Math 32A Problem 15 Solution

Sections 15.8

December 8, 2018

Problem 15. Find the maximum and minimum of $f(x, y, z) = xy + xz$ with the constraint $x^2 + y^2 + z^2 = 4$.

Solution: Let $g(x, y, z) = x^2 + y^2 + z^2 - 4 = 0$ be the constraint surface. First we see that the extrema (max and min) both exist because the surface $x^2 + y^2 + z^2 = 4$ is closed and bounded (it is a sphere of radius 2). We have the Lagrange equations:

\begin{align*}
y + z &= 2x\lambda \quad (1) \\
x &= 2y\lambda \quad (2) \\
x &= 2z\lambda \quad (3)
\end{align*}

From (2) and (3) we see that $y\lambda = z\lambda$. Note that we cannot just cancel $\lambda$ both sides because it can be zero. So we have two cases:

Case 1: $\lambda \neq 0$. Note that from (2), we also have $x \neq 0$. So from (2) and (3) we can solve

\begin{equation}
y = z = \frac{x}{2\lambda} \quad (4)
\end{equation}

Plug this into (1):

\begin{equation}
x = 2x\lambda.
\end{equation}

Since $x \neq 0$, we can cancel $x$ both sides, yielding $\frac{1}{\lambda} = 2\lambda$. This can be solved for $\lambda = \pm \frac{1}{\sqrt{2}}$ (Do not forget $\pm$). So (4) becomes

\begin{equation}
y = z = \pm \frac{x}{\sqrt{2}} \quad (5)
\end{equation}

Plug this into the constraint $x^2 + y^2 + z^2 = 4$, we find:

\begin{equation}
\frac{x^2}{2} + \frac{x^2}{2} + x^2 = 4.
\end{equation}
Thus we find $x = \pm \sqrt{2}$. Plug this into (5) we find four critical points:

\[
\begin{align*}
x &= \sqrt{2}, \quad y = z = 1 \\
x &= \sqrt{2}, \quad y = z = -1 \\
x &= -\sqrt{2}, \quad y = z = 1 \\
x &= -\sqrt{2}, \quad y = z = -1
\end{align*}
\]

**Case 2:** $\lambda = 0$. Then from (2), we have $x = 0$. Thus from (1) and the constraint, we have the system:

\[
\begin{align*}
y + z &= 0 \\
y^2 + z^2 &= 4
\end{align*}
\]

You can solve this system, say by substitution or Vietta’s theorem, to find $(y, z) = (\sqrt{2}, -\sqrt{2})$ or $(y, z) = (-\sqrt{2}, \sqrt{2})$. So we find two more critical points:

\[
\begin{align*}
x &= 0, \quad y = \sqrt{2}, \quad z = -\sqrt{2} \\
x &= 0, \quad y = -\sqrt{2}, \quad z = \sqrt{2}
\end{align*}
\]

Compare the value of $f$ at these six critical points, we see that

\[
\max f = 2\sqrt{2}, \quad \min f = -2\sqrt{2}.
\]

**Some remarks:**

- You need to argue the existence of max/min first. This can be either from using Theorem 3 page 816 (like in this example) or some geometric properties of the domain or constraint. One typical example is to find extrema of, say $f(x, y) = x^2 + y^2$, with constraint $x + 2y = 1$. Geometrically, the constraint is just a line $x + 2y = 1$ and $f(x, y)$ is the distance from the origin to the point $(x, y)$ on that line. We see that, as $(x, y)$ is far away, the distance, $f(x, y)$, can be arbitrarily large. Therefore there is no global maximum for $f$. However, there is a global minimum, which is just the minimum distance from the origin to the line (by drawing perpendicular projection). Therefore we can conclude Lagrange multiplier method will give you the global minimum for $f$.

- There is no general algorithm to determine the existence of global max/min.

- Do not just assume $\lambda \neq 0$. In other words, do not divide by $\lambda$ if you are not sure it is zero or not. If you wish to divide something, always make sure the denominator is non-zero.

- If $x^2 = 1$, then $x = \pm 1$. Don’t forget the $\pm$ sign!