Ch 5.3

3. (a) Each digit has 3 choices, {3,5,7}, thus there are 3^7 7-digit numbers formed with 3,5 and 7.
(b) The answers is $\frac{\text{\# of arrangements of three 3s, two 6s, and two 7s}}{3^7} = \frac{7!/(3!2!2!)}{3^7} = \frac{7!}{3!2!^2 3^7}$

5. Let $x_1, x_2, x_3,$ and x_4 be the number of picked pennies, nickels, dimes and quarters, respectively. Then $x_1 + x_2 + x_3 + x_4 = 8$ where $x_i \geq 0$ for $i = 1, 2, 3, 4$. # of integer solutions is $C(8+4-1, 8) = C(11, 8)$. OR by *Theorem* 2, the number of selections with repetition of r objects chosen from n types of objects is$C(r+n-1, r)$. For this problem $r = 8$ and $n = 4$.

21. First consider TH as one letter. Then the # arrangements of two Ms, two As, one TH, one T, one E, one I, one C and one S is $\frac{10!}{2!2!}$. So # of arr. of the letters in MATHEMATICS in which TH appear together is $\frac{10!}{2!2!}$. Now consider that THE is one another letter. Then the # of arr. of two Ms, two As, one THE, one E, one I, one C and one S is $\frac{9!}{2!2!}$ which is equal to the # of arr. of the letters in MATHEMATICS in which THE appear together. Note that we need to subtract the second result from the first one. Then the answers is $\frac{10!}{2!2!} - \frac{9!}{2!2!}$.

23. (a) We are asked to find the # of arr. of the letters in MISSISSIPPI which contain MI, that is # of arr. of four Ss, three Is, two Ps, and one MI, i.e. $\frac{101}{4!3!2!}$.
(b) We are asked to find the # of arr. of the letters in MISSISSIPPI which contain MI or IM. We should sum the # of arr. that contain MI and # of arr. that contain IM, and then subtract # of arr. that contain IMI since we would count such arrangements twice(in both case MI and IM).In a similar way, # of arr. that contain IM is again $\frac{101}{4!3!2!}$. But # of arr. that contain IMI is equal to the # of arr. of four Ss, two Is, two Ps,and one IMI,that is $\frac{91}{4!2!2!}$. Therefore the answer is 2x$\frac{101}{4!3!2!} - \frac{91}{4!2!2!}$.

Ch 5.4

1. (a) Let x_i denote the # of jelly beans that ith child gets. Then we have $x_1+x_2+x_3+x_4 = 36$ where $x_i \geq 0$ for each i. So the answer is $C(36+4-1, 36) = C(39, 36)$.
(b) Since the jelly beans are identical each child gets 9 jelly beans in a unique way.
(c) For this part we again have $x_1 + x_2 + x_3 + x_4 = 36$ but this time $x_i \geq 1$ for each i. Write $y_i = x_i - 1 \Longrightarrow y_1 + y_2 + y_3 + y_4 = 32$ and $y_i \geq 0$ for all i. So we get $C(32 + 4 - 1, 32) = C(35, 32)$.

12. (a)The # of nonnegative integer solutions to $x_1 + x_2 + x_3 + x_4 + x_5 = 28$ is $C(28 + 5 - 1, 28) = C(32, 28)$.
(b) $x_i > 0$ implies that $x_i \geq 0$, then substitute $y_i = x_i - 1$. Then we have $y_1 + y_2 + y_3 + y_4 + y_5 = 28 - 1 - 1 - 1 - 1 - 1 = 23$ where $y_i \geq 0$ for each i. The answer is $C(23 + 5 - 1, 23) = C(27, 23)$.
(c) Write $y_i = x_i - i$ for all $i \Longrightarrow y_1 + y_2 + y_3 + y_4 + y_5 = 28 - 1 - 2 - 3 - 4 - 5 = 13$ where $y_i \geq 0$ for all i. hence the answer is $C(13 + 5 - 1, 13) = c(17, 13)$.

