Parametric equation:

- **Parametrization of a line:**
  \[
  \mathbf{r}(t) = \mathbf{O} + t \mathbf{v}
  \]

- **Parametrization of the circle:**
  \[
  x = a + R \cos \theta, \quad y = b + R \sin \theta, \quad 0 \leq \theta < 2\pi
  \]
  (center at \((a, b)\) and radius \(R\))

- **Parametrization of the ellipse:**
  \[
  \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1
  \]
  \[
  x(t) = a \cos t, \quad y(t) = b \sin t, \quad 0 \leq \theta \leq 2\pi
  \]

**Question:** how to know which geometric figure the parametric equation represents?

- If it does not match the above equations, we can try solve for \(t\) (in terms of \(x\) and \(y\)) and find the relationship between \(x\) and \(y\) (without \(t\) in it).
Ex: $x(t) = 1-t$, $y(t) = t^2$

$\rightarrow$ We can solve $t = 1-x$. Plug into the second equation gives

$y = (1-x)^2$.

This is the equation of a parabola.

**Question:** Is the parametrization of a curve unique?

$\rightarrow$ No! We have many choices.

**Slope of the tangent line.**

If $x = x(t)$, $y = y(t)$ are differentiable, then

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

where $dx/dt \neq 0$ and continuous

. **Note:** $dy/dt$, $dx/dt$ denote derivatives, not quotients.

. **Remark:** The easiest way to remember this rule is to «heuristically» see $dx/dt$, $dy/dt$ as quotients.
Area under a parametric curve:
Suppose we have a "nice" curve $c(t) = (x(t), y(t))$. The shaded area is

$$A = \int_{t_0}^{t_1} y(t) x'(t) \, dt.$$ 

**Question:** How nice is "nice"?
- $x(t)$, $y(t)$ are continuous and differentiable.

**Limit of vector valued function:***

$$\lim_{t \to t_0} \overrightarrow{r}(t) = \overrightarrow{u} \quad \text{if} \quad \lim_{t \to t_0} \| \overrightarrow{r}(t) - \overrightarrow{u} \| = 0.$$ 

1. Algebraically, we can compute:

$$\lim_{t \to t_0} \overrightarrow{r}(t) = \left< \lim_{t \to t_0} x(t), \lim_{t \to t_0} y(t), \lim_{t \to t_0} z(t) \right>$$

2. $\overrightarrow{r}(t)$ is continuous at $t_0$ if

$$\lim_{t \to t_0} \overrightarrow{r}(t) = \overrightarrow{r}(t_0).$$

**Ex:** $\overrightarrow{r}(t) = \left< \frac{1}{t}, 1, t \right>$ is not continuous at $t=0$ but continuous at every other point.
**Derivatives:**

\[
r'(t) = \frac{d}{dt} r(t) = \lim_{h \to 0} \frac{r(t+h) - r(t)}{h}
\]

Can also compute:

\[
r'(t) = \langle x'(t), y'(t), z'(t) \rangle
\]

**Ex:**

\[
\frac{d}{dt} \langle 2t, t^2, 1-t \rangle = \langle \frac{d}{dt} (2t), \frac{d}{dt} (t^2), \frac{d}{dt} (1-t) \rangle
\]

\[
= \langle 2, 2t, -1 \rangle
\]

**Differentiation rules:**

- \((\vec{r}_1 + \vec{r}_2)' = \vec{r}_1' + \vec{r}_2'\)

- \((\lambda \vec{r})' = \lambda \vec{r}'\) \quad (\lambda \text{ is a scalar})

- \((f \cdot \vec{r})' = f' \cdot \vec{r} + f \cdot \vec{r}'\) \quad (f \text{ is a differentiable scalar valued function})

- \(\frac{d}{dt} \vec{r}(g(t)) = \vec{r}'(g(t)) \cdot g'(t)\) \quad (Chain rule)

**Product rules:**

\(\vec{r}_1, \vec{r}_2\) are differentiable

- \(\frac{d}{dt} (\vec{r}_1(t) \cdot \vec{r}_2(t)) = \vec{r}_1'(t) \cdot \vec{r}_2(t) + \vec{r}_1(t) \cdot \vec{r}_2'(t)\)

- \(\frac{d}{dt} (\vec{r}_1(t) \times \vec{r}_2(t)) = (\vec{r}_1''(t) \times \vec{r}_2(t)) + (\vec{r}_1(t) \times \vec{r}_2''(t))\)
The derivative as a tangent vector:

Suppose we have a curve \( \vec{r}(t) \) parametrized by \( \vec{r}(t) \). Then the vector \( \vec{r}'(t_0) \) represents a tangent vector to \( \vec{r}(t) \) at the point \( t = t_0 \).

If \( \vec{r}(t) \) represents an orbit of a moving particle, then \( \vec{r}'(t_0) \) is the velocity of the particle at time \( t = t_0 \).

Therefore we also call \( \vec{r}'(t) \) the velocity vector.

Equation of tangent line:

Suppose we need to find the equation of the tangent line \( L(t) \) at \( \vec{r}(t_0) \). We know that \( \vec{r}'(t_0) \) points to the same direction as \( L \). From the previous lectures, we obtain:

\[
L(t) = \vec{r}(t_0) + t \cdot \vec{r}'(t_0).
\]
**Theorem:** If \( \| \vec{r}(t) \| = \text{constant} \), then \( \vec{r}(t) \perp \vec{r}'(t) \) at all points.

\[ 0 = \frac{d}{dt} \| \vec{r}(t) \|^2 = \frac{d}{dt} (\vec{r}(t) \cdot \vec{r}(t)) = 2 \vec{r}(t) \cdot \vec{r}'(t). \]

This shows that \( \vec{r}(t) \cdot \vec{r}'(t) = 0 \); equivalently we have \( \vec{r}(t) \perp \vec{r}'(t) \).

**Integration:** is also component-wise.

\[ \int_a^b \vec{r}(t) \, dt = \left< \int_a^b x(t) \, dt, \int_a^b y(t) \, dt, \int_a^b z(t) \, dt \right> \]

**Theorem:** If \( \vec{r}_1 \) and \( \vec{r}_2 \) are differentiable, and \( \vec{r}_1' = \vec{r}_2' \), then

\[ \vec{r}_1 = \vec{r}_2 + \vec{C} \]

for some constant vector \( \vec{C} \).
 Fundamental theorem of calculus:

If \( \vec{r}(t) \) is continuous on \([a, b]\); \( \vec{R}(t) \) is an anti-derivative of \( \vec{r}(t) \), then

\[
\int_a^b \vec{r}(t) \, dt = \vec{R}(b) - \vec{R}(a).
\]

- **Rmk:** In other words,

\[
\int_a^b \frac{d}{dt} \vec{R}(t) \, dt = \vec{R}(b) - \vec{R}(a).
\]

- **Rmk:** Choice of \( \vec{R}(t) \) does not matter thanks to the above theorem. (Why?)