct nonempty basic open sets $\langle U_s \mid s \in 2^{<\omega} \rangle$ such that

- (a) $\text{diam}(U_s) \leq \frac{1}{2^{|s|}}$
- (b) $\overline{U_{s \smallfrown 0}}, \overline{U_{s \smallfrown 1}} \subseteq U_s, \overline{U_{s \smallfrown 0}} \cap \overline{U_{s \smallfrown 1}} = \emptyset$.
- (c) if $s \neq t, |s| = |t| = n + 1$, then

$$(U_s \times U_t) \cap F_n = \emptyset.$$

Assume inductively that U_s is defined, $|s| = n$. Let V_t , $|t| = n + 1$ satisfy (a), (b).

Notice ()*: If $W \times V$ is open, then there exist basic open $W' \times V' \subseteq W \times V$ such that $(W' \times V') \cap F_n = \emptyset$.

Let s_0, \dots, s_{k+1} , $k = 2^n$ enumerate 2^n . Apply (*) $\binom{k}{2}$ times.

Let $f : 2^\omega \rightarrow X$ be given by $f(x) =$ unique member of $\bigcap_{n \in \omega} U_{x|n}$. \square

Definition. A **graph** $\mathcal{G} \subseteq X \times X$ is a symmetric, irreflexive relation.

Let $I = \{s_0, \dots, s_n, \dots\} \subseteq 2^{<\omega}$ such that $|s_n| = n$ and such that I is dense ($\forall t \exists n t \subseteq s_n$). Then the KST-graph $\mathcal{G}_0 \subseteq 2^\omega \times 2^\omega$ consists of all edges

$$(s_n \widehat{i} x, s_n \bar{i} x)$$

where $\bar{0} = 1, \bar{1} = 0$.

Observation. \mathcal{G}_0 is Borel.

Fact. \mathcal{G}_0 is acyclic, and the connected components are precisely the E_0 -equivalence classes (exercise).

Proposition. *Suppose A is non-meager and has the Baire Property. Then $(A \times A) \cap \mathcal{G}_0 \neq \emptyset$.*

Proof. Let $s \in 2^{<\omega}$ be such that $A \cap N_s$ is comeager in N_s . Let $s \subseteq s_n$. It's enough to show there exists x such that $s_n \widehat{0} x, s_n \widehat{1} x \in A$. We have that

$$\{x \mid s_n \widehat{0} x \notin A\}, \{x \mid s_n \widehat{1} x \notin A\}$$

are meager (because e.g. $x \mapsto s_n \widehat{0} x$ is a homeomorphism of 2^ω into $N_{s_n \widehat{0}}$). Take x in neither set. \square

Corollary. *There's no $c : 2^\omega \rightarrow \omega$ a Borel ω -coloring of \mathcal{G}_0 (meaning $\langle x, y \rangle \in \mathcal{G}_0 \implies c(x) \neq c(y)$).*

Proof. Otherwise since $2^\omega = \bigcup_{n \in \omega} c^{-1}\{n\}$, some $c^{-1}\{n\}$ is nonmeager (and it's Borel, so has the Baire property). \square

Theorem (Kechris-Solecki-Todorćevic). *Let $\mathcal{G} \subseteq X \times X$ be a Borel graph. Then exactly one of*

- (i) \mathcal{G} is ω -colorable by a Borel function.
- (ii) There's a continuous map $\pi : 2^\omega \rightarrow X$ a homomorphism of \mathcal{G}_0 into \mathcal{G} , meaning

$$\langle x, y \rangle \in \mathcal{G}_0 \implies \langle \pi(x), \pi(y) \rangle \in \mathcal{G}.$$

Proof later.

Proof of Silver's Theorem. Let E be a Borel equivalence relation. Let $\mathcal{G} = E^c$. If $c : X \rightarrow \omega$ were a Borel coloring of \mathcal{G} , then $x \not E y$, $c(x) \neq c(y)$. So there can only be countably many classes.

The other possibility is there's $\pi : 2^\omega \rightarrow X$ a continuous homomorphism of \mathcal{G}_0 into \mathcal{G} . Use π to "reduce" E , i.e. let $F \subseteq 2^\omega$ be given by $\langle x, y \rangle \in F \iff \langle \pi(x), \pi(y) \rangle \in E$. $F \leq_B E$. Enough to show F is meager, because then by Mycielski, $\text{id}_{2^\omega} \leq_B F \leq_B E$.

By Kuratowski-Ulam, it's enough to show that each "slice" $F_x = [x]$ is meager. Otherwise some $[x]_F$ is non-meager, and has the Baire Property. We get $y, z \in [x]_F$ such that $\langle y, z \rangle \in \mathcal{G}_0$. Then $\langle \pi(y), \pi(z) \rangle \in \mathcal{G}$, i.e. $\pi(y) \not E \pi(z)$. That's a contradiction, because $\pi(y) E \pi(z)$ (both E -equivalent to $\pi(x)$). \square

Theorem (Kechris-Solecki-Todorćevic). *Let $\mathcal{G} \subseteq X \times X$ be a Borel graph. Then exactly one of the following holds:*

- (i) \mathcal{G} is Borel ω -colorable.
- (ii) There's a continuous $\pi : 2^\omega \rightarrow X$ such that $\langle x, y \rangle \in \mathcal{G}_0 \implies \langle \pi(x), \pi(y) \rangle \in \mathcal{G}$.

Definition. Let $R \subseteq X \times X$. Say $A_0, A_1 \subseteq X$. Call (A_0, A_1) **R -discrete** if $(A_0 \times A_1) \cap R = \emptyset$. Given $A \subseteq X$, we say A is **R -discrete** if (A, A) is R -discrete.

Proposition. *Suppose $R \subseteq X \times X$ is analytic, and (A_0, A_1) R -discrete analytic sets. Then there are Borel $A_0 \subseteq B_0$, $A_1 \subseteq B_1$, (B_0, B_1) R -discrete.*

Proof. First let $A'_0 = \{x_0 \in X \mid (\exists x_1 \in A_1)(x_0, x_1) \in R\} = \text{proj}_0[R \cap (X \times A_1)]$. Then A'_0 is analytic, and $A_0 \cap A'_0 = \emptyset$, so there exists a Borel $B_0 \supseteq A_0$, $B_0 \cap A'_0 = \emptyset$; it follows that (B_0, A_1) is R -discrete. By a symmetric argument, we obtain a Borel $B_1 \supseteq A_1$ so that (B_0, B_1) is R -discrete. \square

Corollary. *If $\mathcal{G} \subseteq X \times X$ analytic graph, $A \subseteq X$ \mathcal{G} -discrete analytic set, then there's Borel $B \supseteq A$ which is \mathcal{G} -discrete.*

Proof. Let $B_0, B_1 \supseteq A$, (B_0, B_1) \mathcal{G} -discrete. Let $B = B_0 \cap B_1$. \square

Fix $\mathcal{G} \subseteq X \times X$ Borel. $\text{dom}(\mathcal{G}) = \{x \mid (\exists y)\langle x, y \rangle \in \mathcal{G}\} \cup \{y \mid (\exists x)\langle x, y \rangle \in \mathcal{G}\}$. Fix continuous maps

$$\begin{aligned} \varphi_G : \omega^\omega &\rightarrow X \times X && \text{such that } \mathcal{G} = \varphi_G[\omega^\omega] \\ \varphi_X : \omega^\omega &\rightarrow X && \text{such that } \text{dom}(\mathcal{G}) = \varphi_X[\omega^\omega] \end{aligned}$$

Definition. A **global n -approximation** is a pair $p = (u^p, v^p)$ with

$$u^p : 2^n \rightarrow \omega^n, \quad v^p : 2^{<n} \rightarrow \omega^{<n}.$$

$G_n =$ all global n -approximations.