18. (a) For the first ball there are n options, for the second ball there are $n-1$ choices because we can not put it into the box containing the first ball. The third ball has $n-2$ options and so on. After $k-1$ moves there are $n-k+1$ empty boxes, thus the kth ball has $n-k+1$ options. Hence the answer is $n(n-1)(n-2)...(n-k+1) = P(n,k)$. OR First choose k boxes that will contain a ball, then arrange k distinct balls among these boxes.So we get $C(n,k)k! = P(n,k)$. OR If we consider the ith box as ith place in an arrangement of $n-k$ zeros, $B_1, B_2, ..., B_k$, where B_j denotes the jth ball we get $\frac{(n-k+k)!}{k!} = \frac{n!}{k!} = P(n,k)$
(b) We just need to choose k boxes , and then we will put one ball in each of them. So the answers is $C(n,k)$.OR since each box contains at most one ball, the box is empty (E) or nonempty (N). Then we may assume ith box as the ith place in an arrangement of $n-k$ Es and k Ns. then we get $\frac{n!}{(n-k)!k!} = C(n,k)$. OR Let x_i denote the number of balls that ith box contains. Then $Xx_1 + x_2 + ... + x_n = k$ where $0 \leq x_i \leq 1$ for each i. generating function of this problem is $(1+x)^n$ and we want to find a_k, the coefficient of x^k. By Binomial Theorem $a_k = C(n,k)$.

37. (a) Any number between 0 and 9,999 can be considered as $x_1x_2x_3x_4$ where $9 \geq x_i \geq 0$. We are asked to find the # of nonnegative solutions to $x_1+x_2+x_3+x_4 = 7$, that is $C(7+4-1, 7) = C(10,7)$.
(b) If we write $x_5 = 7 - x_1 - x_2 - x_3 - x_4$, it is easy to see that there is a one to one correspondence between nonnegative solutions of $x_1 + x_2 + x_3 + x_4 \leq 7$ and $x_1 + x_2 + x_3 + x_4 + x_5 = 7$. thus we get $C(7+5-1, 7) = C(11,7)$ solutions. Note that also 10,000 satisfies the condition, hence the answer is $C(11,7)+1$.
(c) $x_1 + x_2 + x_3 + x_4 = 13$ and $0 \leq x_i \leq 9$ for each i. Then the generating function is $(1+x+x^2+...+x^9)^4$ and we need to find a_{13}, the coefficient of x^{13}. $(1+x+x^2+...+x^9)^4 = [\frac{1-x^{10}}{1-x}]^4 = \frac{1-4x^{10}+6x^{20}-4x^{30}+x^{40}}{(1-x)^4}$
$= (1-4x^{10}+6x^{20}-4x^{30}+x^{40})(1+C(1+3,1)x+C(2+3,3)x^2+...+C(k+3,k)x^k+...))$
Then that is not difficult to see that $a_{13} = 1\text{x}C(13+3,13) + (-4)\text{x}C(3+3,3) = C(16,3) - 4C(6,3)$.

Ch 5.5

7. We know that $C(m,k) + C(m,k-1) = C(m+1,k)$. substituting $m = 2n$ and $k = n$ yields $C(2n,n) + C(2n,n-1) = C(2n+1,n)$. Now letting $m = 2n+1$ and $k = n+1$ gives $C(2n+1,n+1) + C(2n+1,n) = C(2n+2,n+1)$. On the other hand since $n + (n+1) = 2n+1$ we have $C(2n+1,n+1) = C(2n+1,n)$. Then, $C(2n,n)+C(2n,n-1) = C(2n+1,n) = \frac{C(2n+1,n+1)+C(2n+1,n)}{2} = \frac{1}{2}C(2n+2,n+1)$

19. By identity (6), $C(m,0)+C(m,1)+C(m,2)+...+C(m,m-1)+C(m,m) = 2^m$. Thus $[C(n,0)+C(n,1)+...+C(n,n)]^2 = (2^n)^2 = 2^{2n}$. On the other hand by substituting $m = 2n$ we get $C(2n,0) + C(2n,1) + ... + C(2n,2n) = 2^{2n}$ which proves the assertion.