Radius of Convergence

(Throughout these notes I'll only refer to power series centered at 0 for convenience; I don't know whether or not you covered power series centered at arbitrary points in class. If you did, you are responsible for knowing about them.)

There seem to be two sources of confusion regarding radii of convergence of power series; there's confusion about the *definition* of the radius of convergence, and there is confusion about how to compute it. Needless to say, if you don't know what you're looking for, you'll probably have a hard time finding it, so we'll start with the definition. For motivation, think about it this way:

A power series $\sum c_n x^n$ is a function of x. We'd like to know where is is defined (i.e., not equal to ∞), and the following theorem/definition tells us just this:

Theorem 1 (Radius of Convergence)**.** *Given a power series*

$$\sum_{n=0}^{\infty} c_n x^n,$$

there exists a number R in $[0, \infty]$ such that $\sum c_n x^n$ converges absolutely for all x with $|x| < R$ and diverges for all x with $|x| > R$. This R is called the radius of convergence *of the series.*

A couple seconds' thought should convince you that $R = 0$ means that the power series does not converge for ANY x, while $R = \infty$ means that the power series converges for ALL x. You should also note that the theorem doesn't tell us anything about what happens when $|x| = R$; the series could either converge or diverge, and we have to test this on a case-by-case basis.

Heuristically, the radius of convergence is a measure of how large we can make x and still have the power series converge.

Now that we've got definitions out of the way, we want to know how to compute the radius of convergence. To do this, we simply note that a power series is, as the name suggests, a series:

$$\sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} a_n(x),$$

where $a_n(x) = c_n x^n$. We want to know for which x this series converges. Applying the Root Test or Ratio Test to the series $\sum a_n(x)$ can tell us exactly what we need. Here is an example.

The power series expansion for e^x is given by

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = \sum_{n=0}^{\infty} a_n(x), \;\; a_n(x) = \frac{x^n}{n!}$$

(see your homework). To figure out the radius of convergence for this power series, we use the Ratio Test. The series converges absolutely if

$$\lim_{n\to\infty} |\frac{a_{n+1}(x)}{a_n(x)}| \; < \; 1$$

But

$$\lim_{n\to\infty} |\frac{a_{n+1}(x)}{a_n(x)}| = \lim_{n\to\infty} |\frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}}| = \lim_{n\to\infty} |\frac{x}{n+1}| \;\; = 0 < 1$$

for ALL x. So the radius of convergence is ∞.

Note that it does not make sense to say something like "radius of convergence = all x"; the radius of convergence is a *number*. Also note that if the series converges absolutely for all x, the radius of convergence is definitely not 0. It's probably a good idea to check out further examples in your textbook.