Given $p \in G_m, q \in G_n, m \leq n$, say $p \sqsubseteq q$ (read “ q extends p ”) if:

$$\begin{aligned} (\forall s \in 2^m)(\forall s' \in 2^n)s \subset s' &\implies u^p(s) \subseteq u^q(s'), \\ (\forall t \in 2^{<m})(\forall t' \in 2^{<n})|t'| - |t| = n - m, t \subseteq t' &\implies v^p(t) \subseteq v^q(t'). \end{aligned}$$

Definition. A **local n -approximation** is a pair $\ell = (f^\ell, g^\ell)$ with $f^\ell : 2^n \rightarrow \omega^\omega, g^\ell : 2^{<n} \rightarrow \omega^\omega$ such that whenever $t \in 2^{n-(k+1)}$ we have

$$\varphi_G(g^\ell(t)) = (\varphi_X(f^\ell(s_k \widehat{0} \smallfrown t)), \varphi_X(f^\ell(s_k \widehat{1} \smallfrown t))).$$

Say $L_n =$ all local n -approximations.

Definition. $\ell \in L_n$ is **compatible** with $p \in G_n$ if

$$\begin{aligned} u^p(s) &\subseteq f^\ell(s) \text{ for } s \in 2^n \\ v^p(t) &\subseteq g^\ell(t) \text{ for } t \in 2^{<n}. \end{aligned}$$

Definition. Say $\ell \in L_n$ is **compatible** with $Y \subseteq X$ if $\varphi_X(f^\ell[2^n]) \subseteq Y$. Given $p \in G_n, Y \subseteq X$, let $L_n(p, Y) =$ set of all local n -approximations compatible with both p and Y .

Definition. $p \in G_n$ is Y -**terminal** if $L_{n+1}(q, Y) = \emptyset$ whenever q is a one-step extension of p .

$$\begin{aligned} T(Y) &= \text{all } Y\text{-terminal global approximations;} \\ A(p, Y) &= \{\varphi_X(f^\ell(s_n)) \mid \ell \in L_n(p, Y)\} \subseteq Y. \end{aligned}$$

Lemma 1. *If $p \in G_n$ and $A(p, Y)$ is not \mathcal{G} -discrete, then $p \notin T(Y)$.*

Proof. Let $\ell_0, \ell_1 \in L_n(p, Y)$ with

$$(\varphi_X(f^{\ell_0}(s_n)), \varphi_X(f^{\ell_1}(s_n))) \in \mathcal{G}.$$

Take $x \in \omega^\omega$ such that $\varphi_G(x)$ is [this guy \uparrow].

Define $\ell \in L_{n+1}$ by: ($i \in \{0, 1\}$)

$$\begin{aligned} f^\ell(s \frown i) &= f^{\ell_i}(s) \\ g^\ell(t \frown i) &= g^{\ell_i}(t) \\ g^\ell(\emptyset) &= x. \end{aligned}$$

This *is* actually in L_{n+1} , because

$$\begin{aligned} \varphi_G(g^\ell(t \frown i)) &= \varphi_G(g^{\ell_i}(t)) = (\varphi_X(f^{\ell_i}(s_k \frown 0 \frown t)), \varphi_X(f^{\ell_i}(s_k \frown 1 \frown t))) \\ &= (\varphi_X(f^\ell(s_k \frown 0 \frown t \frown i)), \varphi_X(f^\ell(s_k \frown 1 \frown t \frown i))); \\ \varphi_G(g^\ell(\emptyset)) &= (\varphi_X(f^{\ell_0}(s_n)), \varphi_X(f^{\ell_1}(s_n))) \\ &= (\varphi_X(f^\ell(s_n \frown 0)), \varphi_X(f^\ell(s_n \frown 1))). \end{aligned}$$

□

So if $p \in T(Y)$, then $A(p, Y)$ is discrete, so we can get $B(p, Y) \supseteq A(p, Y)$ discrete and Borel.

Given Y , let $Y' = Y \setminus \bigcup\{B(p, Y) \mid p \in T(Y)\}$. Then if Y is Borel, so is Y' .

If $c : Y^c \rightarrow \omega \cdot \alpha$ (where $\alpha < \omega_1$) is a Borel coloring of $\mathcal{G} \cap (Y^c \times Y^c)$, then there's $c' : (Y')^c \rightarrow \omega \cdot (\alpha + 1)$ a Borel coloring of $\mathcal{G} \cap (Y'^c \times Y'^c)$.

Lemma 2. *Suppose $p \in G_n$, and every one-step extension of p is in $T(Y)$. Then $p \in T(Y')$.*

Proof. Suppose for contradiction that q is a one-step extension of p , and $\ell \in L_{n+1}(q, Y') \subseteq L_{n+1}(q, Y)$. Then $\varphi_X(f^\ell(s_{n+1})) \in B(q, Y)$.

$B(q, Y) \cap Y' = \emptyset$ as $q \in T(Y)$.

$\varphi_X(f^\ell(s_{n+1})) \notin Y'$, ridiculous. $\rightarrow\leftarrow$

□

Now recursively define (Y_α, c_α) ($\alpha < \omega_1$) where Y_α Borel, $c_\alpha : Y_\alpha^c \rightarrow \omega \cdot \alpha$ is a Borel coloring of $(Y_\alpha \times Y_\alpha) \cap \mathcal{G}$.

$$(Y_\alpha, c_\alpha) = \begin{cases} (X, \emptyset) & \text{if } \alpha = 0 \\ (Y'_\beta, c'_\beta) & \text{if } \alpha = \beta + 1 \\ \left(\bigcap_{\beta < \alpha} Y_\beta, \bigcup_{\beta < \alpha} c_\beta\right) & \text{if } \alpha \text{ a limit.} \end{cases}$$

Note. $Y \subseteq Z \implies T(Z) \subseteq T(Y)$. So $T(Y_0) \subseteq T(Y_1) \subseteq \dots \subseteq T(Y_\alpha) \subseteq \dots$ stabilizes. So get $T(Y_\alpha) = T(Y_{\alpha+1})$ for some α .

Let $p^0 = (\emptyset, \emptyset) \in G_0$ (the unique element of G_0).

Note. $\text{dom}(\mathcal{G}) \cap Y_\alpha \subseteq A(p^0, Y_\alpha)$ because if $z \in \text{dom}(\mathcal{G})$, let $z = \varphi_X(x)$. Set $f^\ell(\emptyset) = x; g^\ell = \emptyset$; then $\ell \in L(p^0, Y_\alpha)$, so $z \in A(p^0, Y_\alpha)$.

Case 1: $p^0 \in T(Y_\alpha)$. Then $A(p^0, Y_\alpha)$ is discrete; then $\text{dom}(\mathcal{G}) \cap Y_\alpha$ is, and easily extend c_α to a full coloring. \checkmark

Case 2: $p^0 \notin T(Y_\alpha)$. Then p^0 has a 1-step extension p^1 which isn't in $T(Y_{\alpha+1}) = T(Y_\alpha)$. Then p^1 has a one-step extension p^2 , etc.

Let $p^n = (u^n, v^n)$.

Define $\pi : 2^\omega \rightarrow \omega^\omega$ by $\pi(x) = \lim_{n \rightarrow \omega} u^n(x \upharpoonright n)$. Then one can check that $\varphi_X \circ \pi : 2^\omega \rightarrow X$ is the desired homomorphism.

