Fefferman’s Disc Multiplier Counterexample

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In this note we will consider the disc multiplier operators

\[ S_R f(x) = \int_{|\xi| \leq R} \hat{f}(\xi) e^{2\pi i x \cdot \xi} \, d\xi, \]

where \( f \in \mathcal{S} \) is a Schwartz function in \( \mathbb{R}^d \). The operator \( S_R f \) can be considered as a “partial sum,” or more precisely, “partial integral” of the Fourier transforms \( \hat{f}(\xi) \). Therefore, a natural question to ask is whether

\[ S_R f \to f \text{ in } L^p \text{ norm} \]

for \( f \in L^p \) as \( R \to \infty \). In dimension one, we have the \( L^p \) convergence for any \( 1 < p < \infty \). This follows from the \( L^p \) boundedness of the Hilbert transform. For dimension two or higher, the convergence only happens when \( p = 2 \). To see why it is true for \( p = 2 \), let us consider \( f \in L^2 \) and denote \( \mathcal{F} \) the Fourier transform operator. Then \( \mathcal{F}(f - S_R f) = \chi_{|\xi| > R} \hat{f} \). From Parseval’s theorem,

\[ \lim_{R \to \infty} \| S_R f - f \|_2 = \lim_{R \to \infty} \| \chi_{|\xi| > R} \hat{f} \|_2 = 0 \]

because \( \hat{f} \in L^2 \). In the late 70s, Fefferman provided a counterexample to show that the convergence does not hold for any \( p \neq 2 \). Before introducing the counterexample, let’s do some reductions to the problem.

**Part I. Reductions**

In this sequel, we always assume the dimension \( d > 1 \) and \( p \neq 2 \). Note that we have the convergence when \( f \) is a test function. If \( f \) is any \( L^2 \) function, we can find a test function \( \phi \) such that \( \| f - \phi \|_2 < \varepsilon \). Then we have

\[ \limsup_{R \to \infty} \| S_R f - f \|_p \leq \limsup_{R \to \infty} \| S_R (f - \phi) \|_p + \| f - \phi \|_p \leq \sup_{R} \| S_R \|_{L^p \to L^p} \cdot \varepsilon + \varepsilon. \]

So if \( \sup_{R} \| S_R \|_{L^p \to L^p} < \infty \), then taking \( \varepsilon \to 0 \) implies the convergence. Since each \( S_R \) has the same norm by scale invariance, this is equivalent to \( S_1 \) is bounded. Conversely, if \( S_1 \) is
unbounded, then none of the $S_R$ is bounded. By the Uniform boundedness principle, there is a $f \in L^2$ such that $\|S_R f\| = \infty$ for every $R > 0$. If follows from the triangle inequality that $\|S_R f - f\|_p = \infty$. Thus our first reduction is to determine whether $S_1$ is bounded.

The second reduction follows from the theorem of de Leeuw:

**Theorem 1: (de Leeuw)** If the operator $S_1$ is bounded on $L^p(\mathbb{R}^d)$, then it is bounded on $L^p(\mathbb{R}^{d-1})$ for any $d \geq 3$.

From this, we only consider the two dimensional disc multiplier $S_1$. From duality, we may further assume that $p > 2$. Indeed if $p < 2$ and $p^*$ is its conjugate, then we have

$$
\|S_1\|_{L^p \to L^{p^*}} = \sup_{\|f\|_p = 1} \sup_{\|g\|_{p^*} = 1} \langle S_1 f, g \rangle = \sup_{\|f\|_p = 1} \langle f, S_1 g \rangle = \|S_1\|_{L^{p^*} \to L^p}.
$$

The above equality follows from the fact that $S_1$ is self-adjoint. In summary, we are going to give a counterexample to the boundedness of $S_1$ in $\mathbb{R}^2$ with $p > 2$.

