**Def:** A continuous function is **locally linear** if the graph looks flatter as we zoom in and becomes indistinguishable from the tangent plane.

**Equation of tangent plane:**

If \( f(x,y) \) is locally linear at \((a,b)\), then the equation of tangent plane is

\[
z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b).
\]

The normal vector to the tangent plane is

\[
\vec{n} = \langle f_x(a,b), f_y(a,b), -1 \rangle.
\]

Define \( L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) \)

\[ e(x,y) = f(x,y) - L(x,y) \quad \text{(error term)} \]
**Def:** \( f(x, y) \) is defined in a disc \( D \) containing \((a, b)\) and \( f_x(a, b) \) and \( f_y(a, b) \) exist.

\( f(x, y) \) is **differentiable** at \((a, b)\) if it is locally linear; i.e.

\[
\lim_{(x, y) \to (a, b)} \frac{e(x, y)}{ \sqrt{(x-a)^2 + (y-b)^2} } = 0.
\]

**Theorem:**

If \( f_x(x, y), f_y(x, y) \) exist and are continuous on a disc \( D \), then \( f(x, y) \) is differentiable on \( D \).

**Linear approximation:**

\( f(x, y) \) is differentiable at \((a, b)\). Then

\[
\begin{align*}
\Delta f &:= f(x, y) - f(a, b) \\
\end{align*}
\]

\[
\begin{align*}
\Delta f &\approx f_x(a, b) \Delta x + f_y(a, b) \Delta y
\end{align*}
\]
Def: When $f$ is differentiable, define the differential of $f$
\[ df = f_x \, dx + f_y \, dy. \]

Remark: Can generalize to 3D
\[ df = f_x \, dx + f_y \, dy + f_z \, dz. \]

Gradient (Generalize derivative in 1D)
\[ \nabla f = \langle f_x, f_y \rangle \]
or \[ = \langle f_x, f_y, f_z \rangle \]

- $\nabla (f + g) = \nabla f + \nabla g$
- $\nabla(cf) = c \, \nabla f$
- Gradient is a vector
- Chain rule: $F(t)$ is differentiable, then
\[ \nabla F(f(x,y,z)) = F'(f(x,y,z)) \cdot \nabla f \]
Chain rule of paths

If \( f \) and \( \vec{r}(t) \) are differentiable, then

\[
\frac{d}{dt} f(\vec{r}(t)) = \nabla f_{\vec{r}(t)} \cdot \vec{r}'(t)
\]

Directional derivation

\( \vec{u} = \langle h, k \rangle \), then

\[
D_{\vec{u}} f(a, b) = \lim_{t \to 0} \frac{f(a+ht, b+kt) - f(a, b)}{t}
\]

- \( D_{\vec{u}} f(P) \) is the rate of change of \( f \) per unit change in the horizontal direction of \( \vec{u} \).

- \( D_x f(x, y) = f_x(x, y) \)

- \( D_y f(x, y) = f_y(x, y) \)

So partial derivatives are just special cases of directional derivative.

- If \( \vec{u} \) is a unit vector,

\[
D_{\vec{u}} f(P) = \nabla f_p \cdot \vec{u}
\]

\( \nabla f_p \) this is dot product.
**Theorem**

Let \( P = (a, b, c) \) be a point on the surface given by equation \( F(x, y, z) = k \), and \( \nabla F_P \neq 0 \). Then \( \nabla F_P \) is the normal vector of the tangent plane at \( P \).

So the tangent plane has the equation:

\[
F_x(a, b, c)(x-a) + F_y(a, b, c)(y-b) + F_z(a, b, c)(z-c) = 0
\]

This generalizes the formula before when \( z = f(x, y) \).

Since we can define

\[
F(x, y, z) = z - f(x, y)
\]

and

\[
F(x, y, z) = 0.
\]

So \( \nabla F = \langle F_x, F_y, F_z \rangle = \langle -f_x, -f_y, 1 \rangle \).