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Theorem (Lusin-Novikov). *Let X, Y be Polish spaces, $B \subseteq X \times Y$ Borel. If $\forall x \in X B_x$ is countable, then there exist Borel $G_n \subset X \times Y$ with $B = \bigcup_{n \in \omega} G_n$ and $(x, y), (x, y') \in G_n \implies y = y'$ (the G_n are graphs of Borel partial functions).*

Proof. Define \mathcal{G} on B by

$$\langle (x, y), (x', y') \rangle \in \mathcal{G} \iff x = x' \text{ and } y \neq y'.$$

Suppose $\exists c : B \rightarrow \omega$ a Borel ω -coloring of \mathcal{G} . Then let $G_n = c^{-1}[n]$ for all $n \in \omega$. Clearly $B = \bigcup_{n \in \omega} G_n$, and $(x, y), (x, y') \in G_n \implies c(x, y) = c(x, y') \implies \langle (x, y), (x, y') \rangle \notin \mathcal{G} \implies y = y'$.

Suppose such a c does not exist. By KST, there's a continuous $\pi : 2^\omega \rightarrow B$ a homomorphism of \mathcal{G}_0 to \mathcal{G} .

Let $\varphi = \text{proj}_X \circ \pi$. Then $\varphi : 2^\omega \rightarrow X$ is continuous and constant on connected components of \mathcal{G}_0 ; hence is constant on E_0 -equivalence classes.

Fact: If $\varphi : 2^\omega \rightarrow X$ is constant on E_0 classes, then it is a constant function (on all of 2^ω).

Let x_0 be the constant value of φ . Then $\pi(f) \in B_{x_0}$ for all $f \in 2^\omega$. Since π is a homomorphism, we must have that $\langle f, g \rangle \in \mathcal{G}_0$ implies $\pi(f) \neq \pi(g)$. But then π furnishes a Borel ω -coloring of \mathcal{G}_0 , a contradiction. \square

Remark. Under the hypotheses of the theorem, B admits a Borel uniformization, and $\text{proj}_X(B)$ is Borel.

Theorem (Feldman-Moore). *Let E be a countable Borel equivalence relation on X Polish. Then there is a countable group G such that E is the orbit equivalence relation of a Borel action $G \curvearrowright X$.*

Proof. By Lusin-Novikov, $E = \bigcup_{n \in \omega} D_n$ where each D_n is the graph of a Borel partial function. Let

$$H_{mn} = D_n \cap \{(y, x) \mid (x, y) \in D_m\}.$$

Then $E = \bigcup_{m, n \in \omega} H_{mn}$ and each H_{mn} is the graph of an *injective* Borel partial function.

Let $\{U_k \times V_k\}_{k \in \omega}$ be a basis for $\{(x, y) \in X \times Y \mid x \neq y\}$ (note that then $U_k \cap V_k = \emptyset$ all k). Put

$$g_{mnk}(x) = \begin{cases} y & \text{if } (x, y) \text{ or } (y, x) \text{ is in } H_{mn} \cap (U_k \times V_k) \\ x & \text{otherwise.} \end{cases}$$

Let G be the group of automorphisms of X generated by the g_{mnk} . This group is countable and E is the orbit space of its action on X . \square

Definition. Given a Polish space X and countable group G , let X^G denote the set of maps from G to X with the product Borel structure. Let $G \curvearrowright X^G$ by

$$(g \cdot p)(h) = p(g^{-1}h)$$

for $p \in X^G$, and denote the corresponding orbit equivalence relation on X by E_G^X . Put

$$E_\omega := E_{\mathbb{F}_\omega}^{2^{\mathbb{N}}}, \quad E_\infty := E_{\mathbb{F}_2}^2$$

Definition. A countable Borel equivalence relation E is **universal** if $F \leq_B E$ for every countable Borel equivalence relation F .

Fact 1. E_ω is universal.

Proof. Let X be Polish and E a countable Borel equivalence relation on X . By Feldman-Moore, E is induced by the Borel action of a countable group G . Let $\varphi : \mathbb{F}_\omega \rightarrow G$ be a surjective homomorphism. Then E is equal to the orbit space of the induced action of \mathbb{F}_ω on X .

Let $\{U_i\}_{i \in \omega}$ be a sequence of Borel sets separating points in X . Define $f : X \rightarrow (2^{\mathbb{N}})^{\mathbb{F}_\omega}$ by $x \mapsto f_x$,

$$f_x(h)(i) = 1 \iff x \in h(U_i).$$

Then f is injective and

$$\begin{aligned} (g \cdot f_x)(h)(i) = 1 &\iff f_x(g^{-1}h)(i) = 1 \\ &\iff x \in g^{-1}h(U_i) \\ &\iff g \cdot x \in h(U_i) \\ &\iff f_{g \cdot x}(h)(i) = 1. \end{aligned}$$

Thus if $g \cdot x = y$, we have $g \cdot f_x = f_y$; conversely, if $g \cdot f_x = f_y$, then by injectivity $g \cdot x = y$. \square

In what follows, G, H are countable groups and X is a Polish space.

Fact 2. If $\varphi : H \rightarrow G$ is a surjective group homomorphism, then $E_G^X \leq_B E_H^X$.

Proof. Map $p \in X^G \mapsto p^* \in X^H$ by

$$p^* = p \circ \varphi.$$

Then $h \cdot p^* = (\varphi(h) \cdot p)^*$, so this is a Borel reduction. \square

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Fact 3. If $G \subseteq H$, then $E_G^X \leq_B E_H^X$.

Proof. Fix some $x_0 \in X$. Map $p \in X^G \mapsto p^* \in X^H$ by

$$p^*(h) = \begin{cases} p(h) & \text{if } h \in G \\ x_0 & \text{otherwise.} \end{cases}$$

If $g \in G$ then $g^{-1}h \in G \iff h \in G$, so $(g \cdot p)^* = g \cdot p^*$.

Suppose $a \notin G$ and $a \cdot p^* = q^*$. Then

$$h \in G \implies q^*(h) = a \cdot p^*(h) = p^*(a^{-1}h) = x_0 \implies q^* \equiv x_0 \equiv p^*.$$

So $p = q$. □

Fact 4. $E_G^{2^{\mathbb{Z}-\{0\}}} \leq_B E_{G \times \mathbb{Z}}^3$.

Proof. Let $p \in (2^{\mathbb{Z}-\{0\}})^G \mapsto p^* \in 3^{G \times \mathbb{Z}}$ by

$$p^*(g, n) = \begin{cases} p(g)(n) & \text{if } n \neq 0 \\ 2 & \text{if } n = 0. \end{cases}$$

Then $g \cdot p = q$ implies $(g, 0) \cdot p^* = q^*$. Conversely, if $(g, 0) \cdot p^* = q^*$, then $g \cdot p = q$.

Suppose $(g, n) \cdot p^* = q^*$, $n \neq 0$. Then $q(g_0)(n) = q^*(g_0, n) = (g, n) \cdot p^*(g_0, n) = p^*(g^{-1}g_0, 0) = 2$, a contradiction. □

Fact 5. $E_G^3 \leq_B E_{G \times \mathbb{Z}_2}^2$.

Proof. Let $p \in 3^G \mapsto p^* \in 2^{G \times \mathbb{Z}_2}$ by

$$p^*(g, i) = \begin{cases} 0 & \text{if } p(g) = 0 \\ 0 & \text{if } p(g) = 1, i = 0 \\ 1 & \text{if } p(g) = 1, i = 1 \\ 1 & \text{if } p(g) = 2. \end{cases}$$

It's easy to see $g \cdot p = q \iff (g, 0) \cdot p^* = q^*$.

Also, if $(g, 1) \cdot p^* = q^*$ and $q \in \{0, 2\}^G$, then clearly $g \cdot p = q$.