**Part II. The counterexample**

Let us consider the parameters $1 \ll a \ll R$ be large numbers. Let $\psi$ be the bump function supported in $[0, 1] \times [0, 1]$ and is identically equal to 1 in $[1/3, 2/3] \times [1/3, 2/3]$. Consider the function

$$
f(x_1, x_2) = e^{2\pi i x_1} \psi \left( \frac{x_1}{R^2}, \frac{x_2}{aR} \right).
$$

Then $f$ is supported on $[0, R^2] \times [0, aR]$ and the Fourier transform of $f$ is

$$
\hat{f}(\xi) = aR^3 \hat{\psi}(R^2(\xi_1 - 1), aR\xi_2).
$$

Since $\hat{\psi}$ is a Schwartz function, we see that $\hat{f}$ is concentrated in a small rectangle centered at $(1, 0)$. Therefore

$$
\widehat{S_1 f}(\xi) = \chi_{|\xi| \leq 1}(\xi) \hat{f}(\xi)
$$

is concentrated in the intersection of the unit ball and the small rectangle. This would suggest a splitting

$$
\widehat{S_1 f}(\xi) = \chi_{\xi_1 \leq 1}(\xi) \hat{f}(\xi) - \chi_{\xi_1 \leq 1, |\xi| \geq 1}(\xi) \hat{f}(\xi).
$$

Therefore we have

$$
S_1 f(x) = \underbrace{\int_{\xi_1 \leq 1} \hat{f}(\xi) e^{2\pi i x_1} d\xi}_{I} - \underbrace{\int_{|\xi| \geq 1} \hat{f}(\xi) e^{2\pi i x_1} d\xi}_{J}.
$$

Here $I$ is the main contribution and $J$ is an error term. For the error term, we estimate crudely by taking absolute value and splitting into two parts:

$$
J \leq \underbrace{\int_{\xi_1 \leq 0, |\xi| > 1} |\hat{f}(\xi)| d\xi}_{J_1} + \underbrace{\int_{0 < \xi_1 \leq 1, |\xi| > 1} |\hat{f}(\xi)| d\xi}_{J_2}.
$$
Since $\hat{\psi}$ is a Schwartz function, for $N$ arbitrarily large:

$$J_1 \lesssim \int_{-\infty}^{0} \int_{|\xi_2| > \sqrt{1-\xi_1^2}} \frac{aR^3}{(R^4(\xi_1 - 1)^2 + a^2R^2\xi_2^2)^N} \, d\xi_2 \, d\xi_1.$$  

Using AM-GM inequality for the denominator of the integrand and notice that

$$\int_{-\infty}^{0} \int_{|\xi_2| > \sqrt{1-\xi_1^2}} \frac{1}{(\xi_1 - 1)^N \xi_2^N} \, d\xi_2 \, d\xi_1 < \infty,$$

we see that $J_1 \lesssim \frac{1}{aR}$ (here we also use $1 < a < R$). Now for the second part of the error term, observe that if $0 \leq \xi_1 \leq 1$ and $|\xi| > 1$, then $\xi_1 > 1 - \xi_2^2$. Therefore we have

$$J_2 \leq \int_{1-\xi_2^2 \leq \xi_1 \leq 1} aR^3 |\hat{\psi}(R^2(\xi_1 - 1), aR\xi_2)| \, d\xi.$$  

Using change of variables $x = R^2(\xi_1 - 1)$, $y = aR\xi_2$ we obtain

$$J_2 \lesssim \int_{-\frac{2}{a} \leq x \leq 0} |\hat{\psi}(x, y)| \, dx \, dy.$$  

Using the fact that $\hat{\psi}$ is Schwartz again, so $|\hat{\psi}(x, y)| \lesssim \frac{1}{(1+y^2)^N}$ for large $N$. We can see from this that $J_2 \lesssim \frac{1}{a}$. It follows that the error term $J$ goes to 0 as $a, R \to \infty$.

For the main term, we look at its Fourier transform

$$\chi_{\xi_1 \leq 1}(\xi) \hat{f}(\xi) = aR^3 (\hat{\psi} \chi_H)(R^2(\xi_1 - 1), aR\xi_2),$$

where $H = \{(\xi_1, \xi_2) : \xi_1 \leq 1\}$. We take the inverse transform:

$$\mathcal{F}^{-1}(\chi_{\xi_1 \leq 1}(\xi) \hat{f}(\xi)) = e^{2\pi i x_1} (\psi * \mathcal{F}^{-1} \chi_H)(\frac{x_1}{R^2}, \frac{x_2}{aR}).$$

