Ch 6.1

#4. (a) Let x_i denote the number of objects in the i-th box. Then $x_1+x_2+x_3+x_4+x_5=r$ and $0 \le x_i \le 5$ for each i.

$\Rightarrow f(x) = (1+x+\cdots+x^5)^5$.

(b) $x_1+x_2+x_3+x_4=r$ and $3 \le x_i \le 6$.

$\Rightarrow f(x) = (x^3+x^4+x^5+x^6)^4$

#9. Let x_i denote the number of i-th element in the collection. Then we have $x_1+x_2+\cdots+x_n=r$ and $x_i \ge 0$ for all i. $\Rightarrow f(x) = (1+x+x^2+x^3+\cdots)^n = \dfrac{1}{(1-x)^n}$

#13. Let x_i denote the number on the top face of the i-th dice. Then $x_1+x_2+\cdots+x_n=r$ and $1 \le x_i \le 6$.

$\Rightarrow f(x) = (x+x^2+\cdots+x^6)^n$.

#16. Let x_i denote the i-th digit of the number. Note that any number between 0 and 999,999 has six digits $x_1, x_2, \ldots, x_6$ where $0 \le x_i \le 9$. $x_1+x_2+\cdots+x_6=r$

$\Rightarrow f(x) = (1+x+x^2+\cdots+x^9)^6$.

Ch 6.2

#4. $(x+x^2+\cdots+x^5)(x^2+x^3+x^4+\cdots)^5 = x(1+x+\cdots+x^4) \cdot x^{10}(1+x+x^2+\cdots)^5$

$= x^{11} \cdot \dfrac{1-x^5}{1-x} \cdot \dfrac{1}{(1-x)^5} = x^{11} \cdot \dfrac{1-x^5}{(1-x)^6} = x^{11}(1-x^5)\left(1+\binom{6}{1}x+\cdots+\binom{k+5}{k}x^k+\cdots\right)$

The coefficient of x^{18} is $\binom{7+5}{7} - \binom{2+5}{2} = \binom{12}{7} - \binom{7}{2}$.

#12. (a) $1+x^4+x^8+\cdots+x^{24} \overset{y=x^4}{=\!=} 1+y+y^2+\cdots+y^6 = \dfrac{1-y^7}{1-y} \overset{y=x^4}{=\!=} \dfrac{1-x^{28}}{1-x^4}$

(b) $x^{20}+x^{40}+\cdots+x^{180} \overset{y=x^{20}}{=\!=} y+y^2+\cdots+y^9 = y(1+y+\cdots+y^8) = y \cdot \dfrac{1-y^9}{1-y}$

$\overset{y=x^{20}}{=\!=} \dfrac{x^{20} \cdot (1-x^{180})}{1-x^{20}}$

#13. $(x^2+x^3+x^4+x^5)^2 = x^{10}(1+x+x^2+x^3)^2$. Each term has degree at least 10, thus there is no x^9 which implies that the coefficient of x^9 is 0.

#20. Let x_1, x_2, x_3, x_4 and x_5 denote the number of rooms painted green, blue, red, black and white, respectively. Then we get $x_1+x_2+x_3+x_4+x_5 = 10$ and $x_4, x_5 \geq 0$
$3 \geq x_1, x_2, x_3 \geq 0$.

$$\Rightarrow f(x) = (1+x+x^2+x^3)^3 \cdot (1+x+x^2+x^3+\cdots)^2 = \left(\frac{1-x^4}{1-x}\right)^3 \cdot \frac{1}{(1-x)^2} = \frac{(1-x^4)^3}{(1-x)^5}$$

$$= \left(1 - \binom{3}{1}x^4 + \binom{3}{1}x^8 - \binom{3}{3}x^{12}\right)\left(1 + \binom{5}{1}x + \cdots + \binom{k+4}{k}x^k + \cdots\right)$$

Then the coefficient of x^{10} is $1 \cdot \binom{10+4}{10} - \binom{3}{1}\cdot\binom{6+4}{6} + \binom{3}{2}\cdot\binom{2+4}{2}$

$$= \binom{14}{10} - 3\cdot\binom{10}{6} + 3\cdot\binom{6}{2}$$

Ch 6.3

#1. (a) 4, 3+1, 2+2, 2+1+1, 1+1+1+1

(b) 6, 5+1, 4+2, 4+1+1, 3+3, 3+2+1, 3+1+1+1, 2+2+2, 2+2+1+1, 2+1+1+1+1, 1+1+1+1+1+1

#3. Let x_i denote the number of is in the sum. Then for each i, $0 \leq x_i \leq 3$. $x_1 + 2x_2 + 3x_3 + \cdots + kx_k + \cdots = r$.

$$\Rightarrow f(x) = (1+x+x^2+x^3)(1 + x^2 + (x^2)^2 + (x^2)^3)\cdots(1 + x^k + (x^k)^2 + (x^k)^3)\cdots$$