Suppose then that $(g, 1) \cdot p^* = q^*$ and that $q(g_0) = 1$ for some $g_0 \in G$. Then

$$p^*(g^{-1}g_0, 0) = (g, 1) \cdot p^*(g_0, 1) = q^*(g_0, 1) = 1,$$

so that we must have $p(g^{-1}g_0) = 2$; and

$$p^*(g^{-1}g_0, 1) = (g, 1) \cdot p^*(g_0, 0) = q^*(g_0, 0) = 0,$$

so that $p(g^{-1}g_0) = 0$, a contradiction. □

Theorem. E_∞ is universal.

Proof. Let E be a countable Borel equivalence relation. Then

$$\begin{aligned} E &\leq_B E_\omega = E_{\mathbb{F}_\omega}^{2^{\mathbb{N}}} && \text{(Fact 1)} \\ &\cong_B E_{\mathbb{F}_\omega}^{2^{\mathbb{Z}-\{0\}}} \\ &\leq_B E_{\mathbb{F}_2}^{2^{\mathbb{Z}-\{0\}}} && \text{(Fact 3)} \\ &\leq_B E_{\mathbb{F}_2 \times \mathbb{Z}}^3 && \text{(Fact 4)} \\ &\leq_B E_{\mathbb{F}_2 \times \mathbb{Z} \times \mathbb{Z}_2}^2 && \text{(Fact 5)} \\ &\leq_B E_{\mathbb{F}_\omega}^2 && \text{(Fact 2)} \\ &\leq_B E_{\mathbb{F}_2}^2 = E_\infty && \text{(Fact 3)} \end{aligned}$$

□

6. OCTOBER 27 - ANUSH TSERUNYAN

In what follows, “ $G \curvearrowright X$ ” means G acts on X in a *continuous* manner.

Definition. Let E be an equivalence relation on X . E is **smooth** if $E \leq_B \text{id}(2^\omega)$.

Example. Let E be the matrix similarity relation. Then E is smooth and it’s witnessed by Jordan canonical form.

Counterexample. E_0 on 2^ω .

Definition. A set is **E -invariant** if it is a union of E -equivalence classes. E is called **generically ergodic** if any Borel E -invariant set is either meager or comeager.

Lemma. *If E is generically ergodic and doesn’t have comeager orbits, then E is not smooth.*

Proof. Let $f : X \rightarrow 2^\omega$ witness the smoothness of E . Let $x \in 2^\omega$ be such that $f^{-1}(N_{X \upharpoonright n})$ is comeager for all $n \in \mathbb{N}$. But then $f^{-1}(x)$ is an equivalence class which is comeager (since it is the countable intersection of comeager sets), contradiction. \square

Lemma. *E_0 is generically ergodic.*

Proof. Let $A \subseteq 2^\omega$ be a Borel invariant non-meager set. We show A is comeager. If not, there exist N_s, N_t such that $N_s \Vdash A$ and $N_t \Vdash \sim A$. Let

$$\varphi(x) = \begin{cases} x & \text{if } s \sqsubseteq x, t \sqsubseteq x \\ t \hat{\ } y & \text{if } x = s \hat{\ } y \\ s \hat{\ } y & \text{if } x = t \hat{\ } y \end{cases}$$

φ is a homeomorphism. φ is E_0 -invariant, i.e. $x E_0 \varphi(x)$. $\varphi[A] = A$. $\varphi[N_s] = N_t$, $N_t = \varphi[N_s] \Vdash \varphi[A] = A \implies N_t \Vdash A$, contradicting $N_t \Vdash \sim A$ (Baire Category Theorem) \square

Theorem (Glimm). *If G is locally compact Polish, $G \curvearrowright X$, then either E_G is smooth or $E_0 \leq_c E_G$.*

Theorem (Effros). *If G is a Polish group, $G \curvearrowright X$, and E_G is F_σ , then $E_0 \leq E_G$ or E_G is smooth.*

Example. Closed $F \subseteq K \times X$, $\text{proj}(F)$ is closed. $a : G \times X \rightarrow X$.

$$\begin{aligned} x E_G y &\iff \exists g \, gx = y \\ &\iff (x, y) \in \text{proj}(\text{graph}(a)) \\ &\iff \exists n(x, y) \in \text{proj}(\text{graph}(a) \cap (K_n \times X)) \end{aligned}$$

(the latter closed).

Theorem (Becker-Kechris). *If $G \curvearrowright X$ such that every G_δ orbit is also F_σ , then E_G is smooth or $E_0 \leq_c E_G$.*

Note the F_σ condition in the above theorem cannot be dropped.

Theorem (Harrington-Kechris-Louveau). *If E is a Borel equivalence relation on X , then E is smooth or $E_0 \leq_c E$.*

Note: not true for Σ_1^1 relations.

Lemma. *If $G \curvearrowright X$ and satisfies the Becker-Kechris Thm, then the Vaught conjecture is true of $G \curvearrowright X$.*

Proof. $E_0 \leq E_G \implies \text{id}(2^\omega) \leq_c E_0 \leq_c E_G$. E_G is smooth \implies Burgess's Theorem: $\exists s : X \rightarrow X$ Borel selector for E . $B = s[X]$. But then the Perfect Set Theorem holds for Σ_1^1 . \square

Definition. A group is **cli** if it admits a complete left-invariant metric.

Theorem (Becker). *If G is a cli Polish group, $G \curvearrowright X$, then there exists a Polish topology on X (having the same Borel sets) keeping the action continuous, with all G_δ orbits being closed.*

\implies Vaught conjecture is true for cli groups.

7. NOVEMBER 3 - ANUSH TSERUNYAN

[Why do we care about G_δ sets?] By the definition of meager,

- (1) A is comeager $\iff A$ contains a dense G_δ subset.
- (2) If X is Polish, $Y \subseteq X$, then Y is Polish $\iff Y$ is G_δ .

Example. \mathbb{Q} is F_σ and dense, but not G_δ (by the Baire Category Theorem).

In general, if A is dense F_σ and A^c is dense, then A isn't G_δ .

Proposition 1. *Let $G \curvearrowright X$ so that every orbit is G_δ . Then E_G^X is smooth.*

Proof. $F(X) = \{F \subseteq X : F \text{ closed}\}$. Let τ be the topology generated by $\{F \subseteq X : F \cap U \neq \emptyset, U \text{ open}\}$.

By a result of Effros, there exists a Polish topology σ so that $B(\tau) = B(\sigma)$; τ is called **Borel-Effros Space**.

Any Polish X is isomorphic to some closed subset of $\mathbb{R}^{\mathbb{N}}$, say $F(X)$. Let $f : X \rightarrow F(X)$ by $x \mapsto \overline{[x]}$.

Claim. *f is a reduction from E_G^X to $\text{id}(F(X))$.*

Proof. Let $[x] \neq [y]$ and assume for contradiction that $\overline{[x]} = \overline{[y]}$. But then $[x], [y]$ are G_δ dense and disjoint, contradiction. \square

Claim. *f is Borel.*

Proof. Let U be open and $\theta = \{F \subseteq X : F \cap U \neq \emptyset\}$. Then

$$x \in f^{-1}[\theta] \iff \overline{[x]} \cap U \neq \emptyset \iff \cap [x] \cap U \neq \emptyset \iff x \in [U] = \bigcup_{g \in G} gU \text{ open.}$$

\square

[This completes the proof of Proposition 1.] \square

[Note - could also prove this in a fashion similar to showing every Polish space embeds into the Hilbert cube, i.e. fix a basis $\{\theta_n\}_{n \in \omega}$ and map $x \mapsto \{n \mid x \in \theta_n\}$.

Proposition 2. $G \curvearrowright X$ so that no orbit is G_δ and there exists a dense orbit. Then $E_0 \sqsubseteq_c E_G^X$.