Write $\chi_H(\xi_1, \xi_2) = \frac{1}{2} - \frac{1}{2} \text{sgn}(\xi_1)$, we can take its inverse Fourier transform:

$$\mathcal{F}^{-1} \chi_H(x_1, x_2) = \frac{1}{2} \delta(x_1)\delta(x_2) - \frac{i}{2\pi x_1} \delta(x_2)$$

in the sense of distribution. Therefore we have

$$\psi * \mathcal{F}^{-1} \chi_H(x_1, x_2) = \frac{1}{2} \psi(x_1, x_2) - \frac{i}{2\pi} \text{p.v.} \int_\mathbb{R} \psi(x_1 - y, x_2) \frac{dy}{y}.$$  

Since $\psi$ is equal to 1 on $[1/3, 2/3] \times [1/3, 2/3]$ and supported on the unit square, we see that

$$|\psi * \mathcal{F}^{-1} \chi_H(x_1, x_2)| \sim 1$$
on the rectangle $[10, 11] \times [1/3, 2/3]$. So $I$ is comparable to 1 on the rectangle $[10R^2, 11R^2] \times [aR/3, 2aR/3]$. It follows that $|S_1f| \sim 1$ on this rectangle.

By rotation and translation the constructed function $f$ we obtain

**Lemma 1.** Let $T$ be any $R^2 \times aR$ rectangle in the plane. Define the *shifted rectangle* of $T$ to be a rectangle $T'$ with length $R^2$, width $\frac{aR}{3}$, obtained by shifting $T$ over by $10R^2$ units in the direction along its length. Then there is a smooth function $T$ supported on $T$ such that $|f_T| \leq 1$ on $T$ and $|S_1f_T| \sim 1$ on $T'$.

The key idea leads to this construction is the following lemma.

**Lemma 2.** (Besicovitch) For any $M > 0$, there is an $R > 0$ and a finite collection of $R^2 \times aR$ rectangles $T$ such that all the rectangles $T$ are disjoint and

$$\left| \bigcup T' \right| \leq M^{-1} \left| \bigcup T \right|.$$ 

That is, we want to rectangle $T'$ to overlap as much as possible. Note that $M, R, a$ can be rescaled to be larger so that the lemma still holds. From Lemma 1, we define the finite collection of smooth functions $f_T$ corresponding to Lemma 2. Let $\{\varepsilon_T\}$ be a finite collection of random signs random variables (i.e. $\varepsilon_T = 1$ or $-1$). Establish the sum

$$f = \sum_T \varepsilon_T f_T.$$ 

Since the rectangles $T$ are so spread out, we expect the $L^p$ norm of $f$ to be small. On the other hands, the rectangles $T'$ clump together, so the $L^p$ norm of $S_1f$ can be large. This is in fact a case. Since $f_T$ have disjoint supports and $\leq 1$ on $T$, we have

$$\|f\|_p = \left( \sum_T \|f_T\|_p \right)^{1/p} \lesssim \left| \bigcup_T \right|^{1/p}.$$ 

On the other hand, by Khinchin’s inequality:

$$E(\|S_1f\|_p^p) \sim \left( \sum_T |S_1f_T|^2 \right)^{1/2} \left| \bigcup_T \right|^{1/2}.$$ 

So there is at least one choice of $\varepsilon_T$’s such that we have $\gtrsim \text{sign}$. Since $|S_1f_T| \sim 1$ on $T'$, the RHS is comparable to

$$\| \sum_T \chi_{T'} \|_{p/2}.$$
By Holder’s inequality,
\[
\left\| \sum_T \chi_{T'} \right\|_1 \leq \left\| \sum_T \chi_{T'} \right\|_{p/2} \left\| \sum_T \chi_{T'} \right\|_{p/(p-2)}
\]

Note that \( p > 2 \). Using the hypothesis that \( \sum_T |T'| \sim |\bigcup_T T| \), we obtain the bound
\[
\left\| \sum_T \chi_{T'} \right\|_{p/2} \gtrsim M^{p/2-1} \left| \bigcup_T T \right|.
\]

It follows that
\[
\| S_1 f \|_p \gtrsim M^{p/2-1} \| f \|_p.
\]
Since \( M \) can be arbitrarily large, we are done.

**References**
