Problem 1. Find the following limits or show it does not exist:

(a) \( \lim_{(x,y) \to (0,0)} \frac{x^3}{x^3 + y^3} \)

(b) \( \lim_{(x,y) \to (0,-1)} \frac{xy^2}{(y+1)^2} \)

(c) \( \lim_{(x,y) \to (0,0)} \left[ (x^2 + y^2) \sin \left( \frac{x^5y^9 - \frac{1}{2}x^2\sin y}{xy} \right) \right] \)

(d) \( \lim_{(x,y) \to (0,0)} \frac{x^2 - y^2}{\sqrt{x^2 + y^2}} \)

(e) \( \lim_{(x,y) \to (0,0)} \frac{x^2 - y^2}{x^2 + y^2} \)

Problem 2.

(a) Show that the function

\[ f(x, y) = \begin{cases} \frac{xy}{x^2 + y^2}, & (x, y) \neq 0 \\ 0, & (x, y) = 0 \end{cases} \]

is NOT continuous at \((0, 0)\). Can we change the value of \(f(0, 0)\) so that it is continuous?

(b) Show that the function

\[ g(x, y) = \begin{cases} \frac{x^2y}{x^2 + y^2}, & (x, y) \neq 0 \\ 0, & (x, y) = 0 \end{cases} \]

is continuous at \((0, 0)\).
Problem 3.

(a) Show that for any number \(a, b \in \mathbb{R}\) we have
\[
a^2 + b^2 \geq 2|ab|.
\]
This is called AM-GM inequality (arithmetic mean - geometric mean).

(b) Use the inequality proved in part (a) to find the following limit
\[
\lim_{(x,y) \to (0,0)} \frac{x^3y^4}{x^2 + y^2}.
\]

Problem 4. Find the following limits

(a) \(\lim_{(x,y) \to 0} \frac{x^3 + y^3}{x + y}\)

(b) \(\lim_{(x,y) \to (1,2)} e^{-\frac{1}{(x-1)^2 + (y-2)^2}}\)

(c) \(\lim_{(x,y) \to (0,0)} e^{\frac{1}{x^2 + y^2}} \sqrt{x^2 + y^2}.\) (Hint: for \(r \geq 0\) we have \(e^r \geq r\))

Problem 5. Find the following partial derivatives

(a) \(f_{wwz}\) with \(f(z, w) = z^2 + \sin(w - 3e^{-w}) + w^2z.\)

(b) Let \(f(x, y) = \log(x^2 + y^2).\) This function is differentiable everywhere but \((0, 0)\). Show that it satisfies
\[
f_{xx} + f_{yy} = 0
\]
everywhere but \((0, 0)\). We say \(f\) solves the Laplace equation (usually we use \(\Delta f\) to denote \(f_{xx} + f_{yy}\)).