Theorem* (Becker-Kechris). If E_G^X is meager and there exists a dense orbit, then $E_0 \sqsubseteq_c E_G^X$.

Theorem (Effros). Let $G \curvearrowright X$, $x \in X$. $[x]$ is G_δ iff $[x]$ is non-meager in its relative topology.

If $[x]$ is not G_δ , then $[x]$ is meager in itself; so $[x]$ is meager in X .

Proof of Proposition 2. Since no orbit is G_δ , every orbit is meager, by Effros's Theorem. By Kuratowski-Ulam, E_G^X is meager; then by Theorem*, $E_0 \sqsubseteq_c E_G^X$. \square

Proof of Becker-Kechris Theorem. Enough to prove either all orbits are G_δ or $E_0 \sqsubseteq_c E_G^X$.

Assume $[x]$ is not G_δ for some $x \in X$. We reduce to Proposition 2.

Let $Y = \{y \in X : \overline{[y]} = \overline{[x]}\}$; clearly invariant.

Claim. Y is G_δ .

Proof. Fix a countable base \mathcal{B} .

$$\begin{aligned} y \in Y &\iff \forall U \in \mathcal{B} [[x] \cap U \neq \emptyset \iff [y] \cap U \neq \emptyset] \\ &\iff (\forall U \in \mathcal{B} \text{ s.t. } [x] \cap U \neq \emptyset) y \in [U] = \bigcup_y gU \\ &\quad \& (\forall U \in \mathcal{B} \text{ s.t. } [x] \cap U = \emptyset) y \notin [U] \\ &\implies Y \text{ is } G_\delta. \end{aligned}$$

\square

Restrict the action of G to Y . $E_G^Y \sqsubseteq_c E_G^X$. So it's enough to show $E_0 \sqsubseteq_c E_G^Y$. Note that still $[x]$ is not G_δ .

Claim. No orbit of E_G^Y is G_δ .

Proof. If $[y]$ is G_δ then $[y] \cap [x] = \emptyset$. But by the hypothesis, $[y]$ is F_σ and $[y]$ is dense in Y (because $Y \subseteq \overline{[y]}$), also $[x]$ is dense in $Y \implies [y]^c$ is dense in Y . But then $[y]$ and $[y]^c$ are both comeager, contradiction. \square

$[x]$ is dense in Y and no orbit is G_δ ; so by Proposition 2, $E_0 \sqsubseteq_c E_G^Y \sqsubseteq_c E_G^X$. This completes the proof of Becker-Kechris. \square

Proof of Theorem.* Note that since E_G^X is meager, X is perfect. Let F_n be increasing nowhere dense closed such that $E \subseteq \bigcup_n F_n$. Construct $(U_s)_{s \in 2^{<\omega}}$, $(g_{s,t})_{s,t \in \omega^{<\omega}} \subseteq G$ such that

- (i) $U_{s \smallfrown 0} \cap U_{s \smallfrown 1} = \emptyset$, $\overline{U_{s \smallfrown 0}}, \overline{U_{s \smallfrown 1}} \subseteq U_s$, $\text{diam}(U_s) \leq 2^{-|s|}$.
- (ii) If $n = |s| = |t|$ and $s(n-1) \neq t(n-1)$, then $U_s \times U_t \cap F_n = \emptyset$.
- (iii) $U_t = g_{s,t} U_s$,
- (iii)' $g_{s,s} = 1_G$, $g_{s,t}^{-1} = \circ g_{t,s}$, $g_{s,t} = g_{u,t} \circ g_{s,u}$.
- (iv) If n is the last place where s and t differ, then $g_{s,t} = g_{s|n,t|n}$, $g_{s \smallfrown u, t \smallfrown u} = g_{s,t}$.

Claim. Granted this, we're done.

Proof. $f : 2^\omega \rightarrow X$ by $y \mapsto u$ where $[u] = \bigcap_{n \in \mathbb{N}} U_{y \upharpoonright n}$. Clearly f is continuous. Assume $x \not\equiv_0 y$. Then there exist infinitely many n s.t. $x \upharpoonright n \neq y \upharpoonright n \implies U_{x \upharpoonright n} \times U_{y \upharpoonright n} \cap F_n \neq \emptyset$. Since the F_n 's are increasing we have for all n , $U_{x \upharpoonright n} \times U_{y \upharpoonright n} \cap F_n \neq \emptyset \implies (f(x), f(y)) \notin E$. Assume

$$\begin{aligned} x \equiv_0 y &\implies \exists n \forall m \geq n \ s(m) = y(n) \\ &\implies \text{by (iv), } g_{x \upharpoonright m, y \upharpoonright m} = g_{x \upharpoonright n, y \upharpoonright n} =: g. \\ \forall m \geq n \ U_{y \upharpoonright m} &= g U_{x \upharpoonright m} \ni f(x) \\ &\implies \forall m \geq n \ g f(x) \in U_{y \upharpoonright m} \\ &\implies g f(x) \in \bigcap_{m \geq n} U_{y \upharpoonright m} = \{f(y)\} \\ &\implies g f(x) = f(y). \end{aligned}$$

□

□

8. NOVEMBER 10 - BILL CHEN

Notation. Let \equiv_T denote the Turing equivalence relation; \mathcal{P} the set of Turing degrees (\equiv_T -classes).

Properties. \equiv_T is a countable Borel equivalence relation.

\leq_T has countable initial segments; $|\mathcal{P}| = 2^{\aleph_0}$.

\leq_T is not a linear order (K-U).

For $x, y \in 2^\omega$, the **join** of x and y is $x \oplus y := (x(0), y(0), x(1), y(1), \dots)$; it is the \leq_T -least upper bound of x, y .

Question (Kechris). Is \equiv_T a universal countable Borel equivalence relation?

Definitions. $f : 2^\omega \rightarrow 2^\omega$ is **degree-invariant** if $x \equiv_T y \implies f(x) \equiv_T f(y)$.

A **cone** is $C_x := \{y \in 2^\omega \mid x \leq_T y\}$.

f is **constant on a cone** if $\exists x \in 2^\omega$ s.t. $\forall y \in C_x \ f(x) \equiv_T f(y)$.

f is **increasing on a cone** if $\exists x \in 2^\omega$ s.t. $\forall y \in C_x \ y \leq_T f(y)$.

Martin's Conjecture. *If $f : 2^\omega \rightarrow 2^\omega$ is Borel, degree-invariant, either*

- (i) f is constant on a cone
- (ii) f is increasing on a cone

Remark. Martin's Conjecture implies \equiv_T is not universal.

Proof. $\equiv_T^{X_1} \sqcup \equiv_T^{X_2} \not\leq_B \equiv_T$. Suppose f a Borel set. $f|_{X_1} : X_1 \rightarrow 2^\omega$; $f|_{X_1}$ is degree invariant \implies increasing on a cone. By

Theorem (Martin). *If $P \subseteq 2^\omega$ is Borel, degree-invariant, then either P contains a cone or P^c contains a cone.*

$f[X_1]$ is Borel, since f is countable one-to-one (by Lusin-Novikov). $[f[X_1]]$ (the saturation of $f[X_1]$; the smallest degree invariant B such that $B \supseteq f[X_1]$) is Borel, since \equiv_T is countable.

So $[f''X_1]$ (and similarly $[f''X_2]$) both contain a cone, contradicting f being a Borel reduction. □

Slaman-Steel (88):

Theorem 1. $f : 2^\omega \rightarrow 2^\omega$ Borel, degree-invariant, $f(x) <_T x$ on a cone. Then f is constant on a cone.

Theorem 2. Martin's Conjecture holds for f "uniformly degree-invariant".

Definition. Let $T \subseteq 2^\omega$ be a tree. T is **perfect** if any member of T has two incompatible extension (i.e., if $[T]$ is a perfect set).

T is **pointed** if it is perfect and $T \leq_T X$ for any $x \in [T]$.

Remark. T pointed $\implies T$ contains branches of every degree above T .

Theorem (Martin). $P \subseteq 2^\omega$ degree-invariant, Borel, cofinal (contains arbitrarily high Turing degrees). Then there exists a pointed tree T such that $[T] \subseteq P$.

Proof.
$$\frac{\text{I} \mid x(0) \quad x(1) \quad \dots \quad x \in 2^\omega}{\text{II} \mid y(0) \quad y(1) \quad \dots \quad y \in 2^\omega}$$
 II wins if $x \leq_T y$ and $y \in P$.

Claim. I does not have a winning strategy.

Proof of claim. Suppose otherwise; that s is a winning strategy for I. II plays $y \in P$ such that $s \leq_T y$, I plays $s(y)$. Then $s(y) \leq_T s \oplus y \equiv_T y$, contradicting s winning for I. \square

(By Borel determinacy,) II has a winning strategy; call it σ . Play against any x , choose x so that σ has to split. \square

9. NOVEMBER 17 - BILL CHEN

Definition. $x \equiv_T y$ via (i, j) if $\{i\}^x = y$ and $\{j\}^y = x$. $f : 2^\omega \rightarrow 2^\omega$ is **uniformly degree invariant** if (it's degree-invariant and) there exists $t : \omega^2 \rightarrow \omega^2$ and T a pointed tree such that if $x, y \in [T]$ and $x \equiv_T y$ via (i, j) , then $f(x) \equiv_T f(y)$ via $t(i, j)$.

Examples. Identity, constant, Turing jump.

$\varphi : 2^{<\omega} \rightarrow 2^{<\omega}$ is monotone if $s \subseteq t \implies \varphi(s) \subseteq \varphi(t)$.

Definition. Given φ a strategy for game, let $\varphi^*(x) := \bigcup_{n \in \omega} \varphi(x \upharpoonright n) \in 2^\omega$. φ is **proper** if $\lim_n \varphi(x \upharpoonright n) = \infty$ for all $x \in 2^\omega$.

Proposition. φ proper monotone $\implies \varphi^*$ is continuous.

$f : 2^\omega \rightarrow 2^\omega$ is continuous \implies there exists a proper monotone φ s.t. $f = \varphi^*$.

f is recursive if there is a φ such that $\varphi^* = f$ is recursive.

Definition. $x \leq_T y$ if there exists a recursive f such that $f(y) = x$.

$\sigma^*(x) \leq_T x, \sigma$ [?]

Proof. Theorem 2 (Assume Theorem 1): Fix f , let T be pointed witnessing f uniformly degree invariant. Consider the following game:

$$\frac{\text{I} \mid x(0) \quad x(1) \quad \dots \quad x \in \omega^\omega}{\text{II} \mid y(0) \quad y(1) \quad \dots \quad y \in \omega^\omega}$$

$x(0) = \langle e, n \rangle$; $x^-(i) = x(i+1)$. I loses unless $x^- \in [T]$ and $\{e\}^x = y$. If those happen, II wins iff $x \equiv_T y$ and $f(x^-)(n) = 0$ iff $y(y(0)) = 0$.

Suppose I has a winning strategy s . Let $z \in 2^\omega$ be such that $T \oplus s \leq_T z$. Will show: $z \leq_T f(z)$.

Let $\langle e_0, n_0 \rangle$ be the first play of I according to s . II plays $(m)^\frown z := y_m$. $y_m(y_n(0)) = y_m(m) = z(m-1)$. HOPE: $z(m-1) = 0$ iff $f(x_m)(n) \neq 0$.

$$\begin{array}{c|c} \text{I} & \langle e_0, n_0 \rangle^\frown x_m \\ \hline \text{II} & (m)^\frown z \end{array}$$

Let $x_m \in 2^\omega$ be such that $s(y_m) = \langle e_0, n_0 \rangle^\frown x_m$.

Want: $F : \omega^2 \rightarrow \omega^2$ by $m, n \mapsto f(x_m)(n)$ is recursive in $f(z)$.

Because $z(m-1) = f(x_m)(n_0) = F(m, n_0)$

Know: $\{e_0\}^{x_m} = y_m$ since s is winning. ...Find (a, b) s.t. $x_m \equiv_T x_{m+1}$ via (a, b) . (x_m, z) are all \equiv_T $f(x_m) \equiv_T f(x_{m+1})$ via $t(a, b)$; $(f(x_m))$ is recursive in $f(z)$, $z \leq_T f(z)$.

Suppose II has a winning strategy. Then take $z \in 2^\omega$, $\sigma, \tau \leq_T z$.

Show $f(z) \leq_T Z$.

I plays $\langle e, n \rangle^\frown z$, II responds with $y \leq_T z$. Suppose we can get I to meet first condition. Then $f(z)(n) = 0$ iff $y_n(y_n(0)) = 0$ (σ is winning) (This gives $f(z) \leq_T z$). By recursion theorem relative to z , find effectively $\{e\}^z = \sigma(\langle e, n \rangle^\frown z) = y_n$ \square

10. NOVEMBER 24 & DECEMBER 1 - QUINN MAURMANN HYPERFINITE BOREL EQUIVALENCE RELATIONS

All content is from Jackson-Dougherty-Kechris.

Definition. A Borel equivalence relation F on a Polish space X is *finite* if all its equivalence classes are finite; a Borel equivalence relation E is *hyperfinite* if it is the increasing union of a sequence of finite Borel equivalence relations, $E = \bigcup_{n \in \omega} F_n$ for $F_{n+1} \subseteq F_n$.

Remark 1. $F_{n+1} \subseteq F_n$ means F_{n+1} is “coarser,” i.e. larger equivalence classes, i.e. more points get identified.

Remark 2. That the union be increasing is necessary “by the proof of Feldman-Moore.”

Example 1. E_0 , the relation of eventual agreement on 2^ω .

Proof. Put $x F_n y$ iff x and y agree after place n , i.e. $(\forall m > n)x(m) = y(m)$. \square

Example 2. The orbit equivalence of $\mathbb{Z} \curvearrowright \mathbb{R}$ by shift.

Proof. For $x, y \in [-n, n[$, put $x F_n y$ iff $x - y \in \mathbb{Z}$; outside of this interval, all F_n -equivalence classes are singletons (we need to cut off equivalence classes to keep them finite). \square

Example 3. Generalization: if E admits a Borel selector B (meets each equivalence class once), then E is hyperfinite. (think of $B = [0, 1)$ in Example 2.)

Proof. Write E as the orbit-equivalence of a countable group $G = \{g_n\}_{n \in \omega}$ action. For each $x \in X$, let $N(x)$ be the least n such that $g_n \cdot x \in B$. Note that $x E y$ if and only if $g_{N(x)} \cdot x = g_{N(y)} \cdot y$. Put $x E_n y$ if and only if $x = y$ or $N(x), N(y) < n$ and $g_{N(x)} \cdot x = g_{N(y)} \cdot y$. \square

Remark. But note that Borel selectors can only exist for smooth relations! For E on X with Borel selector B , put $f : X \rightarrow X$ by $f(x) = y$ iff $x E y$ and $y \in B$, so that f reduces E to id_X .

Example 4. The orbit equivalence of $\mathbb{Q} \curvearrowright \mathbb{R}$ by shift (the ‘‘Vitali equivalence relation’’).

Proof. For $x, y \in [-n, n]$, put $x F_n y$ iff $x - y \in \mathbb{Z}/n!$ (we need to bound denominators to keep equivalence classes finite). \square

The rest of this talk will be dedicated to proving the following main theorem:

Theorem 3. *For a countable Borel equivalence relation E on X , the following are equivalent:*

- (1) E is hyperfinite.
- (2) E is the orbit equivalence of some Borel \mathbb{Z} -action on X , i.e. there is a Borel automorphism T of X such that $x E y$ iff $(\exists n \in \mathbb{Z})T^n(x) = y$.
- (2') There is a Borel relation on X^3 , denoted $x <_z y$ such that $<_z$ is a linear order of $[z]_E$ with order type either finite or \mathbb{Z} .
- (3) $E \leq_B E_0$ (thus by Harrington-Kechris-Louveau, all nonsmooth hyperfinite relations are bi-reducible with E_0).

Lemma 3. *Let E be a countable Borel equivalence relation on X .*

- (a) *If B is a Borel subset of $X^2 \times Y$, then the set*

$$A = \{(x, y) \in X \times Y : (\exists z \in [x]_E)(x, z, y) \in B\}$$

is also Borel.

- (b) *The sets $\{x \in X : [x]_E \text{ is finite}\}$ and $\{x \in X : [x]_E \text{ is infinite}\}$ are Borel.*

Proof. (a) Basically follows from the fact (Luzin-Novikov) that the countable-to-one Borel image of a Borel set is Borel. Adjoin an isolated point 0 to X , and define $f : X^2 \times Y \rightarrow X \times (X \cup \{0\}) \times Y$ by

$$f(x, z, y) = \begin{cases} (x, 0, y) & \text{if } z E x \\ (x, z, y) & \text{otherwise} \end{cases}$$

The set A can be identified with $f[B] \cap (X \times \{0\} \times Y)$, where $f[B]$ a countable-to-one image.

- (b) The condition ‘‘ $[x]_E$ has at least n elements’’ can be written

$$(\exists y_1 \in [x]_E) \dots (\exists y_n \in [x]_E)(y_i \neq y_j \text{ for } i \neq j).$$

\square

Proof of (2) \rightarrow (2'). Take $T : X \rightarrow X$ a Borel automorphism whose orbit-equivalence relation is E . If all equivalence classes are infinite, we can just put

$$x <_z y \iff x E z \wedge y E z \wedge (\exists n > 0)T^n(x) = y,$$

and then all classes have order type \mathbb{Z} . To fix things if there are finite equivalence classes, let \prec be *some* Borel linear order on X (e.g. by pushing through a Borel isomorphism $\mathbb{R} \cong X$), then put

$$x <_z y \iff x E z \wedge y E z \wedge \left(([z]_E \text{ finite} \wedge x \prec y) \vee ([z]_E \text{ infinite} \wedge (\exists n > 0) T^n(x) = y) \right),$$

which is Borel by part (b) of the preceding lemma. \square

Proof of (2') \rightarrow (2). Again, if all equivalence classes are infinite and so each $([x]_E, <_x)$ has order type \mathbb{Z} , then just let T shift inside each equivalence class,

$$\begin{aligned} T(x) = y &\iff y \text{ is the } <_x \text{-successor of } x \text{ in } [x]_E \\ &\iff x <_x y \wedge (\forall z \in [x]_E) \neg(x <_x z <_x y), \end{aligned}$$

so that the graph of T is Borel by part (a) of the preceding lemma. (If there is a finite equivalence class, just handle the wraparound in the obvious way.) \square

Proof of (1) \rightarrow (2'). Let E be the increasing union $\bigcup_{n \in \omega} F_n$ of finite Borel equivalence relations. Our immediate goal is to define a sequence of Borel relations $x <_z^n y$ such that $<_z^n$ is a linear order on $[z]_{F_n}$, and such that $([z]_{F_n}, <_z^n)$ is a “middle segment” of $([z]_{F_{n+1}}, <_z^{n+1})$.

Once we have this goal, we can define the Borel relation $x <_z y$ by taking the increasing union of the $x <_z^n y$ relations, and then $<_z$ is a linear order of $[z]_E$ with order type either finite, \mathbb{Z} , ω , or reversed- ω . Since successor-within-equivalence classes is a Borel function (by earlier parts of this proof), we can rearrange those classes with order type ω or reversed- ω , so that all infinite classes have type \mathbb{Z} . Thus our goal will yield (2').

For $n = 0$, let \prec be a Borel linear ordering of X , and let $<_z^0$ be $\prec \upharpoonright_{[z]_{F_0}}$ for all z .

Inductively, we use a lexicographic order on $[z]_{F_{n+1}}$.

Write $[z]_{F_{n+1}}$ as a disjoint union $[a_1]_{F_n} \cup \dots \cup [a_k]_{F_n}$, where the representative a_i is taken to be \prec -least in $[a_i]_{F_n}$, arranged so that $a_1 \prec \dots \prec a_k$. Now let $<_z^{n+1}$ rank elements of $[z]_{F_{n+1}}$ first by which class $[a_i]_{F_n}$ they appear in and second by $<_{a_i}^n$. Note that $[z]_{F_n}$ is a middle segment of $[z]_{F_{n+1}}$. Also, quantification over equivalence classes preserves being Borel, so we can freely speak of \prec -least elements. \square

Proof of (2') \rightarrow (1). Remark: if we had a Borel selector, we'd be done by Example 3; we'll try to do the next best thing.

By taking a Borel isomorphism, we can assume X is Cantor space 2^ω . Let \prec be the obvious lexicographic ordering on Cantor space; identifying 2^ω with Cantor's middle-thirds subset of \mathbb{R} , we see that \prec is the restriction of the usual ordering of \mathbb{R} to a closed set. Since the Cantor set is closed in \mathbb{R} (so that infimums can be taken in the Cantor set), \prec -infimums always exist in 2^ω .

Moreover, the map $z \mapsto \inf_{\prec} [z]_E$ is Borel, because

$$\inf_{\prec} [z]_E = x \iff (\forall y \in [z]_E) (x \preceq y) \wedge (\forall n > 0) (\exists y' \in [z]_E) (d(x, y') < 1/n).$$

Put $\alpha_z = \inf_{\prec} [z]_E$.

We now treat equivalence classes case-by-case (making sure the distinction between cases is Borel).

Case 1: For any equivalence class $[z]_E$ with $\alpha_z \in [z]_E$, we have identified in a Borel way a point from the class, so just let T send things in $[z]_E$ to their successors (if the class is finite, wrap around).

Now suppose $\alpha_z \notin [z]_E$; we can still *sometimes* pick a point from $[z]_E$ in a Borel way. Consider the sequence of basic open neighborhoods $N_{\alpha_z \upharpoonright n}$ shrinking to α_z , and let $S_n^z = N_{\alpha_z \upharpoonright n} \cap [z]_E$. Note that $S_0^z = [z]_E$, and that they shrink down to “nothing” $\bigcap_{n \in \omega} S_n^z = \{\alpha_z\} \cap [z]_E = \emptyset$.

Case 2: If there is any n such that S_n^z is $<_z$ -bounded above or below in $[z]_E$, then by choosing the least such n , and then taking the greatest/least element, we select a point out of $[z]_E$ in a Borel manner.

Case 3 (the general case): for all n , S_n^z is $<_z$ -unbounded above and below in $[z]_E$. The half-intervals $[a, b)$ (relative to $<_z$) with endpoints in S_n^z partition $[z]_E$, and *because* S_n^z is *unbounded*, all classes in the partition are finite. Because the S_n^z 's are nested downwards, shrinking to \emptyset , classes increase to $[z]_E$, so use these as the classes for F_n . \square

For (3) \rightarrow (1), we just need part (b) of the following lemma:

Lemma 4. *For E on X and F on Y ,*

- (a) *If $A \subseteq X$ is Borel and meets every E -class and $E \upharpoonright_A$ is hyperfinite, then E is hyperfinite (generalizes Borel selectors).*
- (b) *If $F \leq_B E$ and E is hyperfinite, then F is hyperfinite. (Note: if the reduction is finite-to-one, then the conclusion is obvious.)*

Proof. (a) Write $E = E_G$, for countable group action $G = \{g_n\}_{n \in \omega}$, and for each $x \in X$, let $N(x)$ be the least n such that $g_n \cdot x \in A$. Note that $x E y$ if and only if $g_{N(x)} \cdot x E \upharpoonright_A g_{N(y)} \cdot y$. Exhibiting $E \upharpoonright_A = \bigcup_{n \in \omega} F_n$ as hyperfinite, we put

$$x E y \iff N(x), N(y) \leq n \wedge g_{N(x)} \cdot x F_n g_{N(y)} \cdot y.$$

(b) Let $f : Y \rightarrow X$ be a Borel reduction of F to E . Note that f is countable-to-one, so $f[X]$ is Borel, and meets every E -class. Note $E \upharpoonright_{f[X]}$ is hyperfinite (it's a smaller relation). By Luzin-Novikov (?), take a “Borel inverse” $g : f[X] \rightarrow Y$ i.e. $g(x) \in f^{-1}(x)$ for each x . Then $g[f[X]]$ meets every F -class, is Borel (1-1 image of $f[X]$), and $F \upharpoonright_{g[f[X]}}$ is hyperfinite just because $E \upharpoonright_{f[X]}$ is. Then by (a), F is hyperfinite. \square

Proof of (1) \rightarrow (3). By (1) \rightarrow (2), any hyperfinite relation is the orbit relation of a Borel \mathbb{Z} -action on X (generated by a Borel automorphism $T : X \rightarrow X$), and hence is Borel reducible to the orbit equivalence relation $\mathbb{Z} \curvearrowright (2^\omega)^\mathbb{Z}$ by $k \cdot (x_n)_{n \in \mathbb{Z}} = (x_{n-k})_{n \in \mathbb{Z}}$. For the reduction, enumerate the basic open sets $(U_i)_{i \in \omega}$, and then map $a \in X$ to $(y_n)_{n \in \mathbb{Z}}$, where

$$y_n(i) = \begin{cases} 1 & \text{if } T^{-n}(a) \in U_i \\ 0 & \text{otherwise.} \end{cases}$$

(More generally any $G \curvearrowright X$ for countable G is Borel reducible to $G \curvearrowright (2^\omega)^G$ by roughly the same proof). Thus for (3) \rightarrow (1), it suffices to reduce the orbit relation E of $\mathbb{Z} \curvearrowright (2^\omega)^\mathbb{Z}$ to E_0 .

Note that we can break the problem into pieces: if there's a Borel E -invariant set $X \subseteq (2^\omega)^\mathbb{Z}$ such that $E \upharpoonright_X$ and $E \upharpoonright_{X^c}$ both admit Borel reductions f and g to E_0 , then we can reduce

E to E_0 by modifying f so that its image is in $(0, *, 0, *, \dots)$ and modifying g so that its image lies in $(1, *, 1, *, \dots)$. (In particular, for Borel E -invariant sets with Borel selectors.)

We think of an element $x = (x_n)_{n \in \mathbb{Z}}$ of $(2^\omega)^\mathbb{Z}$ as an array of 0's and 1's on the right half-lattice $\omega \times \mathbb{Z}$, with $(x_n)_{n \in \mathbb{Z}} E (y_n)_{n \in \mathbb{Z}}$ whenever there's a vertical shift of rows taking one to the other. Let $(2^n)^n$ be the set of $n \times n$ arrays of 0's and 1's, and say that $w \in (2^n)^n$ occurs in $x \in (2^\omega)^\mathbb{Z}$ (notation: $w \subseteq x$) if it appears as a left-aligned subarray of x .

Define orderings $<_n$ on $(2^n)^n$, which is coherent with restriction to the "initial" subarrays, i.e. for $k < n$ if $(w_1 \upharpoonright_k, \dots, w_k \upharpoonright_k) <_k (v_1 \upharpoonright_k, \dots, v_k \upharpoonright_k)$, then $(w_1 \upharpoonright_n, \dots, w_n \upharpoonright_n) <_n (v_1 \upharpoonright_n, \dots, v_n \upharpoonright_n)$. Let $f_n(x)$ be $<_n$ -least $w \in (2^n)^n$ which occurs in x ; then $f_{n+1}(x)$ always extends $f_n(x)$, so we can define their common extension $f(x) \in (2^\omega)^\omega$.

Consider the set of $x \in (2^\omega)^\mathbb{Z}$ such that there's some $w \in \bigcup_{n \in \omega} (2^n)^n$ which occurs in x but the set of positions (rows) of the occurrences is bounded either above or below. It is Borel, E -invariant (shift-invariant), and admits a Borel selector. Similarly for the set of x such that $f(x)$ occurs in x but the positions at which it does so are bounded below. For the set of x such that $f(x)$ occurs in x and the set of positions is unbounded below, we have that x is periodic in rows, hence $[x]_E$ is finite, so this set is also smooth and E -invariant with Borel selector.

Thus we can restrict our attention to the complement Y of these three parts, where:

- (a) "everything that happens happens unboundedly often," and
- (b) $f(x) \not\subseteq x$ i.e. "nothing that happens happens all at once."

Now define a sequence of positions

$$\begin{aligned} k_0^x &= 0 \\ k_{2n+1}^x &= \text{least } k > k_{2n}^x \text{ such that } f_{2n+1}(x) \text{ occurs at } k \\ k_{2n+2}^x &= \text{greatest } k < k_{2n+1}^x \text{ such that } f_{2n+2}(x) \text{ occurs at } k \end{aligned}$$

This is possible by (a). Note by coherence of the $f_n(x)$'s that

$$\dots \leq k_4^x \leq k_2^x < k_1^x \leq k_3^x \leq \dots,$$

and by (b), the two sides go off to $\pm\infty$.

We are (almost) ready to define the reduction of $E \upharpoonright_Y$ to E_0 . Suppose for $x, y \in Y$ that $x E y$, so $(x_{m+i})_{i \in \mathbb{Z}} = (y_i)_{i \in \mathbb{Z}}$ for some shift $m \in \mathbb{Z}$. Note that eventually both $|k_{2n}^x|$ and $|k_{2n+1}^x|$ are larger than $|m|$, at which point we have $k_{2n}^x = m + k_{2n}^y$ and $k_{2n+1}^x = m + k_{2n+1}^y$. Moreover for such n , the finite sequences

$$(x_{k_{2n}^x} \upharpoonright_n, \dots, x_{k_{2n+1}^x} \upharpoonright_n) \quad \text{and} \quad (y_{k_{2n}^y} \upharpoonright_n, \dots, y_{k_{2n+1}^y} \upharpoonright_n)$$

are equal. The converse is also true: if for $x, y \in Y$ the above two sequences are equal for all large n , then $x E y$.

Then define $G : Y \rightarrow (2^{<\omega})^\omega$ by $G(x) = (x_{k_{2n}^x} \upharpoonright_n, \dots, x_{k_{2n+1}^x} \upharpoonright_n)_{n \in \omega}$, so that $x E y$ if and only if $G(x)$ and $G(y)$ eventually agree. Finally, define $H : Y \rightarrow \mathcal{P}(\omega)$ so that $H(x) \subseteq \omega$ is the set of codes of the finite sequences appearing as entries in $G(x)$, so that $x E y$ iff $H(x) \Delta H(y)$ is finite. Identifying $\mathcal{P}(\omega)$ with 2^ω , H reduces $E \upharpoonright_Y$ to E_0 . \square